UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

GENERALIZATIONS OF BUMBY'S THEOREM TO PURE-INJECTIVE AND RD-INYECTIVE MODULES

PRESENTA

Luz María Gurrola Ramos

PARA OBTENER EL GRADO DE MAESTRA EN CIENCIAS CON OPCIÓN MATEMÁTICAS APLICADAS

TUTOR

Dr. Jorge Eduardo Macías-Díaz

COMITÉ TUTORAL

Dr. José Villa Morales
Dr. Manuel Ramírez Aranda

Aguascalientes, Ags., Agosto de 2016

UNIVERSIDAD AUTONOMA
DE AGUASCALIENTES

LUZ MARÍA GURROLA RAMOS
MAESTRÍA EN CIENCIAS: CON OPCIÓN
A LAS MATEMÁTICAS APLICADAS
P R E S E N T E.

Estimada alumna:

Por medio de este conducto me permito comunicar a Usted que habiendo recibido los votos aprobatorios de los revisores de su trabajo de tesis y/o caso práctico titulado: **"GENERALIZATIONS OF BUMBY'S THEOREM TO PURE-INJECTIVE AND RD-INYECTIVE MODULES",** hago de su conocimiento que puede imprimir dicho documento y continuar con los trámites para la presentación de su examen de grado.

Sin otro particular me permito saludarle muy afectuosamente.

A T E N T A M E N T E
Aguascalientes, Ags., a 28 de julio de 2016
"Se lumen proferre"
EL DECANO

M. en C. JOSÉ DE JESÚS RUÍZ GALLEGOS

c.c.p.- Archivo.

UNIVERSIDAD AUTONOMA
DE AGUASCALIENTES

FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
DECANO DEL CENTRO DE CIENCIAS BÁSICAS
PRESENTE

Por medio de la presente, en mi calidad de tutor designado del estudiante **LUZ MARIA GURROLA RAMOS** con ID 128620 quien realizó la tesis titulada: **GENERALIZATIONS OF BUMBY'S THEOREM TO PURE-INJECTIVE AND RD-INYECTIVE MODULES**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y, sin otro particular por el momento, me permito enviarle un cordial saludo.

ATENTAMENTE
"Se Lumen Proferre"
Aguascalientes, Ags., a 26 de Julio de 2016

Dr. Jorge Eduardo Macías-Díaz

c.c.p.- Interesado
c.c.p.- Secretaría de Investigación y Posgrado
c.c.p.- Jefatura del Depto. de Matemáticas y Física
c.c.p.- Consejero Académico
c.c.p.- Minuta Secretario Técnico

UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
DECANO DEL CENTRO DE CIENCIAS BÁSICAS
PRESENTE

Por medio de la presente, en mi calidad de tutor designado del estudiante **LUZ MARIA GURROLA RAMOS** con ID 128620 quien realizó la tesis titulada: **GENERALIZATIONS OF BUMBY'S THEOREM TO PURE-INJECTIVE AND RD-INYECTIVE MODULES**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y, sin otro particular por el momento, me permito enviarle un cordial saludo.

ATENTAMENTE
"Se Lumen Proferre"
Aguascalientes, Ags., a 26 de Julio de 2016

Dr. José Villa Morales

c.c.p.- Interesado
c.c.p.- Secretaría de Investigación y Posgrado
c.c.p.- Jefatura del Depto. de Matemáticas y Física
c.c.p.- Consejero Académico
c.c.p.- Minuta Secretario Técnico

UNIVERSIDAD AUTONOMA
DE AGUASCALIENTES

FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
DECANO DEL CENTRO DE CIENCIAS BÁSICAS
PRESENTE

Por medio de la presente, en mi calidad de tutor designado del estudiante **LUZ MARIA GURROLA RAMOS** con ID 128620 quien realizó la tesis titulada: **GENERALIZATIONS OF BUMBY'S THEOREM TO PURE-INJECTIVE AND RD-INYECTIVE MODULES**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y, sin otro particular por el momento, me permito enviarle un cordial saludo.

ATENTAMENTE
"Se Lumen Proferre"
Aguascalientes, Ags., a 26 de Julio de 2016

Dr. Manuel Ramírez Aranda

c.c.p.- Interesado
c.c.p.- Secretaría de Investigación y Posgrado
c.c.p.- Jefatura del Depto. de Matemáticas y Física
c.c.p.- Consejero Académico
c.c.p.- Minuta Secretario Técnico

Agradecimientos

Primeramente agradezco a **DIOS** por darme la fortaleza de terminar este importante proyecto de vida.

Quiero agradecer a mis padres, por ser los principales promotores de mis sueños, han creído en mí y me han mostrado apoyo incondicional a lo largo de toda mi vida, lo hicieron cuando decidí que estudiaría matemáticas, también lo hicieron al inicio de este proyecto. Estoy profundamente agradecida por sus enseñanzas, su paciencia, sus consejos, por siempre animarme a seguir adelante y estar siempre que los necesito. Porque sé que gracias a ustedes pude culminar esta meta.

Quiero agradecer muy especialmente a mi tutor, el Dr. Jorge Eduardo Macías Díaz, por ser un gran guía a lo largo de este proyecto, su paciencia, confianza, consejos, conocimientos, enseñanzas que compartió conmigo para la realización de mi tesis, por creer en mí y mostrarme su apoyo incondicional a lo largo de todo el posgrado y más aún en la estancia en el extranjero. Gracias a él he aprendido de varias áreas de matemáticas, no solo en el posgrado sino también desde la licenciatura, que Dios lo bendiga a él y a su familia, que todos sus proyectos se realicen con satisfacción.

Quiero Agradecer muy especialmente al Dr. Thomas Kucera, por su apoyo invaluable en mi estancia en el extranjero, sus consejos, sus enseñanzas y su guía en esa bonita aventura que me toco emprender.

Agradezco al Dr. José Villa Morales y al Dr. Manuel Ramírez Aranda por su disponibilidad, atención y sugerencias en cada seminario.

Agradezco a la Universidad Autónoma de Aguascalientes, en particular al Decano y personal del Centro de Ciencias Básicas y a la coordinación del posgrado por el apoyo para la presentación de ponencias nacionales y en el extranjero, así como la estancia realizada.

También quiero agradecer al Departamento de Matemáticas y Física, así como a todos los docentes que me apoyaron en este camino.

También quiero agradecer a CONACYT por el apoyo económico, pues sin ese apoyo no hubiera podido realizar la maestría.

Además, quiero agradecer a Ramiro Abad por haberme acompañado en este, algunas veces difícil, camino, pero que gracias a él siempre se tornó más sencillo, me ayudaste, estuviste ahí para mí, creíste en mí y me apoyaste. Sin duda este posgrado me deja muchas cosas buenas, como las enseñanzas y conocimiento académico, o las experiencias vividas a lo largo de estos dos años, pero lo mejor que me deja es haberte conocido.

A mis padres, María de la Luz y Francisco, quienes me han apoyaron en cada momento.

Contents

Resumen

En esta tesis se presentó la generalización del teorema de Bumby para categorías de Grothendieck, ademas cabe destacar que dicha generalización también es valida para los casos de módulos puros inyectivos así como para módulos RD-inyectivos.

Abstract

In this thesis, we present a generalization of Bumby's theorem for Grothendieck categories, and also is important to mention that this generalization is also valid for pure-injective modules and RD- injective modules.

Chapter 1

Introduction

1.1 Background

In this thesis, we consider a fixed (though arbitrary) ring R with an identity element 1, which satisfies $1 \neq 0$ for the sake of non-triviality. In this work, the notation $\hom(A, B)$ will represent the set of all morphisms from an object A to an object B within some specific category. Throughout this section, however, the objects and morphisms considered will all belong to the category of left R-modules.

The motivation and point of departure of this work is the following result, which is an algebraic extension of the famous Cantor-Bernstein-Schröder's theorem on the cardinality of sets [7].

Theorem 1.1.1 (Bumby)**.** *Two injective modules are isomorphic if they are isomorphic to submodules of each other.*

The next results are straightforward consequences of Theorem 1.1.1. We refer to [1] for the proofs of Bumby's theorem and its corollaries.

Corollary 1.1.2 (Bumby)**.** *Two modules which are isomorphic to submodules of each other have isomorphic injective hulls.*

A module is **quasi-injective** if it is a fully invariant submodule of every injective module. Alternatively, the module M is quasi-injective if every homomorphism of any submodule N of M into M extends to an endomorphism of M. Injective modules are clearly quasi-injective, and every module is contained as a submodule in a smallest quasi-injective module (called its **quasi-injective hull**) which is unique up to canonical isomorphism [12].

Corollary 1.1.3 (Bumby)**.** *Two modules which are isomorphic to submodules of each other have isomorphic quasi-injective hulls.*

The following discussion will be crucial in order to state the problem under investigation. A submodule N of the R-module M is **relatively divisible** if the solubility in M of equations of the form $rx = a \in N$, with $r \in R$, implies their solubility in N. We say that the submodule N

of the R-module M is **pure** if for each pair of positive integers m and n, any finite system of equations of the form

$$\sum_{j=1}^{m} r_{ij} x_j = a_i \in N \quad (i = 1, \ldots, n),$$

with $r_{ij} \in R$, is soluble in N if it is soluble in M. An ***RD*-morphism** (respectively, a **pure-morphism**) is any monomorphism from a module A to a module B under which the image of A is a relatively divisible (respectively, pure) submodule of B.

Purity implies relative divisibility, and both conditions coincide for modules over Prüfer domains [14], that is, integral domains in which finitely generated ideals are projective. In fact, Prüfer domains are the only integral domains for which relative divisibility and purity are equivalent [2]. As it is the case with the property of projectivity [8], the condition of injectivity is crucial in some studies on the structure of modules and it possesses various generalizations [3, 4, 10]. Two of such generalizations are quoted next.

A module Q is ***RD*-injective** (respectively, **pure-injective**) if for each RD-morphism (respectively, pure-morphism) $\alpha \in \hom(A, B)$ and each $\phi \in \hom(A, Q)$ there exists $\psi \in \hom(B, Q)$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & B \arrow[dl, "\psi"] \\ & Q & \end{tikzcd} \tag{1.1}$$

Injectivity implies RD-injectivity which, in turn, implies pure-injectivity. It is worth mentioning that these conditions share many properties. For instance, they are closed with respect to the construction of direct products and with respect to the formation of direct summands [9]. The existence of minimal injective modules which contain a given module as an 'essential' submodule (injective hulls), is also a property that is common to all the conditions on injectivity mentioned above [5, 6]. Moreover, the following versions of Bumby's criterion for RD-injective and pure-injective modules are easy to prove.

Theorem 1.1.4. *Two RD-injective (respectively, pure-injective) modules are isomorphic if they are isomorphic to relatively divisible (respectively pure) submodules of each other.*

The next proposition is an extension of Corollaries 1.1.2 and 1.1.3. Its validity is readily established as a consequence of the last theorem.

Corollary 1.1.5. *Two modules which are isomorphic to relatively divisible (respectively, pure) submodules of each other have isomorphic RD-injective (respectively, pure-injective) hulls.*

1.2 Aims and scope

In view of these remarks, many questions arise in the investigation of conditions under which two modules are isomorphic whenever they are isomorphic to submodules of each other. For instance, is there a general criterion for the isomorphism of modules which extends Bumby's theorem and which contemplates the cases of RD-injective and pure-injective modules as particular scenarios? In the present work, we establish an affirmative answer to this question, and derive in the way several results that generalize well-known properties shared by all the conditions on injectivity quoted so far.

This work is sectioned as follows. Chapter 2 provides a brief introduction on module theory. We introduce therein various constructions on modules, including factor modules, submodules and the useful isomorphism theorems. Introduces the crucial notion of injective objects with respect to a family of morphisms. Our definition is a generalized form of the categorical definition of injectivity presented in [11], and it has been motivated by [13]. Some properties on generalized injective objects and injective hulls are established in Chapter 4. The main result is presented in Chapter 4 along with several immediate corollaries, including Bumby's theorem, as well as a generalized forms of the corollaries. This work closes with a section of concluding remarks and perspectives of future investigation.

Chapter 2

Modules

It's important to establish that in all this work R is a ring, it may or may not have an identity element. Clarified that point, we can introduce the next concept: A left R- **module** is an abelian group M endowed with a left action $(r, x) \to rx$ of R on M, and the structure is such that satisfy the next properties:

1. $r(sx) = (rs)x$
2. $(r + s)x = rx + sx$, $r(x + y) = rx + ry$ for all $r, s \in R$ and $x, y \in M$.

If R has an identity element, then a left R-module M is unital when

3. $1x = x$ for all $x \in M$

Analogously, we can define the right R- **module**, it is an abelian group M endowed with a right action $(x, r) \to xr$ of R on M, and the structure is such that satisfy the next properties:

1. $(xr)s = x(rs)$
2. $x(r + s) = xr + xs$, $(x + y)r = xr + yr$ for all $r, s \in R$ and $x, y \in M$.

If R has an identity element, then a right R-module M is unital when

3. $1x = x$ for all $x \in M$

It's easy to see that if R is commutative, then every left R-module is a right R-module, and conversely.

Moreover, if R isn't commutative it is clear that for every property of left R-modules there exists an analogy property of right R-modules, that's the reason why we will omit the word "left" or "right" when we talk about R-modules, in addition, in the beginning we mention that R is any ring, then we can omit it too, thus from here we will call them only modules and in all the proofs we will work with left modules because for right modules is analogous.

Another important structure that we need to introduce is a submodule, this is a subset of the module which inherited a module structure, this is: A **submodule** of a module M is an additive subgroup A of M such that $x \in A$ implies $rx \in A$ for all $r \in R$. With this concept we can formulate the next proposition:

Proposition 2.0.1. *Let M be a module. Every intersection of submodules of M is a submodule of M. The union of a nonempty directed family of submodules of M is a submodule of M.*

Proof. Let M be a module. Let $\{A_\alpha\}_{\alpha \in I}$ a family of submodules.

For the first part, we know that $\forall \alpha \in I$ A_α is a subgroup of M, and every intersection of subgroups is a subgroup. Then $\cap_{\alpha \in I} A_\alpha$ is a subgroup. Let $x \in \cap_{\alpha \in I} A_\alpha$ and $r \in R$. Furthermore we have that $x \in A_\alpha$ $\forall \alpha \in I$ but, A_α is a submodul. Then $rx \in A_\alpha$ $\forall \alpha \in I$. Thus $rx \in \cap_{\alpha \in I} A_\alpha$. In conclusion $\cap_{\alpha \in I} A_\alpha$ is a submodule.

For the second part, we know that $\forall \alpha \in I$ A_α is a subgroup of M, and the union of this family is a subgroup of M, then $\cup_{\alpha \in I} A_\alpha$ is a subgroup. Let $x \in \cup_{\alpha \in I} A_\alpha$ then exists $\alpha \in I$ such that $x \in A_\alpha$ thus as A_α is a submodule $rx \in A_\alpha$ $\forall r \in R$, then $rx \in \cup_{\alpha \in I} A_\alpha$. In conclusion $\cup_{\alpha \in I} A_\alpha$ is a submodule. □

The last proposition indicates that there is for every subset A of a module M a smallest submodule of M that contains A, which is called the **submodule of M generated by A**. And also the **sum** of a family $(A_i)_{i \in I}$ of submodules is defined:

$\sum_{i \in I} A_i = \{\sum_{i \in I} a_i \mid a_i \in A_i$ for all i, and $a_i = 0$ for almost all $i\}$

It's easy to see that $\sum_{i \in I} A_i$ is the submodule generated by the union $\cup_{i \in I} A_i$.

Now we can define module homomorphisms, let A and B be left modules with the same ring R. A **homomorphism** $\varphi : A \to B$ of left modules is a mapping $\varphi : A \to B$ such that $\varphi(x+y) = \varphi(x) + \varphi(y)$ and $\varphi(rx) = r\varphi(x)$, for all $x, y \in A$ and $r \in R$. Analogous, we can define homomorphisms of right modules.

It's important to define some particular kind of homomorphisms: An **endomorphism** of a module A is a module homomorphism of A onto A, If a module homomorphism is injective is called **monomorphism** of modules, a module homomorphism which is surjetive is called **epimorphism** of modules, an isomorphism of modules is a bijective homomorphism of modules, thus the inverse bijection is also an **isomorphism**.

We will see some properties for homomorphism of modules in the following propositions, but also it is important to clarify that in all the proofs we will suppose a left R-module, but the proof of right R-module is analogous.

Proposition 2.0.2. *The identity mapping on any module M is a module homomorphism.*

Proof. Let M a module and 1_M the identity module. Let $x, y \in M$ and $r \in R$.

$$1_M(x+y) = x + y = 1_M(x) + 1_M(y)$$

$$1_M(rx) = rx = r1_M(x)$$

In conclusion, 1_M is a homomorphism. □

Proposition 2.0.3. *Module homomorphisms composition is a homomorphism.*

Proof. Let A,B and C modules, also $\varphi : A \to B$ and $\psi : B \to C$ be homomorphism of modules. Let $x, y \in A$ and $r \in R$.

$$\begin{aligned}(\psi \circ \varphi)(x+y) &= \psi(\varphi(x+y)) = \psi(\varphi(x)+\varphi(y)) = \psi(\varphi(x)) + \psi(\varphi(y)) \\ &= (\psi \circ \varphi)(x) + (\psi \circ \varphi)(y)\end{aligned}$$

$$(\psi \circ \varphi)(rx) = \psi(\varphi(rx)) = \psi(r\varphi(x)) = r\psi(\varphi(x)) = r(\psi \circ \varphi)(x)$$

Then $\psi \circ \varphi$ is an homomorphisms. □

It's important to clarify that homomorphisms can be added pointwise, let $\varphi, \psi : A \to B$ homomorphisms of left modules, we define $\varphi + \psi : A \to B$ by $(\varphi+\psi)(x) = \varphi(x) + \psi(x)$ $\forall x \in A$, and also it is a homomorphism of left modules. For right modules we have an analogous definition.

For the next proposition it‘s important to mention the following notation:

Proposition 2.0.4. *Let $\varphi : A \to B$ be a module homomorphism. If C is a submodule of A, then $\varphi(C) = \{\varphi(x) | x \in C\}$ is a submodule of B. If D is a submodule of B then $\varphi^{-1}(D) = \{x \in A | \varphi(x) \in D\}$ is a submodule of A.*

Proof. Let $\varphi : A \to B$ be module homomorphism and let C be a submodule of A. Then we know that $\varphi(C)$ is a subgrup. So let $r \in R$ and $x \in \varphi(C)$. Thus exits $c \in C$ such that $x = \varphi(c)$. Then $rx = r\varphi(c) = \varphi(rc)$ but C is a submodule, furthermore $rc \in C$ that implies $rx \in \varphi(C)$, in conclusion $\varphi(C)$.

Let D a submodule of B, we know that $\varphi^{-1}(D)$ is a subgroup, then let $r \in R$ and $x \in \varphi^{-1}(D)$ then $\varphi(x) \in D$, but D submodule that implies $r\varphi(x) = \varphi(rx) \in D$ thus $rx \in \varphi^{-1}(D)$. □

In the last proposition, $\varphi(C)$ is called the **direct image** and $\varphi^{-1}(D)$ is called the **preimage** or **inverse image** but is important to clarify that the use of the notation $\varphi^{-1}(D)$ not imply that φ has an inverse function.

Let $\varphi : A \to B$ be a module homomorphism, we can define the **image** or **range** of φ is $Im(\varphi) = \{\varphi(x) | x \in A\} = \varphi(A)$, and also the **kernel** of φ is $Ker(\varphi) = \{x \in A | \varphi(x) = 0\} = \varphi_{-1}(0)$.

Now, let L, M, N be a modules, an epimorphism $\varphi : M \to N$ is said to be **split** if there exists an homomorphism $\alpha : N \to M$ with $\varphi \circ \alpha = 1_N$, in this case α is called a splitting map for φ. Analogously, a monomorphism $\alpha : L \to M$ is said to be **split** if there exists an homomorphism $\varphi : M \to L$ such that $\varphi \circ \alpha = 1_L$. In this case φ is called a splitting map for α.

Let $\{M_i\}_{i \in I}$ a family of modules, the **direct product** of the modules $\{M_i\}_{i \in I}$ is denoted $\Pi_{i \in I} M_i$, is the Cartesian product of M_i like sets, with operations defined componentwise.

And also, the **external direct sum** of the modules $\{M_i\}_{i \in I}$ is denoted by $\bigoplus_{i \in I} M_i$, it is the submodule of the direct product which consists of the elements (m_i) for which are zero for all but a finite number of $i \in I$.

It's immediate that the direct product and the external direct sum of modules are also modules. In particular, a finite external direct sum is the same as finite direct product. If the set I is infinite, in general the direct product and the external direct sum are different, and also they can't be isomorphic.

Let M a module, and $\{M_i\}_{i\in I}$ a family of submodules of M, M is the **internal direct sum** of the submodules $\{M_i\}_{i\in I}$, denoted by $M = \bigoplus_{i\in I} M_i$, if every element of M can be expressed uniquely like a sum of elements of M_i. In this case, we say that each M_i is a direct summand of M, and it's denoted by $Mi|M$.

It's clear that every module has as direct summand itself and the zero submodule.

Proposition 2.0.5. *Let M, N be a modules.*

a) Let $\varphi : M \to N$ and $\alpha : N \to M$ be homomorphisms such that $\varphi \circ \alpha = 1_N$, then $M = Ker(\varphi) \oplus Im(\alpha)$

b) A monomorphism $\alpha : N \to M$ splits if and only if $Im(\alpha)$ is a direct summand of M.

c) An epimorphism $\varphi : M \to N$ split if and only if $Ker(\varphi)$ is a direct summand of M.

Proof.

a) Assume that $\varphi : M \to N$ and $\alpha : N \to M$ are homomorphisms with $\varphi \circ \alpha = 1_N$. If $m \in Ker(\varphi) \cap Im(\alpha)$, then there exist $x \in N$ with $\alpha(x) = m$ and so $x = (\varphi \circ \alpha)(x) = \varphi(m) = 0$, then $m = 0$ because $m = \alpha(0) = 0$. Thus $Ker(\varphi) \cap Im(\alpha) = \{0\}$.

Given any $m \in M$, consider the element $m - (\alpha \circ \varphi)(m)$. We haver $(\alpha \circ \varphi)(m) \in Im(\alpha)$ and $m - (\alpha \circ \varphi)(m) \in Ker(\varphi)$ since $\varphi(m - (\alpha \circ \varphi)(m)) = \varphi(m) - (\varphi \circ \alpha)(\varphi(m)) = 0$. Thus

$$m = (m - (\alpha \circ \varphi)(m)) + ((\alpha \circ \varphi)(m)) \in Ker(\varphi) + Im(\alpha)$$

It follows that $M = Ker(\varphi) \oplus Im(\alpha)$.

b) Assume that $\alpha : N \to M$ is a monomorphism. If α has a splitting map $\varphi : M \to N$ such that $\varphi \circ \alpha = 1_N$, then $M = Im(\alpha) \oplus Ker(\varphi)$ by part *a*).

Conversely, assume that α is a monomorphism and $Im(\alpha)$ is a direct summand of M, in other words $M = Im(\alpha) \oplus M_2$. Then each $m \in M$ is uniquely represented as $\alpha(x) + y$ for some $x \in N$ and some $y \in M_2$. Since α is one to one, the element x is uniquely determined by m. To show this, suppose that $\alpha(x_1) + y_1 = \alpha(x_2) + y_2$. Since $Im(\alpha) \cap M_2 = \{0\}$, this implies that $\alpha(x_1) = \alpha(x_2)$, and so $x_1 = x_2$ since α is one to one. Thus we have a well-defined function $\varphi : M \to N$ given by $\varphi(m) = x$, for all $m \in M$. Since φ is well defined, it follows easily that φ is an homomorphism, and it's clear that $\varphi \circ \alpha = 1_N$

c) Assume that $\varphi : M \to N$ is an epimorphism that has a splitting map $\alpha : N \to M$ with $\varphi \circ \alpha = 1_N$. Applying part *a*) gives $M = Ker(\varphi) \oplus Im(\alpha)$.

Conversely, assume that $M = Ker(\varphi) \oplus M_2$ for some submodule $M_2 \subseteq M$. Define $\gamma : M_2 \to N$ by letting $\gamma(x) = \varphi(x)$, for all $x \in M_2$. Then γ is an onto mapping, since if $y \in N$ then there exists $m \in M$ with $\varphi(m) = y$, where $m = m_1 + m_2$ for some $m_1 \in Ker(\varphi)$ and $m_2 \in M_2$, ans so $y = \varphi(m) = \varphi(m_2) = \gamma(m_2)$. Furthermore, γ is one to one, since $Ker(\gamma) = Ker(\varphi) \cap M_2 = \{0\}$. It is clear that γ is an homomorphism, and if we let $\alpha = \gamma^{-1}$, then $\varphi \circ \alpha = 1_N$ □

Proposition 2.0.6. *Let M a module and let A be a submodule of M. The quotient group M/A is module, in which $r(x + A) = rx + A$ for all $r \in R$ and $x \in M$. If M is unital, then M/A is unital. The projection $x \mapsto x + A$ is a homomorphism of modules, whose kernel is A.*

Proof. Let M be module and let A be a submodule, then M is an abelian group and A a subgroup of M, A is normal because every subgroup of an abelian group is normal, thus there is M/A the quotient group, in which cosets of A are added as subsets. Then $\forall x, y \in M$

$$\begin{aligned}(x + A) + (y + A) &= \{a + b \mid a \in x + A \text{ and } b \in y + A\} \\ &= \{a + b \mid a = x + z_1,\ b = y + z_2 \text{ with } z_1, z_2 \in A\} \\ &= \{x + y + z_1 + z_2 \mid x, y \in M,\ z_1, z_2 \in A\} \\ &= \{x + y + z \mid x, y \in M,\ z \in A\} \\ &= (x + y) + A.\end{aligned}$$

Let $r \in R$ and $x \in M$, then:

$$\begin{aligned}r(x + A) &= r\{x + a \mid a \in A\} \\ &= \{r(x + a) \mid a \in A\} \\ &= \{rx + ra \mid a \in A\} \\ &= \{rx + b \mid b \in A\} \\ &= rx + A.\end{aligned}$$

Then both are well define. Now we will check the properties 1 and 2 of the definition:

Let $r, s \in R$ and $x, y \in M/A$

$$\begin{aligned}r(s(x + A)) &= r(sx + A) \\ &= r(sx) + A \\ &= (rs)x + A \\ &= (rs)(x + A).\end{aligned}$$

$$\begin{aligned}(r + s)(x + A) &= (r + s)x + A \\ &= (rx + sx) + A \\ &= (rx + A) + (sx + A) \\ &= r(x + A) + s(x + A).\end{aligned}$$

$$\begin{aligned}r((x + A) + (y + A)) &= r((x + y) + A) \\ &= r(x + y) + A \\ &= (rx + ry) + A\end{aligned}$$

$$
\begin{aligned}
&= (rx + A) + (ry + A) \\
&= r(x + A) + r(y + A).
\end{aligned}
$$

Then M/A is a module. And also, if M is an unital module then $1 \in R$

$$
\begin{aligned}
1(x + A) &= 1x + A \\
&= x + A.
\end{aligned}
$$

then M/A is also unital.

Now, consider the projection $\pi(x) = x + A$, it has the next properties:

$$\pi(x + y) = (x + y) + A = (x + A) + (y + A) = \pi(x) + \pi(y).$$

$$\pi(rx) = rx + A = r(x + A) = r(\pi(x)) = r\pi(x).$$

then the projection is a homomorphism.

Clearly, $[0] \in M/A$ is $[0] = A$, and remember that A is a submodule, then $x + A = A$ for all $x \in A$. Then $\pi(x) = x + A = A = [0]$ if and only if $x \in A$, in conclusion, $Ker(\pi) = A$ ☐

As we have seen in the last proposition, if M is a module and A a submodule of it, then the module of all cosets of A is the **quotient** M/A of M by A.

An important property of the quotients module is that submodules of a quotients module M/A are quotients of submodules of M.

Proposition 2.0.7. *Each submodule of quotient module M/A has the form C/A for some submodule C, with $A \leq C \leq M$.*

Proof. Let a submodule $K \leq M/A$ then every $k \in K$ can be written like $k = x + A$ with $x \in M$, then Let $P = \{x \in M \mid x + A = K\}$, we will prove that P is a submodule of M.

Let $k_1, k_2 \in K$, then $\exists x_1, x_2 \in P$ such that $k_1 = x_1 + A$ and $k_2 = x_2 + A$ we have that $k_1 + k_2 \in K$ then

$$k_1 + k_2 = (x_1 + A) + (x_2 + A) = (x_1 + x_2) + A \in K.$$

Then $x_1 + x_2 \in P$, P is closed, also the associativity and commutativity follows of M is an abelian group.

Since K is a submodule of M, the identity element is in K, that is, $[0] = 0 + A \in K$ that implies that $0 \in P$. And also we have that for all $k_1 \in K$ there are $k_2 \in K$ such that $k_1 + k_2 = [0]$ but $\exists x_1, x_2 \in P$ such that $k_1 = x_1 + A$ and $k_2 = x_2 + A$ and we have

$$
\begin{aligned}
&x_1 + A + x_2 + A = 0 + A \\
&\Leftrightarrow (x_1 + x_2) + A = 0 + A,
\end{aligned}
$$

that means that for every x_1 there exists x_2 such that $x_1 + x_2$ is in the same equivalence class than 0, then x_2 is the inverse of x_1. In conclusion P is a subgroup and also by the construction, clearly $K = P/A$. ☐

Theorem 2.0.8 (Factorization Theorem)**.** *Let A a module and $B \leq A$, if $\varphi : A \to C$ is a module homomorphism whose kernel contains B, then φ can be factorized uniquely through the canonical projection $\pi : A \to A/B$ such that $\varphi = \psi \circ \pi$ for some unique module homomorphism $\psi : A/B \to C$, that means, the next diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & A/B \arrow[d, dashed, "\psi"] \\ & C \end{tikzcd}$$

Proof. We will use the formal definition of a mapping $\psi : A/B \to C$ as a set of ordered pairs,

$$\psi = \{(x + B, \varphi(x)|x \in A\}$$

We will check that is well define, by definition of A/B for every element in A/B we can find an element in C such that the pair are in the function ψ, Moreover if we have $a_1, a_2 \in A/B$ and $a_1 = a_2$ that means that they are in the same equivalence class, then there is a representative element $a \in A$ such that $a_1 = a_2 = a + B$ that implies $\psi(a_1) = \varphi(a) = \psi(a_2)$, thus ψ is well define and clearly $\psi \circ \pi = \varphi$.

Now, let $a, b \in A/B$ then there are $x, y \in A$ such that $a = x + B$ and $b = y + B$ and $\alpha \in R$ consider

$$\begin{aligned} \psi(x + B + y + B) &= \psi(x + y + B) \\ &= \varphi(x + y) \\ &= \varphi(x) + \varphi(y) \\ &= \psi(x + B) + \psi(y + B). \end{aligned}$$

$$\begin{aligned} \psi(\alpha(x + B)) &= \psi(\alpha x + B) \\ &= \varphi(\alpha x) \\ &= \alpha\varphi(x) \\ &= \alpha\psi(x + B). \end{aligned}$$

then ψ is a homomorphism of modules.

Now, to show that ψ is unique consider $\chi : A/B \to C$ be a homomorphism such that $\chi \circ \pi = \varphi$ then $\chi(x + B) = \chi(\pi(x)) = \varphi(x) = \psi(\pi(x)) = \psi(x + B)$ for all $x + B \in A/B$, in conclusion $\chi = \psi$. □

The next theorem is a useful stronger version of the factorization theorem.

Theorem 2.0.9 (Factorization Theorem)**.** *If $\varphi : A \to B$ and $\rho : A \to C$ are module homomorphism, ρ surjective and $Ker(\rho) \subseteq Ker(\varphi)$, then φ factors uniquely through ρ, that means, the* *next diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\varphi"'] & C \arrow[r] \arrow[dl, dashed, "\psi"] & 0 \\ B & & \end{tikzcd}$$

Proof. Assume that $Ker(\rho) \subseteq Ker(\varphi)$ and let $x', y' \in C$, since ρ is surjective there are $x, y \in A$ such that $\rho(x) = x'$ and $\rho(y) = y'$, now if $x' = y'$, then

$$\rho(x - y) = \rho(x) - \rho(y) = x' - y' = 0.$$

whence $x - y \in Ker(\rho) \subseteq Ker(\varphi)$ and so $\varphi(x) = \varphi(y)$. In other words there is a function $\psi : C \to B$ such that $\psi(\rho(x)) = \varphi(x)$ for all $x \in A$. Is easy to check that ψ is a homomorphism, let $x', y' \in C$ then there are $x, y \in A$ such that $\rho(x) = x'$ and $\rho(y) = y'$ and $\alpha \in R$ consider

$$\begin{aligned} \psi(x' + y') &= \psi(\rho(x) + \rho(y)) \\ &= \psi(\rho(x + y)) \\ &= \varphi(x + y) \\ &= \varphi(x) + \varphi(y) \\ &= \psi(\rho(x)) + \psi(\rho(y)) \\ &= \psi(x') + \psi(y'). \end{aligned}$$

$$\begin{aligned} \psi(\alpha x') &= \psi(\alpha\rho(x)) \\ &= \psi(\rho(\alpha x)) \\ &= \varphi(\alpha x) \\ &= \alpha\varphi(x) \\ &= \alpha\psi(\rho(x)) \\ &= \alpha\psi(x'). \end{aligned}$$

then ψ is a module homomorphism. Now to show that ψ is unique, consider $\chi : C \to B$ be a homomorphism such that $\chi \circ \rho = \varphi$, let $x' \in C$ then there is $x \in A$ such that $\rho(x) = x'$ then $\chi(x') = \chi(\rho(x)) = \varphi(x) = \psi(\rho(x)) = \psi(x')$, in conclusion $\chi = \psi$. $\square$

Theorem 2.0.10 (Homomorphism Theorem)**.** *If $\varphi : A \to B$ is a homomorphism of modules then*

$$A/Ker(\varphi) \cong Im(\varphi)$$

In fact, there is an isomorphism $\theta : A/Ker(\varphi) \to Im(\varphi)$ unique such that $\varphi = \iota \circ \theta \circ \pi$, where $\iota : Im(\varphi) \to B$ is the inclusion homomorphism and $\pi : A \to A/Ker(\varphi)$ is the canonical projection, *that means the next diagram commutes*

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "\pi"'] & B \\
A/Ker(\varphi) \arrow[r, dashed, "\theta"'] & Im(\varphi) \arrow[u, "\iota"']
\end{tikzcd}$$

Proof. Let $\varphi : A \to B$ a homomorphism of modules then let $\theta : A/Ker(\varphi) \to Im(\varphi)$ given by $\overline{x} = x + Ker(\varphi) \mapsto \varphi(x)$. We will check that is well defined. Let $\overline{x}, \overline{y} \in A/Ker(\varphi)$ then if $\overline{x} = \overline{y}$ that means that $\overline{x}$ and $\overline{y}$ are in the same equivalence class, then there is a representative element $x \in A$ such that $\overline{x} = \overline{y} = x + Ker(\varphi)$ then $\theta(\overline{x}) = \varphi(x) = \theta(\overline{y})$, then θ is well define, and since φ is a homomorphism, then so is θ.

Now we will check that θ is an isomorphism. Let $\overline{x}, \overline{y} \in A/Ker(\varphi)$ then there are $x, y \in A$ such that $\overline{x} = x + Ker(\varphi)$ and $\overline{y} = y + Ker(\varphi)$, furthermore $\theta(\overline{x}) = \theta(\overline{y})$ then $\theta(\overline{x}) - \theta(\overline{y}) = 0$ that implies $\varphi(x) - \varphi(y) = 0$ more $\varphi(x - y) = 0$ then $x - y \in Ker(\varphi)$ then $x + Ker(\varphi) = y + Ker(\varphi)$ finally $\overline{x} = \overline{y}$, then θ is injective.

Let $y \in Im(\varphi)$ then there is $x \in A$ such that $\varphi(x) = y$ moreover $\overline{x} = x + Ker(\varphi) \in A/Ker(\varphi)$ and clearly $\theta(\overline{x}) = \varphi(x) = y$, then θ is surjective. We have already seen that θ is an isomorphism.

Let $\iota : Im(\varphi) \to B$ the inclusion homomorphism and $\pi : A \to A/Ker(\varphi)$ the canonical projection, let $a \in A$ then

$$\iota(\theta(\pi(a))) = \iota(\theta(a + Ker(\varphi))) = \iota(\varphi(a)) = \varphi(a).$$

Then for every $a \in A$ we have $\varphi = \iota \circ \theta \circ \pi$.

Now to show that θ is unique, consider $\chi : A/Ker(\varphi) \to Im(\varphi)$ be a isomorphism such that $\varphi = \iota \circ \chi \circ \pi$, and let $a \in A$ then

$$\iota(\chi(\pi(a))) = \varphi(a) = \iota(\theta(\pi(a))).$$

that implies

$$\iota(\chi(a + Ker(\varphi))) = \iota(\theta(a + Ker(\varphi))).$$

and also

$$\chi(a + Ker(\varphi)) = \theta(a + Ker(\varphi)).$$

In conclusion, $\chi = \theta$. □

Theorem 2.0.11 (First Isomorphism Theorem)**.** *If A is a module and $B \supseteq C$ are submodules of A then*

$$A/B \cong (A/C)/(B/C)$$

In fact, there is a unique isomorphism $\theta : A/B \to (A/C)/(B/C)$ such that $\theta \circ \rho = \tau \circ \pi$, where $\pi : A \to A/C$, $\rho : A \to A/B$, and $\tau : A/C \to (A/C)/(B/C)$ are the canonical projections, that *means the next diagram commutes*

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[d, "\rho"'] & A/C \arrow[d, "\tau"] \\
A/B \arrow[r, dashed, "\theta"'] & (A/C)/(B/C)
\end{tikzcd}$$

Proof. Consider $\varphi : A/C \to A/B$ defined by $a + C \mapsto a + B$, we will check that is well define. By the definiton of A/B, for every element $\overline{a} \in A/C$ we can find an element $\overline{b} \in A/B$ such that $\varphi(\overline{a}) = \overline{b}$. Let $\overline{a}, \overline{b} \in A/C$ then if $\overline{a} = \overline{b}$ that means that $\overline{a}$ and $\overline{b}$ are in the same equivalence class, then there is a representative element $x \in A$ such that $\overline{a} = \overline{b} = x + C$ then we have $\varphi(\overline{a}) = x + B = \varphi(\overline{b})$, in conclusion φ is well define, and clearly is a homomorphism of modules.

Clearly is surjective, let $\overline{a} \in A/B$ then there is $a \in A$ such that $\overline{a} = a + B$ then there is $a + C$ such that $\varphi(a + C) = a + B$.

Now we will check that $Ker(\varphi) = B/C$, let $a + C \in Ker(\varphi)$ then $\varphi(a + C) = a + B = B$ that means $a \in B$ therefore $a + C \in B/C$. On the other hand, let $b + C \in B/C$ consider $\varphi(b + C) = b + B = B$ then $b + C \in Ker(\varphi)$.

By the Homomorphism Theorem we obtain

$$(A/C)(B/C) \cong A/B$$

in fact, there is a unique isomorphism $\theta : A/B \to (A/C)/(B/C)$ and the diagram commutes. $\square$

For the following theorem, we need to define the **sum of submodules**, if A and B are submodules of a module, then the sum of them is define by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

Theorem 2.0.12 (Second Isomorphism Theorem)**.** *If A and B are submodules of a module, then*

$$(A + B)/B \cong A/(A \cap B)$$

in fact, there is an isomorphism $\theta : A/(A \cap B) \to (A + B)/B$ unique such that $\theta \circ \rho = \pi \circ \iota$, where $\pi : A + B \to (A + B)/B$ and $\rho : A \to A/(A \cap B)$ are the canonical projections and $\iota : A \to A + B$ is the inclusion homomorphism, that means the next diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\rho"] \arrow[d, "\iota"'] & A/(A \cap B) \arrow[d, dashed, "\theta"] \\
A + B \arrow[r, "\pi"'] & (A + B)/B
\end{tikzcd}$$

Proof. Consider $\varphi : A + B \to A/(A \cap B)$ defined by $\overline{c} = a + b \mapsto a + (A \cap B)$, we will check that is well define. By the definiton of $A + B$, for every element $\overline{a} \in A + B$ we can find an element $\overline{b} \in A/(A \cap B)$ such that $\varphi(\overline{a}) = \overline{b}$. Let $\overline{c}, \overline{d} \in A + B$ then if $\overline{c} = \overline{d}$ that means that there are

$a \in A$ and $b \in B$ such that $\bar{c} = \bar{d} = a + b$ then we have $\varphi(\bar{c}) = \varphi(a+b) = \varphi(\bar{d})$, in conclusion φ is well define, and clearly is a homomorphism of modules.

Clearly is surjective, let $\bar{c} \in A/(A \cap B)$ then there is $a \in A$ such that $\bar{c} = a + (A \cap B)$ then there is $a + 0 \in A + B$ such that $\varphi(a + 0) = a + (A \cap B)$.

Now we will check that $Ker(\varphi) = B$, let $a+b \in Ker(\varphi)$ then $\varphi(a+b) = a+(A \cap B) = A \cap B$ that means $a \in A \cap B$ then $a \in B$ therefore $a + b \in B$. On the other hand, let $b \in B$ consider $0 + b \in A + B$ then $\varphi(0 + b) = 0 + (A \cap B) = A \cap B$ then $b \in Ker(\varphi)$.

By the Homomorphism Theorem we obtain

$$(A + B)/B \cong A/(A \cap B)$$

in fact, there is a unique isomorphism $\theta : A/(A \cap B) \to (A+B)/B$ and the diagram commutes. □

2.1 Universal Properties

Theorem 2.1.1 (Universal Property of Direct Product). *Let $\{M_i\}_{i \in I}$ a family of modules. Denote the direct product of the family by P. For each $j \in I$, the function $\pi_j : P \to M_j$ given by $M_j((m_i)) = m_j$ is a homomorphism of modules and it's called projection to M_j.*

If X is other module with a family of homomorphisms of modules $P_j : X \to M_j$, then there exists a unique homomorphism of modules $P : X \to P$ such that $\pi_j \circ P = P_j$ for all $j \in I$.

Proof. Consider the function $\pi_j : P \to M_j$ for some $j \in I$ and $\alpha \in R$. Clearly it is a homomorphism because

$$\pi_j(x_i + y_i) = \pi_j((x + y)_i) = (x + y)_j = x_j + y_j = \pi_j(x_i) + \pi_j(y_i).$$

$$\pi_j(\alpha x_i) = \pi_j((\alpha x)_i) = (\alpha x)_j = \alpha(x)_j = \alpha\pi_j(x_i).$$

Now, consider X a module and a family of homomorphisms $P_j : X \to M_j$, consider the function $P(x) = (P_i(x))$, this function is an homomorphism because each component so is, and also $\pi_j \circ P = P_j$ for every $j \in I$.

Consider $Q : X \to P$ such that $\pi_j \circ Q = P_j$ for every $j \in I$. Let $x \in X$, and $(\pi_j \circ P)(x) = (\pi_j \circ Q)(x)$ for every $j \in I$ that implies $(P(x))_j = (Q(x))_j$ for every $j \in I$ then $P(x) = Q(x)$ for every $x \in X$, finally $P \equiv Q$, in conclusion P is unique. □

Theorem 2.1.2 (Universal Property of External Direct Sum). *Let $\{M_i\}_{i \in I}$ a family of modules. Denote the external direct sum of the family by S. For each $j \in I$, the function $\lambda_j : M_j \to S$ given by $\lambda_j(m) = (m_i)$ with $m_i = 0$ for $i \neq j$ and $m_j = m$, is a homomorphism of modules, and it's called inclusion of M_j. If X is other module with a family of homomorphisms of modules $l_j : M_j \to X$, then there exists a unique homomorphism of modules $q : S \to X$ such that $q \circ \lambda_j = l_j$ for every $j \in I$.*

Proof. Let $j \in I$, and $x, y \in M_j$, and also $\alpha \in R$ then

$$\begin{aligned}\lambda_j(x+y) &= (0,0,\ldots,x+y,\ldots)\\ &= (0,0,\ldots,x,\ldots)+(0,0,\ldots,y,\ldots)\\ &= \lambda_j(x)+\lambda_j(y).\end{aligned}$$

$$\begin{aligned}\lambda_j(\alpha x) &= (0,0,\ldots,\lambda x,\ldots)\\ &= \alpha(0,0,\ldots,x,\ldots)\\ &= \alpha(\lambda_j(x)).\end{aligned}$$

Now, consider X a module and a family of homomorphisms $l_i : M_i \to X$, consider the function $q(x_i) = \sum_{i\in\Omega} l_i(x_i)$ with $\Omega = \{i \in I \mid x_i \neq 0\}$, is a homomorphism since is a finite sum of homomorphisms and also $q \circ \lambda_j = l_j$ for every $j \in I$.

Consider $\varphi \circ \lambda_j = l_j$ for every $j \in I$ then for every $m_j \in M_j$ $\varphi(\lambda_j(m_j)) = q(\lambda_j(m_j))$.

Let $m \in S$ then $m = \sum_{i\in\Omega} \lambda_i(m_i)$ then

$$\begin{aligned}\varphi(m) &= \varphi(\sum_{i\in\Omega}\lambda_i(m_i))\\ &= \sum_{i\in\Omega}\varphi(\lambda_i(m_i))\\ &= \sum_{i\in\Omega}q(\lambda_i(m_i))\\ &= q(\sum_{i\in\Omega}\lambda_i(m_i))\\ &= q(m).\end{aligned}$$

For every $m \in M$, in conclusion q is unique. $\square$

Chapter 3

Injective Modules

In this moment we can introduce the next concept: Let a finite or infinite sequence

$$\cdots M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} M_{i+2} \cdots$$

of a module homomorphisms is called **exact at position** $i \in \{2, \ldots, n, \ldots\}$ if $Im(\varphi_{i-1}) = Ker(\varphi_i)$. It is called **exact** if it is exact at every position.

In particular, if the source or target of the map is the zero module, so that the map is necessarily the zero homomorphism. And also, If a sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact is called a **short exact sequence**.

Proposition 3.0.1. *We note some particular kinds of exact sequences:*

a) $0 \to A \xrightarrow{\varphi} B$ *is exact if and only if* φ *is injective.*

b) $A \xrightarrow{\varphi} B \to 0$ *is exact if and only if* φ *is surjective.*

c) $0 \to A \xrightarrow{\varphi} B \to 0$ *is exact if and only if* φ *is an isomorphism*

Proof. a) We have by definition $Im(0) = Ker(\varphi)$, but $Im(0) = \{0\}$, that implies φ es injective.

b) We have that $Im(\varphi) = Ker(0)$, but $Ker(0) = B$, that implies $Im(\varphi) = B$, in others words φ is surjective.

c) Follows by the two last.

□

It's important to mention that exact sequences $0 \to A \to B \to C$ are some times called **left exact**, analogously exact sequences $A \to B \to C \to 0$ are some times called **right exact**.

A short exact sequence $0 \to N \to M \to P \to 0$ is called **splitting** when there is an isomorphism $\theta : M \to N \oplus P$ such that the next diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\varphi"] \arrow[d, "1_N"'] & M \arrow[r, "\alpha"] \arrow[d, "\theta"'] & P \arrow[r] \arrow[d, "1_P"'] & 0 \\
0 \arrow[r] & N \arrow[r] & N \oplus P \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

Proposition 3.0.2. *Let M, N, L be modules, the following properties of a short exact sequence*

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\alpha} L \longrightarrow 0$$

are equivalent:

a) The sequence splits.

b) There exists a homomorphism $\beta : N \to M$ such that $\beta \circ \varphi = 1_M$

c) There exists a homomorphism $\psi : L \to N$ such that $\alpha \circ \psi = 1_L$

Proof. We proceed in the following progression $a) \Leftrightarrow b)$ and $b) \Leftrightarrow c)$.

$a) \Rightarrow b)$ Suppose that the sequence splits, let $\theta : N \to M \oplus L$ be the guaranteed isomorphism. Define a map β by $\beta = \pi_M \circ \theta$ where π_M is the natural inclusion, then we have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, shift left] & N \arrow[r, "\alpha"] \arrow[d, "\theta"'] & L \arrow[r] \arrow[d, shift left] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota_M"] \arrow[u, shift left, "1_M"] & M \oplus L \arrow[r, "\pi_L"] \arrow[ul, "\pi_M"'] & L \arrow[r] \arrow[u, shift left, "1_L"'] & 0
\end{tikzcd}$$

where everything is commutative. Since β is the composition of two homomorphisms, it's a homomorphism and also $\beta \circ \varphi = \pi_M \circ \theta \circ \varphi = \pi_m|circ\iota_M \circ 1_M = 1_M$. The conclusion follows.

$b) \Rightarrow c)$ Suppose there is a homomorphism $\beta : N \to M$ such that $\beta \circ \varphi = 1_M$. Define $\psi : L \to N$ as follows. Since α is an epimorphism there exist , for each $l \in L$ some $n \in N$ such that $\alpha(n) = l$, define $\psi(l) = n - \varphi(\beta(n))$.To see that this is well-defined, in the sense that if $\alpha(n') = \alpha(n)$ then $n - \varphi(\beta(n)) = n' - \varphi(\beta(n'))$ and also $n - n' \in Ker(\alpha)$ and so by exactness $n - n' \in Im(\varphi)$ and since $\beta \circ \varphi = 1_M$ this tells us that $\varphi \circ \beta \circ \varphi = \varphi$ and so evidently $\varphi(\beta(n - n')) = n - n'$ and so

$$n - \varphi(\beta(n)) - (n' - \varphi(\beta(n'))) = n - n' - \varphi(\beta(n - n')) = 0.$$

This is a homomorphism since if $l = \alpha(n)$ and $k = \alpha(n')$ then $rl = \alpha(rn)$ and so $rl + k = \alpha(rn + n')$ and so

$$\begin{aligned}
\psi(rl + k) &= rn + n' - \varphi(\beta(rn - n')) \\
&= r(n - \varphi(\beta(n))) + (n' - \varphi(\beta(n'))) \\
&= r\psi(l) + \psi(k).
\end{aligned}$$

And clearly $\alpha \circ \psi = 1_L$.

$c) \Rightarrow b)$ Suppose $\psi : L \to N$ with $\alpha \circ \psi = 1_L$. Let $n \in N$ be arbitrary, note that

$$\alpha(n - \psi(\alpha(n))) = \alpha(n) - \alpha(\psi(\alpha(n))) = \alpha(n) - \alpha(n) = 0.$$

So that $n - \psi(\alpha(n)) \in Ker(\alpha)$ by exactness $n - \psi(\alpha(n)) \in Im(\varphi)$, and since φ is a monomorphism this implies that there exists a unique $m \in M$ with $\varphi(m) = n - \psi(\alpha(n))$, so define $\beta(n) = m$. To see that this is a homomorphism we note that $\varphi(m) = n - \psi(\alpha(n))$ and $\varphi(m') = n' - \psi(\alpha(n'))$ then $r\varphi(m) + \varphi(m') = rn - n' - \psi(\alpha(rn + n'))$ and so $r\varphi(m) + \varphi(m') = \varphi(rm + m')$. Lastly, we need to prove that $\beta \circ \varphi = 1_M$, or equivalently $\varphi(m) = f(m) - \psi(\alpha(\varphi(m)))$ but since $Im(\varphi) = Ker(\alpha)$ this is clear.

$b) \Rightarrow a)$ To do this we define $\theta : N \to M \oplus L$ by $n \mapsto (\beta(n), \alpha(n))$. This is a homomorphism since each coordinate map is a homomorphism. To see that θ is a monomorphism we note that $\theta(n) = \theta(n')$ implies $\beta(n) = \beta(n')$ and $\alpha(n) = \alpha(n')$. From the second of these equalities we gather that $n - n' \in Ker(\alpha)$ and since the sequence is exact this implies that $n - n' \in Im(\varphi)$ and so as previously noted $\varphi(\beta(n - n')) = n - n'$, but $\varphi(\beta(n - n')) = \varphi(\beta(n) - \beta(n')) = \varphi(0) = 0$ and so $n = n'$. To see that θ is surjective we let $(m, l) \in M \oplus L$ be arbitrary. Since α is surjective we know that there exists $n \in N$ such that $\alpha(n) = l$ and since β is surjective (since it has a right inverse) we may find $n' \in N$ with $\beta(n') = m$. So, set $x = n + \varphi(\beta(n' - n))$. Note then that

$$\beta(x) = \beta(n) + \beta(\varphi(\beta(n' - n))) = \beta(n) + \beta(n') - \beta(n) = \beta(n') = m$$

and

$$\alpha(x) = \alpha(n) + \alpha(\varphi(\beta(n' - n))) = \alpha(n) = l$$

Since the sequence is exact. Thus, $\theta(x) = (\beta(x), \alpha(x)) = (m, n)$. Since (m, n) was arbitrary the surjectivity follows. It only remains to show that the relevant diagram commutes. This amounts to showing that $\theta \circ \varphi = \iota_M \circ 1_M$ and $1_L \circ \alpha = \pi_L \circ \theta$. But it is clear since, for example $\theta(\varphi(m)) = (\beta(\varphi(m)), \alpha(\varphi(m))) = (m, 0) = \iota(m)$ and $\pi_L(\theta(n)) = \pi_L(\beta(n), \alpha(n)) = \alpha(n)$. Combining all this gives the desired conclusion.

$\square$

Let $f : M \to L$ and $g : M \to N$ be module homomorphisms:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & L \\ N & \end{tikzcd}$$

Then the **pushout** of these maps is a module F together with homomorphisms $\alpha : L \to F$ and $\beta : N \to F$ such that $\alpha \circ f = \beta \circ g$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & L \arrow[d, "\alpha"] \\ N \arrow[r, "\beta"'] & F \end{tikzcd}$$

And such that the following mapping property holds: Suppose that G is a module and that $\alpha' : L \to G$ and $\beta' : N \to G$ are homomorphism with $\alpha' \circ f = \beta' \circ g$, then there is a unique module homomorphism $\varphi : F \to G$ with $\varphi \circ \alpha = \alpha'$ and $\varphi \circ \beta = \beta'$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g"'] & L \arrow[d, "\alpha"] \arrow[ddr, bend left, "\alpha'"] & \\
N \arrow[r, "\beta"'] \arrow[drr, bend right, "\beta'"'] & F \arrow[dr, dashed, "\varphi"] & \\
 & & G
\end{tikzcd}$$

If f, g are the 0 maps, then the commutativity condition are trivial, and so the mapping property above is exactly that of the coproduct, or direct sum, of L and N. We will use the direct sum to prove the existence of the pushout of any diagram.

Proposition 3.0.3. *Let $f : M \to L$ and $g : M \to N$ be module homomorphisms. If we set $T = \{(f(m), -g(m)) \in L \oplus N : m \in M\}$, then $(L \oplus N)/T$, together with the maps $\alpha(l) = (l, 0) + T$ and $\beta(n) = (0, n) + T$ is the pushout of f and g.*

Proof. With the detinition of α and β, we have $\alpha \circ f = \beta \circ g$ since $(f(x), 0) \equiv (0, g(x)) modT$. We then have to verify the mapping property. Set $F = (L \oplus N)/T$, and suppose there is a module G and maps $\alpha' : L \to G$ and $\beta' : N \to G$ with $\alpha' \circ f = \beta \circ g$. To define $\varphi : F \to G$, we have the canonical map $\alpha' \oplus \beta' : L \oplus N \to G$, given by $(l, n) \mapsto \alpha'(l) + \beta'(n)$, that arises from the mapping property of a coproduct. This map sends $(f(m), -g(m))$ to $\alpha'(f(m)) - \beta'(g(m)) = 0$ since $\alpha' \circ f = \beta' \circ g$. Therefore, $\alpha' \oplus \beta'$ factors through T to give a map $\varphi : F \to G$, defined by $\varphi((l, n) + T) = \alpha'(l) + \beta'(n)$.It is easy to see that $\alpha' = \varphi \circ \alpha$ and $\beta' = \varphi \circ \beta$. Moreover, the definition of φ is forced upon us by the requirement that $\alpha' = \varphi \circ \alpha$ and $\beta' = \varphi \circ \beta$. Thus, F, together with α and β, is a pushout of f, g. □

Proposition 3.0.4. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\beta"'] & P \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & A \arrow[r] \arrow[d, "1_A"'] & 0 \\
0 \arrow[r] & B \arrow[r, "j"'] & X \arrow[r, "\tau"'] & A \arrow[r] & 0
\end{tikzcd}$$

Then X is the pushout of B and P with respect to i and β.

Proof. We first note that $X = \alpha(P) + j(B)$. To prove this, let $x \in X$. Then $\tau(x) = \sigma(p)$ for some $p \in P$. Therefore, $x - \sigma(p) \in Ker(\tau) = im(j)$, so $x - \alpha(p) = j(b)$ for some $b \in B$.

Thus, $x = \alpha(p) + j(b)$, as desired. We have $j \circ \beta = \alpha \circ i$ by the assumption that the diagram is commutative. To verify the mapping property, suppose G is a module with homomorphisms $\alpha' : P \to G$ and $j' : B \to G$ such that $\alpha' \circ i = j' \circ \beta$. We define $\varphi : X \to G$ by $\varphi(\alpha(p) + j(b)) = \alpha'(p) + j'(b)$. If we show that φ is well defined, then we get $\varphi \circ \alpha = \alpha'$ and $\varphi \circ = j'$ by alternatively setting $b = 0$ and $p = 0$ in the definition of φ. Furthermore, 1_ϕ is another map from X to G with

$\phi \circ \alpha = \alpha'$ and $\phi \circ \beta = \beta'$, then

$$\phi(\alpha(p) + j(b)) = \varphi(\alpha(p)) + \varphi(j(b)) = \alpha'(p) + j'(b) = \varphi(\alpha(p) + j(b))$$

showing that $\phi = \varphi$. Also, it is clear that φ will be a homomorphism once we know that its well defined. To see that φ is well defined, it is enough to show that if $\alpha(p) + j(b) = 0$, then $\alpha'(p) + j'(b) = 0$. So, suppose that $\alpha(p) + j(p) = 0$. Applying τ gives $0 = \tau(\alpha(p)) = \sigma(p)$.

Thus, $p = i(m)$ for some $m \in M$. Therefore, $0 = \alpha(i(m)) + j(b) = j(\beta(m)) + j(b)$. Since j is injective, $\beta(m) + b = 0$, or $b = -\beta(m)$. This yields

$$\alpha'(p) + j'(b) = \alpha'(i(m)) + j'(-\beta(m)) = (\alpha' \circ i - j' \circ \beta)(m) = 0$$

as $\alpha' \circ i = j' \circ \beta$. this shows that φ is well defined, and finishes the proof. □

We say that a module J is **injective** when every homomorphism into J can be factored or extended through every monomorphism: if $\varphi : M \to J$ and $\mu : M \to N$ are module homomorphisms, and μ is injective, then $\varphi = \psi \circ \mu$ for some homomorphism $\psi : N \to J$.

$$\begin{tikzcd}
 & J & \\
0 \arrow[r] & M \arrow[u, "\varphi"] \arrow[r, "\mu"'] & N \arrow[ul, dashed, "\psi"']
\end{tikzcd}$$

Worth mentioning that this factorization is not necessarily unique, because ψ does't have to be unique.

Proposition 3.0.5. *If J is an injective module and if A is a direct summand of J, then A is injective*

Proof. We can assume $A \subseteq J$ and that there is an homomorphism $\pi : J \to A$ that is the identity on A. Let $W \subseteq V$ be modules and let $\theta : W \to A$ be given. Then $\theta : W \to J$ and , since J is injective, θ extends to $\theta^* : V \to J$. Finally, $\pi \circ \theta^* : V \to A$ extends the map $\theta : W \to A$. □

Proposition 3.0.6. *For a module J the following condition are equivalent:*

a) J is injective.

b) Every monomorphism $J \to M$ splits

c) Every short exact sequence $0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$ splits.

d) J is a direct summand of every module $M \supseteq J$.

Proof. $a) \Rightarrow b)$ Let J injective and $\mu : J \to M$ a monomorphism, consider 1_J, it can be extended through μ for some $\psi : M \to J$, then we have the next diagram:

$$\begin{tikzcd}
& J & \\
0 \arrow[r] & J \arrow[u, "1_J"] \arrow[r, "\mu"'] & M \arrow[ul, dashed, "\psi"']
\end{tikzcd}$$

In conclusion, $\psi \circ \mu = 1_J$, in other words μ splits.

$b) \Rightarrow a)$ Let $\varphi : M \to N$ be a monomorphism and let $\beta : M \to J$ be a homomorphism. Then we have the pushout defined by

$$M' := \frac{N \oplus J}{\{(\varphi(m), -\beta(m)) : m \in M\}}.$$

We have natural maps from N and J to M', call them ι_1 respectively ι_2. First notice that by construction of M' it follows that $\iota_1 \circ \varphi = \iota_2 \circ \beta$. We claim that ι_2 is injective, indeed if $(0, i) = (f(m), g(m))$ for some $m \in M$, it follows that $m = 0$ and hence $i = 0$ (since φ is monomorphism). By our assumption we obtain a splitting map $\psi : M' \to J$, that is $\psi \circ \iota_2 = 1_I$. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \\
& J \arrow[r, bend left, "\iota_2"] & M' \arrow[l, bend left, "\psi"]
\end{tikzcd}$$

We obtain a map $\lambda := \psi \circ \iota_1 : N \to J$. We just calculate:

$$\lambda \circ \varphi = \psi \circ \iota_1 \circ \varphi = \psi \circ \iota_2 \circ \beta = \beta$$

In conclusion, β can be factored through every monomorphism. In other words, J is injective.

$a) \Rightarrow c)$ Suppose that J is injective and also that we have the next exact sequence $0 \longrightarrow J \xrightarrow{\varphi} B \longrightarrow C \xrightarrow{\beta}$ Note that we have the following diagram

$$\begin{tikzcd}
& J \\
0 \arrow[r] & J \arrow[u, "1_J"] \arrow[r, "\varphi"] & B
\end{tikzcd}$$

So there exists an homomorphism $\rho : B \to J$ such that $1_J = \rho \circ \varphi$. Thus, by the splitting lemma, we may conclude that our sequence splits.

$c) \Rightarrow a)$ Conversely, suppose that every exact sequence $0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$ of modules splits. Let $\beta : A \to B$ be a monomorphism between modules, and let $\varphi : A \to J$ be an homomorphism. Then there is the pushout with $D = (J \oplus B)/W$ where $W =$

$\{(\varphi(a), -\beta(a)) | a \in A\}$; this is shown by the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\varphi"'] & B \arrow[d, "\varphi'"] \\ & J \arrow[r, "\beta'"'] & D \end{tikzcd}$$

Then φ' and β' are homomorphism, given by $\varphi'(b) = (0, b) + W$ and $\beta'(j) = (j, 0) + W$. Note that for every $a \in A$, $(\varphi(a), -\beta(a)) \in W$, so

$$\beta'(\varphi(a)) = (\varphi(a), 0) + W = (0, \beta(a)) + W = \varphi'(\beta(a)).$$

Thus, $\beta' \circ \varphi = \varphi' \circ \beta$. We will now show that β' is one to one. Let $x \in Ker(\beta')$. Then $(x, 0) + W = \beta'(x) = (0, 0) + W$. Thus $(x, 0) \in W$, meaning that there exists $a \in A$ such that $\varphi(a) = x$ and $-\beta(a) = 0$. Since $a \in Ker(\beta)$ and β is one to one, $a = 0$, so $0 = \varphi(0) = \varphi(a) = x$. Then, $x = 0$ and β' is a monomorphism. So we can extend β' to an exact sequence $0 \longrightarrow J \xrightarrow{\beta'} D \xrightarrow{\phi} C \longrightarrow 0$ for some module C. By assumption, there exist an homomorphism $\kappa : D \to M$ such that a $\kappa \circ \beta' = 1_M$. Define $\mu : B \to J$ by $\mu = \kappa \circ \varphi'$, so μ is an homomorphism. Note that for every $a \in A$, $\mu \circ \varphi(a) = \kappa \circ \varphi' \circ \beta(a) = \kappa \circ \beta' \circ \varphi(a) = 1_J \circ \varphi(a) = \varphi(a)$, showing that $\mu \circ \beta = \varphi$. Hence, J is injective.

$a) \Rightarrow d)$ Assume J is injective and J is a submodule of a module M, then we have an exact sequence $0 \longrightarrow J \longrightarrow M \longrightarrow M/J \longrightarrow 0$ which, in view of c) splits. Then $M \cong J \oplus M/J$, in conclusion J is a direct summand of M.

$d) \Rightarrow a)$ Let $\varphi : M \to N$ be a monomorphism and let $\beta : M \to J$ be a homomorphism, then as above we have the pushout

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \\ 0 \arrow[r] & J \arrow[r, "\iota_2"] & M' \end{tikzcd}$$

By our assumption $M' = J \oplus J'$. Consider the canonical projection $\pi_J : J \oplus J' \to J$, by construction we have $\pi_J \circ \iota_2 = 1_J$, define $\theta = \pi_J \circ \iota_1$, then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \arrow[dl, "\theta"'] \\ & J \arrow[r, bend left, "\iota_2"] & M' \arrow[l, bend left, "\pi_J"] \end{tikzcd}$$

Finally consider $\theta \circ \varphi = \pi_J \circ \iota_1 \circ \varphi = \pi_J \circ \iota_2 \circ \beta = 1_J \circ \beta = \beta$.

In conclusion, β can be factored through every monomorphism. In other words, J is injective.

$\square$

Proposition 3.0.7. *Every direct summand of an injective module is injective.*

Proof. Let $J = A \oplus B$ a injective module, let $\varphi : M \to N$ be a monomorphism and also $\alpha : M \to A$ be a homomorphism. Consider the natural map $\iota_A : A \to A \oplus B$ and the canonical projection $\pi_A : A \oplus B \to A$, its clear that $\pi_A \circ \iota_A$. Since J is injective, there is a homomorphism $\beta : N \to J$ such that $\iota_A \circ \alpha = \beta \circ \varphi$. Define $\theta = \pi_A \circ \beta$ and consider

$$\theta \circ \varphi = \pi_A \circ \beta \circ \varphi = \pi_A \circ \iota_A \circ \alpha = \alpha$$

We can summarize the last proof in the next diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & N \arrow[dl, "\theta"'] \arrow[ddl, dashed, "\beta"] \\
& A \arrow[d, bend right, "\iota_A"'] & \\
& J = A \oplus B \arrow[u, bend right, "\pi_A"'] &
\end{tikzcd}$$

In conclusion, α can be factored through every monomorphism. In other words, A is injective. $\square$

Proposition 3.0.8. *Every direct product of modules is injective if and only if every factor is injective.*

Proof. Let $J_{i \in I}$ a family of modules. Suppose $\prod_{i \in I} J_i$ is injective, consider the next commutative diagram where A and B are modules:

$$\begin{tikzcd}
& \prod_{i \in I} J_i & \\
0 \arrow[r] & A \arrow[u, "\varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "\lambda"']
\end{tikzcd}$$

Since $\prod_{i \in I} J_i$ is a direct product we have the next diagram

$$\begin{tikzcd}
\prod_{i \in I} J_i \arrow[d, "\pi_i"'] & B \arrow[l, "\lambda"'] \arrow[dl, "l_i"] \\
J_i &
\end{tikzcd}$$

Then $l_i = \pi_i \circ \lambda$, and using the above diagrams we obtain

$$\begin{tikzcd}
& J_i & \\
0 \arrow[r] & A \arrow[u, "\pi_i \circ \varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "l_i"']
\end{tikzcd}$$

Since $\pi_i \circ \varphi = \pi_i \circ \lambda \circ \beta = l_i \circ \beta$. In conclusion, J_i is injective, for every $i \in I$.

Now suppose that for every $i \in I$, J_i is injective. Then for A and B modules we have the next diagram:

$$\begin{tikzcd}
& J_i & \\
0 \arrow[r] & A \arrow[u, "\gamma"] \arrow[r, "\beta"'] & B \arrow[ul, "\varphi_i"']
\end{tikzcd}$$

where we have $\varphi_i \circ \beta = \gamma$. Also, since $\prod_{i \in I} J_i$ is the direct product of J_i then we have the following diagrams commutes:

$$\begin{tikzcd}
B \arrow[r, "\varphi_i"] \arrow[d, "\varphi"'] & J_i \\
\prod_{i\in I} J_i \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
J_i \arrow[r, "\psi_i"] \arrow[d, "\iota_i"'] & B \\
\prod_{i\in I} J_i \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Because $\psi \circ \iota_i = \psi_i$ and $\pi_i \circ \varphi = \varphi_i$. Now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[dl, bend right, "\varphi_i"'] \arrow[d, bend right, "\varphi"'] \\
& J_i \arrow[r, bend left, "\iota_i"] \arrow[ur, "\psi_i"] & \prod_{i\in I} J_i \arrow[l, bend left, "\pi_i"] \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

Since $\iota_i \circ \pi_i = 1_{\prod_{i\in I} J_i}$ then $\iota_i \circ \varphi_i = \iota_i \circ \pi_i \circ \varphi = \varphi$. Finally, we have $\iota_i \circ \gamma = \iota_i \circ \varphi_i \circ \beta = \varphi \circ \beta$, in conclusion the last diagram commutes, in other words $\prod_{i\in I} J_i$ is injective. $\square$

By the last proposition we can conclude that every finite external direct sum of modules is injective if and only if every module is injective, but this property does not extend to infinite external direct sums.

Theorem 3.0.9 (Baer's Criterion)**.** *An R-module M is injective if and only if every homomorphism $I \to M$, where I is an ideal of R, can be extended to a homomorphism $R \to M$.*

Proof. Suppose that M is an injective R-module. Let $\varphi : I \to M$ be a homomorphism, where I is a left ideal of R. Note that I and R are both R-modules, where the inclusion mapping $\iota : I \to R$ is a monomorphism. Since M is injective, there exists an homomorphism $\alpha : R \to M$ such that $\varphi = \alpha \circ \iota$. Thus, $\alpha(a) = \alpha \circ \iota(a) = \varphi(a)$ for every $a \in I$.

Now, consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"'] & N' \\
& M &
\end{tikzcd}$$

We need to find a map $h : N' \to M$. Consider the set of pairs (N'', h) such that $N \subseteq N'' \subset N'$, $h : N'' \to M$ with the property that $h|_N = g$. This set is non-empty, since it contains (N, g). We order this set by the relations that $(N_1, h_1) \leq (N_2, h_2)$ if $N_1 \subset N_2$ and $h_2|_{N_1} = h_1$. A non-empty chain (S_i, h_i) has an upper bound, namely the 'union' defined as (S, h) where $S = \cup S_i$ and for $s \in S_i$ define $h(x) = h_i(x)$. Zorn's lemma now give a maximal element (N'', h), we claim that $N'' = N'$ and hence h will be an extension of g. Suppose that $N'' \neq N'$ and let $x \in N' \setminus N''$. Let $I = \{r \in R : rx \in N''\} \subset R$, then I is an ideal of R.

Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "i \mapsto h(ix)"'] & R \\ & M & \end{tikzcd}$$

For $i \in I$ we have that $ix \in N''$ and hence $h(ix)$ is defined and this obviously is linear. By the assumption of the theorem, we obtain a map $\varphi : R \to M$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "i \mapsto h(ix)"'] & R \arrow[dl, "\varphi"'] \\ & M & \end{tikzcd}$$

As $x = 1 \cdot x$, it seems natural to define the following map:

$$\begin{aligned} \varphi' :& Rx + N' \to M \\ & rn + n'' \mapsto r\varphi(1) + h(n'') \end{aligned}$$

for $r \in R$ and $n'' \in N$. We check that this maps is well-defined. For this suppose that $rx = n$ there $r \in R$ and $n \in N''$. But this follows since $r\varphi(1) = \varphi(r) = h(rx) = h(n)$. Hence $(Rx+N'', \varphi)$ is a proper extension of (N'', h) contradicting the maximality of (N'', h). Hence $N'' = N'$ and we are done. $\square$

Every abelian group can be considered in a natural sense as a $\mathbb{Z}$-module, it is important to remember that group A is called **divisible** if and only if for every element $z \in \mathbb{Z}$ if $z \neq 0$ then $Az = A$. Also it is important to remember that every epimorphic image of a divisible group is divisible and consequently every factor group of a divisible group is divisible, and also every abelian group can be mapped monomorphically into a divisible group.

Proposition 3.0.10. *If $\varphi : M_{\mathbb{Z}} \to L_{\mathbb{Z}}$ is a monomorphism and if $M_{\mathbb{Z}}$ is divisible, then φ splits*

Proof. By the last property of divisible groups, $Im(\varphi)$ is divisible, so that without loss of generality we can consider $M_{\mathbb{Z}}$ to be a submodule of $L_{\mathbb{Z}}$ and $\varphi = \iota$ to be the inclusion mapping. Let then $T := \{U | U \mapsto L \bigwedge M \cap U = 0\}$.

Since we have $U = 0 \in T$, $T \neq \phi$; since further the union of every totally ordered subset of T (under inclusion) is again an element of T, there is by reason of Zorn's Lemma a maximal element in T, which is again to be denoted by U. As a result we then have $M+U = M \oplus U \mapsto L$, and it is to be shown that $L = M \oplus U$.

For an arbitrary $l \in L$ we consider the ideal $z_0\mathbb{Z}$ consisting if the $z \in \mathbb{Z}$ with $lz \in M+U$. Let $lz_0 = m+u$. Since M is divisible there is a m_0 with $m_0 z_0 = m$ then $(l-m_0)z_0 = u$. Evidently $z_0\mathbb{Z}$ is then also the ideal of the $z \in \mathbb{Z}$ with $(l-m_0)z \in M+U$.

We claim that $M \cap (U+(l-m_0)\mathbb{Z}) = 0$. Assume $m_1 = u_1 + (l-m_0)z_1 \in M \cap (U+(l-m_0)\mathbb{Z})$. Then $(l-m_0)z_1 = m_1 - u1 \in M+U$ and so $z_1 = z_0 t$, $t \in \mathbb{Z}$. Then $(l-m_0)z_0 t = ut = m_1 - u_1$. Then $0 = m_1 - (u_1 + ut)$ finally $m_1 = 0$. From the maximality of U it follows that $(l-m_0)\mathbb{Z} \mapsto U$ then $l - m_0 \in U$ thus $l \in M+U$. Thus we have in fact, $L = M \oplus U$. □

For $R = \mathbb{Z}$ there is characterization and this has also considerable significance for the case of an arbitrary ring R, it is in fact used to show the existence of injective extensions.

Theorem 3.0.11. *A $\mathbb{Z}$-module (an abelian group) is injective if and only if it is divisible.*

Proof. Let $M_\mathbb{Z}$ be divisible, then D is injective. Now let $Q_\mathbb{Z}$ be injective. Let $q_o \in Q$, $0 \neq z_0 \in \mathbb{Z}$; if we consider the homomorphisms

$$\begin{tikzcd} z_0\mathbb{Z} \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \mathbb{Z} \arrow[dl, dashed, "\kappa"] \\ Q & \end{tikzcd}$$

where ι is the inclusion mapping and φ is defined by $\varphi(z_0) := q_0$, then there is a κ with $\varphi = \kappa \circ \iota$, since Q is injective. Thus we have $\kappa(1)z_0 = \kappa(1z_0) = \kappa(z_0) = (\kappa \circ \iota)(z_0) = \varphi(z_0) = q_0$. Since $q_0 \in Q$ was arbitrary, it follows therefore that $Qz_0 = Q$, i.e. Q is divisible. □

Proposition 3.0.12. *If M is divisible (injective) $\mathbb{Z}$-module then $Hom_\mathbb{Z}(R, M)$ is injective as a R-module.*

Proof. Let $\alpha : A \to B$ be an R-monomorphism and let $\varphi : A \to Hom_\mathbb{Z}(R, M)$ be an R-homomorphism. Let σ be the $\mathbb{Z}$-homomorphism defined by $\sigma : Hom_\mathbb{Z}(R, M) \to M$, $f \mapsto f(1)$ then we consider the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & B \arrow[dl, dashed, "\kappa"'] \arrow[ddl, dashed, "\tau"] \\ Hom_\mathbb{Z}(R, M) \arrow[d, "\sigma"'] & \\ M & \end{tikzcd}$$

If we regard α and φ only as $\mathbb{Z}$-homomorphisms, then there is, since M is $\mathbb{Z}$-injective, a $\mathbb{Z}$-homomorphism $\tau : B \to M$ with $\alpha \circ \varphi = \tau \circ \alpha$. Now let $\tau : B \to Hom_\mathbb{Z}(R, M)$ be defined by $\kappa(b)(r) = \tau(br)$, $b \in B$, $r \in R$.

Then for fixed $b \in B$, obviously $\kappa(b) \in Hom_\mathbb{Z}(R, M)$ and we have

$$\kappa(br_1)(r) = \tau(br_1 r) = \kappa(b)(r_1 r) = (\kappa(b)r_1)(r),$$

i.e $\kappa(br_1) = \kappa(b)r_1$, thus κ is an R-homomorphism. Therefore we have

$$\begin{aligned}\kappa \circ \alpha(a)(r) &= \tau(\alpha(a)r) = \tau(\alpha(ar)) = \tau \circ \alpha(ar) = \sigma \circ \varphi(ar)\\ &= \varphi(ar)(1) = (\varphi(a)r)(1) = \varphi(a)(1)\end{aligned}$$

and consequently $\kappa \circ \alpha = \varphi$ □

Proposition 3.0.13. *For every module there is a monomorphism into an injective module.*

Proof. Let M a module. Then there is a $\mathbb{Z}$-monomorphism $\mu : M \to D$ into a divisible abelian group. Thus $Hom_{\mathbb{Z}}(R, D)_R$ is injective as an R-module. If we define $\rho : M \to Hom_{\mathbb{Z}}(R, D)$ by $\rho(m)(r) = \mu(mr)$, $m \in M$, $r \in R$, then ρ is evidently an R-homomorphism and, since μ is a monomorphism, even a monomorphism. □

Lemma 3.0.14. *Let $\rho : M_R \to N_R$ be a monomorphism. Then there is a module N' with $M \to N'$ and an isomorphism $\tau : N' \to N$ that $\rho = \tau \circ \iota$, where ι is the inclusion mapping of M in N'.*

Proof. Let D be a set of the same cardinality as the complement $N \setminus \rho(M)$ if $\rho(M)$ in N with $D \cap M = \phi$ and let $\beta : D \to N \setminus \rho(M)$ be an injective mapping. Then define a set $N' = M \cup D$ and let $\tau : N' \to N$ be the bijective mapping defined by

$$\begin{aligned}\tau(m) &= \rho(m) \quad m \in M\\ \tau(d) &= \beta(d) \quad\ \ d \in D.\end{aligned}$$

In order to make N' into a module containing M_R and to make τ into a module homomorphism, we put:

$$\begin{aligned}x + y &= \tau^{-1}(\tau(x) + \tau(y)) \quad x, y \in N'\\ xr &= \tau^{-1}(\tau(x)r) \qquad\qquad r \in R.\end{aligned}$$

As it is immediately seen, all assertions are then satisfied. □

Theorem 3.0.15. *Every module is a submodule of an injective module.*

This theorem follows from the last lemma, since $Hom_{\mathbb{Z}}(R, D)_R$ and the isomorphic module N' are both injective.

3.1 Injective Hull

Now consider M a module, and let S a submodule of M, we say that S is **essential** when $S \cap T \neq 0$ for every submodule $T \neq 0$ of M. And also a monomorphism $\alpha : A \to B$ is called **essential** if $Im(\alpha)$ is an essential submodule of B.

Proposition 3.1.1. *If $A \subseteq B$ are modules of C, then A is essential in C if and only if A is essential B and B is essential in C*

Proof. Suppose A is essential in C. Thus for every submodule $D \subseteq C$ we have $A \cap D \neq 0$. But remember that $A \subseteq B$. Then we obtain $B \cap D \neq 0$, in other words B is essential in C. Let a subobject $E \subseteq B$, then we have that $E \subseteq C$ thus $A \cap E \neq 0$ in conclusion A is essential in C.

On the other hand, suppose A is essential in B and B is essential in C. Let $D \subseteq C$. Since B is essential, we have $B \cap D \neq 0$. But $B \cap D \subseteq B$ and also A is essential in B, thus $A \cap (B \cap C) \neq 0$ but we need to remember that $A \subseteq B$ that implies that $A \cap B = A$. In conclusion $A \cap D \neq 0$, in other words A is essential in C. □

Proposition 3.1.2. *If $\alpha : A \to B$ and $\beta : B \to C$ are monomorphisms, then $\beta \circ \alpha$ is essential if and only if both α and β are essential.*

Proof. Clearly $Im(\beta \circ \alpha) \subseteq Im(\beta)$ are subobjects of C. Suppose that $\beta \circ \alpha$ is essential, thus $Im(\beta \circ \alpha)$ is essential in C, by the last proposition we have $Im(\beta)$ is essential in C in other words β is a essential monomorphism, and also $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$. Let $N \subseteq B$ then we know that $\beta(N) \subseteq Im(\beta)$ thus $Im(\beta \circ \alpha) \cap \beta(N) \neq 0$ then there is $m \neq 0$ such that $m \in Im(\beta \circ \alpha)$ and $m \in \beta(N)$, then there is $x \in A$ such that $m = (\beta \circ \alpha)(x)$ and also there is $y \in N$ such that $m = \beta(y)$ then since β is a monomorphism then $0 \neq (\beta \circ \alpha)(x) = \beta(y)$ implies $\beta(x) = y \neq 0$ in conclusion $0 \neq y \in Im(\alpha) \cap N$, in other words α is essential.

Now, suppose α and β are essential, let $E \subseteq Im(\beta)$ then there is $F \subseteq B$ such that $\beta(F) = E$ but $Im(\alpha) \cap F \neq 0$ thus there exists $m \neq 0$ such that $0 \neq \beta(m) \in Im(\beta \circ \alpha)$ and $0 \neq \beta(m) \in \beta(F) = E$, in other words $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$. In conclusion we have $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$ and $Im(\beta)$ is essential in D by 3.1.1 $Im(\beta \circ \alpha)$ is essential in C, thus $\beta \circ \alpha$ is a essential monomorphism. □

Proposition 3.1.3. *If μ is an essential monomorphism, and $\varphi \circ \mu$ is injective, then φ is injective.*

Proof. Since $\varphi \circ \mu$ is injective, then $Ker(\varphi) \cap Im(\mu) = 0$, and also μ is an essential monomorphism, hence $Ker(\varphi) = 0$. □

Consider the module A, an **essential extension** of A is defined as a module B such that A is an essential submodule of B, more general, a module B with an essential monomorphism $A \to B$.

Proposition 3.1.4. *If $\alpha : M \to J$ is a monomorphism and J is an injective object, then for every essential monomorphism $\beta : M \to N$ there exists a monomorphism $\gamma : N \to J$ such that $\gamma \circ \beta = \alpha$.*

Proof. Since J is injective, there is a $\gamma : N \to J$ such that $\gamma \circ \beta = \alpha$. Since α is a monomorphism we have $0 = Ker(\alpha) = Ker(\gamma \circ \beta) = Ker(\gamma) \cap Im(\beta)$, but β is essential, that implies $Ker(\gamma) = 0$. In conclusion γ is a monomorphism. □

Proposition 3.1.5. *An module J is injective if and only if every essential monomorphism $\alpha : J \to C$ is an isomorphism.*

Proof. Suppose J is injective, and $\alpha : J \to C$ a essential monomorphism, thus α splits, in conclusion α must be onto.

Now suppose every essential monomophism is an isomorphism. Let $M \supseteq J$, by Zorn's lemma there is a subobject $K \subseteq M$ maximal such that $J \cap K = 0$. Let the canonical projection $\beta : M \to M/K$ and consider $\mu : J \to M/K$ given by $\mu(x) = \beta(x)$ for every $x \in J$. Let $0 \neq S \subseteq M/K$ then there is $L \subseteq M$ such that $S = L/K$.

We have two cases, the first one, suppose $J \cap L = 0$ that implies $L \subseteq K$ and also $L/K \subseteq K$ then $L/K \cap J/K \subseteq K \cap Im(\mu) \neq 0$ in conclusion μ is an essential monomorphism. On the other hand, suppose $J \cap L \neq 0$ then there is $m \neq 0$ such that $m \in J \cap L$ thus $m + K \in J/K \cap L/K = Im(\mu) \cap L/K$, in conclusion μ is an essential monomorphism.

Since μ is essential monomorphism, then is an isomorphism, hence $M = J+K$, since $J\cap K = 0$ thus $M = J \oplus K$, in other words J is injective. □

Proposition 3.1.6. *Let $\mu : M \to N$ and $\nu : M \to J$ be monomorphisms. If μ is essential and J is injective, then $\nu = \kappa \circ \mu$ for some monomorphisms $\kappa : N \to J$.*

Proof. Since J is injective, there exists a homomorphism $\kappa : N \to J$ such that $\nu = \kappa \circ \mu$, which is injective by 3.1.3. □

By the last proposition, every essential extension of a module M is, up to isomorphism, contained in every injective extension of M.

Theorem 3.1.7. *Every module M is an essential submodule of an injective module, which is unique up to isomorphism.*

Proof. M is a submodule of an injective submodule K. Let $\mathcal{S}$ be the set of all submodules $M \subseteq S \subseteq K$ of K in which M is essential. If $(S_i)_{i\in I}$ is a chain in $\mathcal{S}$, then $S = \cup_{i\in I} S_i \in \mathcal{S}$: if $N \neq 0$ is a submodule of S, then $S_i \cap N \neq 0$ for some i, and then $M \cap N = M \cap S_i \cap N \neq 0$ since M is essential in S_i; thus M is essential in S. By Zorn's lemma, $\mathcal{S}$ has a maximal element J. If J has a proper essential extension, then by the last proposition J would have a proper essential extension $J \subsetneqq J' \subseteq K$ and would not be maximal; therefore J is injective by 3.1.5.

Now, assume that M is essential in two injective modules J and J'. The inclusion monomorphisms $\mu : M \to J$ and $\nu : M \to J'$ are essential. By the last proposition there is a monomorphisms $\theta : J \to J'$ such that $\nu = \theta \circ \mu$. Then θ is essential, by the proposition 3.1.2, and is an isomorphisms by 3.1.5. □

The last theorem generates the main definition of this section, that is, the injective hull of a module M is the injective module unique up to isomorphism, in which M is an essential submodule. The injective hull of injective envelope of M is denoted by $E(M)$.

The injective hull of M can be characterized in several ways, $E(M)$ is injective and an essential extension of M, it is a maximal essential extension of M. $E(M)$ is, up to isomorphism, the largest essential extension of M and also $E(M)$ is a minimal injective extension of M.

Lemma 3.1.8. *If $M_i \to M_i'$ $(i = 1, \ldots n)$ are essential monomorphisms then $M_1 \oplus \cdots \oplus M_n \to M_1' \oplus M_n'$ is essential*

Proof. By induction, we only proof for $n = 2$, we show that each of the two monomorphisms $M_1 \oplus M_2 \to M_1' \oplus M_2 \to M_1' \oplus M_2'$ is essential. For the first one consider $\pi : M_1' \otimes M_2 \to M_1'$ denote the canonical projection and suppose $\beta : B \to M_1' \oplus M_2$ is an arbitrary non-zero monomorphism. Then we have two cases, either $\pi \circ \beta = 0$, then $Im(\beta) \subseteq M_2$ and hence $Im(\beta) \cap (M_1 \oplus M_2) \neq 0$ or $\pi \circ \beta \neq 0$ in which case $Im(\pi \circ \beta) \cap M_1 \neq 0$ by $M_1 \to M_1'$ is a essential monomorphism, and hence $Im(\beta) \cap (M_1 \oplus M_2) \neq 0$. In any case this shows that $M_1 \oplus M_2$ is essential in $M_1' \oplus M_2$, that implies $M_1 \oplus M_2 \to M_1' \oplus M_2$ is an essential monomorphism. For the second one is analogous with $\pi : M_1' \otimes M_2' \to M_2'$ and $\beta : B \to M_1' \oplus M_2'$. $\square$

The following proposition is a consequence of the last lemma:

Proposition 3.1.9. *The monomorphism $M_1 \oplus \cdots M_n \to E(M_1) \oplus \cdots E(M_n)$ induces an isomorphism*

$$E(M_1 \oplus \cdots M_n) \cong E(M_1) \oplus \cdots E(M_n)$$

Chapter 4

Category of Modules

4.1 RD-Injective Modules

A submodule M of an module N is called **relatively divisible** (RD-submodule) if $rM = M \cap rN$ for each $r \in R$.

As the inclusion $rM \leq M \cap rN$ holds for all submodules M of N, relative divisibility amouts to the reverse inclusion, i.e. Let M and N be modules such that $M \leq N$. We say that M is **relatively divisible** in N if for each $r \in R$ and each $a \in M$, solubility of the equation $rx = a$ in N implies its solubility in M.

An exact sequence of modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{4.1}$$

is RD**-exact** if M is a relatively divisible submodule of N.

An module M is said to be RD**-inyective** if it has the injective property relative to all RD-exact sequences. That is, For every monomorphism: $\alpha : A \longrightarrow B$ where $Im(\alpha)$ is relatively divisible in B and for every map $\phi : A \longrightarrow M$ there is a homomorphism $\psi : B \longrightarrow M$ such that $\phi = \psi \circ \alpha$.

Proposition 4.1.1. *The module Q is RD-*injective *if and only if it is a direct summand of every module that contains it as a relatively divisible submodule.*

Proof. Let Q be a RD-injective and consider the RD-exact sequence

$$0 \to Q \to N \to N/Q \to 0$$

Then this sequence splits, in other words $N \cong Q \oplus N/Q$.

Let $\varphi : M \to N$ be a monomorphism where $Im(\varphi)$ is relatively divisible in N and let $\beta : M \to$

Q be a homomorphism, then like in the last chapter

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \\ 0 \arrow[r] & Q \arrow[r, "\iota_2"] & M' \end{tikzcd}$$

By our assumption $M' = Q \oplus J'$, consider the canonical projection $\pi_Q : Q \oplus J' \to Q$, by construction we have $\pi_Q \circ \iota_2 = 1_Q$, define $\theta = \pi_Q \circ \iota_1$, then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \arrow[dl, "\theta"'] \\ & Q \arrow[r, bend left, "\iota_2"] & M' \arrow[l, bend left, "\pi_J"] \end{tikzcd}$$

Finally consider $\theta \circ \varphi = \pi_Q \circ \iota_1 \circ \varphi = \pi_Q \circ \iota_2 \circ \beta = 1_Q \circ \beta = \beta$

In conclusion, β can be factored through every monomorphism, in other words Q is RD-injective. $\square$

Clearly, the class of injective modules over R is a full subcategory of the category of *RD*-injective R-modules. The class of *RD*-injective modules have properties similar to that of injective modules.

Proposition 4.1.2. *A direct product of modules is RD-injective if and only if each factor is likewise RD-injective .*

Proof. Let $Q_{i \in I}$ a family of modules. Suppose $\prod_{i \in I} Q_i$ is RD-inyective, consider the next commutative diagram where A and B are modules and $Im(\beta)$ is relatively divisible in B:

$$\begin{tikzcd} & \prod_{i \in I} Q_i & \\ 0 \arrow[r] & A \arrow[u, "\varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "\lambda"'] \end{tikzcd}$$

Since $\prod_{i \in I} Q_i$ is a direct product we have the next diagram

$$\begin{tikzcd} \prod_{i \in I} Q_i \arrow[d, "\pi_i"'] & B \arrow[l, "\lambda"'] \arrow[dl, "l_i"] \\ Q_i & \end{tikzcd}$$

then $l_i = \pi_i \circ \lambda$, and using the above diagrams we obtain

$$\begin{tikzcd} & Q_i & \\ 0 \arrow[r] & A \arrow[u, "\pi_i \circ \varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "l_i"'] \end{tikzcd}$$

Since $\pi_i \circ \varphi = \pi_i \circ \lambda \circ \beta = l_i \circ \beta$. In conclusion, Q_i is RD-injective, for every $i \in I$.
Now suppose that for every $i \in I$, Q_i is RD-injective, then consider A and B modules and $\beta : A \to B$ with $Im(\beta)$ is relatively divisible in B, then we have the next diagram:

$$\begin{tikzcd}
& Q_i & \\
0 \arrow[r] & A \arrow[u, "\gamma"] \arrow[r, "\beta"'] & B \arrow[ul, "\varphi_i"']
\end{tikzcd}$$

where we have $\varphi_i \circ \beta = \gamma$. Also, like $\prod_{i\in I} Q_i$ is the direct product of Q_i then we have the following diagrams commutes:

$$\begin{tikzcd}
B \arrow[r, "\varphi_i"] \arrow[d, "\varphi"'] & Q_i \\
\prod_{i\in I} Q_i \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Q_i \arrow[r, "\psi_i"] \arrow[d, "\iota_i"'] & B \\
\prod_{i\in I} Q_i \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Because $\psi \circ \iota_i = \psi_i$ and $\pi_i \circ \varphi = \varphi_i$. Now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[dl, bend right, "\varphi_i"'] \arrow[d, bend right, "\varphi"'] \\
& Q_i \arrow[ur, "\psi_i"] \arrow[r, "\iota_i"] & \prod_{i\in I} Q_i \arrow[u, bend right, "\psi"'] \arrow[l, bend left, "\pi_i"]
\end{tikzcd}$$

Since $\iota_i \circ \pi_i = 1_{\prod_{i\in I} Q_i}$ then $\iota_i \circ \varphi_i = \iota_i \circ \pi_i \circ \varphi = \varphi$. Finally, we have $\iota_i \circ \gamma = \iota_i \circ \varphi_i \circ \beta = \varphi \circ \beta$. In conclusion the last diagram commutes, in other words $\prod_{i\in I} Q_i$ is RD-injective. $\square$

It can be shown, as in the case of injective modules that every module is a relatively divisible submodule of an *RD*-injective module.

Let M be a relatively divisible submodule of the *RD*-injective module Q. Then there exists a minimal *RD*-injective submodule of Q which contains M as a relatively divisible submodule. Moreover, any two minimal *RD*-injective modules which contain M as a relatively divisible submodule are isomorphic over M. Such module (which, as we said, is unique up to isomorphism over M) is called the ***RD*-injective hull** of M.

4.2 Pure Injective Modules

Let M and N be modules such that $M \leq N$. we say that M is **pure** in N if for every $m, n \in \mathbb{Z}^+$, each system of equations

$$\sum_{j=1}^{m} r_{ij} x_j = a_i \in M \quad (i = 1, \ldots, n)$$

with coefficients $r_{ij} \in R$ is soluble in M whenever it is soluble in N.

Clearly purity implies relative divisibility.

An exact sequence of modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{4.2}$$

is **pure-exact** if M is pure submodule of N.

An module M is called **pure-inyective** if it has the injective property relative to all pure-exact sequences. That is, For every monomorphism: $\alpha : A \longrightarrow B$ where $Im(\alpha)$ is pure in B and for every map $\phi : A \longrightarrow M$ there is a homomorphism $\psi : B \longrightarrow M$ such that $\phi = \psi \circ \alpha$. Equivalently, we say that the module Q is **pure-injective** if it is a direct summand of every module that contains it as a pure submodule.

Clearly, the class of injective modules over R is a full subcategory of the category of RD-injective R-modules which, in turn, is a full subcategory of the category of pure-injective R-modules. The classes of RD-injective and pure-injective modules have properties similar to that of injective modules.

Proposition 4.2.1. *The module P is pure-injective if and only if it is a direct summand of every module that contains it as a relatively pure-submodule.*

Proof. Let P be a pure-injective and consider the pure-exact sequence

$$0 \to P \to N \to N/Q \to 0$$

Then this sequence splits, in other words $N \cong P \oplus N/P$.

Let $\varphi : M \to N$ be a monomorphism where $Im(\varphi)$ is pure in N and let $\beta : M \to P$ be a homomorphism, then like in the last chapter

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \\
0 \arrow[r] & P \arrow[r, "\iota_2"] & M'
\end{tikzcd}$$

By our assumption $M' = P \oplus J'$, consider the canonical projection $\pi_P : P \oplus J' \to P$, by construction we have $\pi_P \circ \iota_2 = 1_P$, define $\theta = \pi_P \circ \iota_1$, then we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[d, "\beta"'] & N \arrow[d, "\iota_1"] \arrow[dl, "\theta"'] \\
 & P \arrow[r, bend left, "\iota_2"] & M' \arrow[l, bend left, "\pi_J"]
\end{tikzcd}$$

Finally consider $\theta \circ \varphi = \pi_P \circ \iota_1 \circ \varphi = \pi_P \circ \iota_2 \circ \beta = 1_P \circ \beta = \beta$

In conclusion, β can be factored through every monomorphism, in other words P is pure-injective. $\square$

Proposition 4.2.2. *A direct product of modules is pure-injective if and only if each factor is likewise pure-injective.*

Proof. Let $P_{i\in I}$ a family of modules. Suppose $\prod_{i\in I} P_i$ is pure-inyective, consider the next commutative diagram where A and B are modules and $Im(\beta)$ is pure in B:

$$\begin{tikzcd}
 & \prod_{i\in I} P_i & \\
0 \arrow[r] & A \arrow[u, "\varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "\lambda"']
\end{tikzcd}$$

Since $\prod_{i\in I} P_i$ is a direct product we have the next diagram

$$\begin{tikzcd}
\prod_{i\in I} P_i \arrow[d, "\pi_i"'] & B \arrow[l, "\lambda"'] \arrow[dl, "l_i"] \\
P_i &
\end{tikzcd}$$

then $l_i = \pi_i \circ \lambda$, and using the above diagrams we obtain

$$\begin{tikzcd}
 & P_i & \\
0 \arrow[r] & A \arrow[u, "\pi_i\circ\varphi"] \arrow[r, "\beta"'] & B \arrow[ul, "l_i"']
\end{tikzcd}$$

Since $\pi_i \circ \varphi = \pi_i \circ \lambda \circ \beta = l_i \circ \beta$. In conclusion, P_i is pure-injective, for every $i \in I$.
Now suppose that for every $i \in I$, P_i is pure-injective, then consider A and B modules and $\beta : A \to B$ with $Im(\beta)$ is pure in B, then we have the next diagram:

$$\begin{tikzcd}
 & P_i & \\
0 \arrow[r] & A \arrow[u, "\gamma"] \arrow[r, "\beta"'] & B \arrow[ul, "\varphi_i"']
\end{tikzcd}$$

where we have $\varphi_i \circ \beta = \gamma$. Also, like $\prod_{i\in I} P_i$ is the direct product of P_i then we have the following diagrams commutes:

$$\begin{tikzcd}
B \arrow[r, "\varphi_i"] \arrow[d, "\varphi"'] & P_i \\
\prod_{i\in I} P_i \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
P_i \arrow[r, "\psi_i"] \arrow[d, "\iota_i"'] & B \\
\prod_{i\in I} P_i \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Because $\psi \circ \iota_i = \psi_i$ and $\pi_i \circ \varphi = \varphi_i$. Now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[dl, bend right, "\varphi_i"'] \arrow[d, bend left, "\varphi"] \\
& P_i \arrow[r, bend left, "\iota_i"] \arrow[r, "\psi_i"'] & \prod_{i\in I} P_i \arrow[l, bend left, "\pi_i"] \arrow[u, bend right, "\psi"']
\end{tikzcd}$$

Since $\iota_i \circ \pi_i = 1_{\prod_{i\in I} P_i}$ then $\iota_i \circ \varphi_i = \iota_i \circ \pi_i \circ \varphi = \varphi$. Finally, we have $\iota_i \circ \gamma = \iota_i \circ \varphi_i \circ \beta = \varphi \circ \beta$. In conclusion the last diagram commutes, in other words $\prod_{i\in I} P_i$ is pure-injective. □

It can be shown, as in the case RD-injective modules that every module is a pure submodule of an pure-injective module.

Let M be a pure submodule of the pure-injective module Q. Then there exists a minimal pure-injective submodule of Q which contains M as a pure submodule. Moreover, any two minimal pure-injective modules which contain M as a relatively divisible submodule are isomorphic over M. Such module (which, as we said, is unique up to isomorphism over M) is called the **pure-injective hull** of M.

4.3 Category

A **category** $\mathcal{C}$ consist of the following data:

1. A class of objects $Ob(\mathcal{C})$, usually denoted by just $\mathcal{C}$.

2. For each $A, B \in \mathcal{C}$, a set of morphisms $Hom^{\mathcal{C}}(A, B)$. An element $f \in Hom^{\mathcal{C}}(A, B)$ is called a morphism between A and B, and will sometimes be denoted by $f : A \to B$ of $A \xrightarrow{f} B$.

3. An associative composition rule for morphisms $f : A \to B$ and $g : B \to C$, i.e. this is a map

$$Hom^{\mathcal{C}}(A, B) \times Hom^{\mathcal{C}}(B, C) \to Hom^{\mathcal{C}}(A, C)$$

   denoted by $(f, g) \mapsto f \circ g$.

4. For each object $A \in \mathcal{C}$ a distinguished identity morphism 1_A, which acts as a two-sided identity for composition of morphisms, i.e. for all $f \in Hom^{\mathcal{C}}(A, B)$, one has $f \circ 1_A = f$, and for all $g \in Hom^{\mathcal{C}}(B, A)$ one has $1_A \circ g = g$.

It's very important to mention some examples of categories:

- The category of sets. In other words, objects of this category are sets; if X and Y are sets, then $Hom(X, Y)$ is the set of all maps form X and Y; and composition of morphisms in the category is the usual compositions of maps.

- The category of groups. The morphisms being group homomorphisms.

- The category of abelian groups. The morphisms being group homomorphisms.

- The category of modules. The morphisms being module homomorphisms.
- The category of vector space. The morphisms being linear maps.
- The category of rings. The morphisms being maps.
- The category of commutative rings. The morphisms being maps.
- The category of topological spaces. The morphisms being continuous maps.

An **abelian category** is a category in which morphisms have structure of abelian group. A morphism $f : X \to Y$is called a **monomorphism** if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : Z \to X$. It is also called a mono or a monic. Dually to monomorphisms, a morphism $f : X \to Y$ is called an **epimorphism** if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : Y \to Z$. It is also called an epi or an epic. A morphism $f : X \to Y$ is called an **isomorphism** if there exists a morphism $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Let $\mathbf{C}$ and $\mathbf{J}$ be categories ($\mathbf{J}$ for index category). The diagonal functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ sends each object c to the constant functor Δc (the functor which has the value c at each object $i \in \mathbf{J}$ and the value $1_{\mathbf{C}}$ at each arrow of $\mathbf{J}$. If $f : c \to c'$ is an arrow of $\mathbf{C}$, Δf is the natural transformation $\Delta f : \Delta c \to \Delta c'$ which has the same value f at each object i of $\mathbf{J}$. Each functor $F : \mathbf{J} \to \mathbf{C}$ is an object of $\mathbf{C}^{\mathbf{J}}$. A universal arrow $\langle r, u\rangle$ from F to Δ is called a **colimit** or **direct limit** diagram for the functor F. It is consists of an object r of $\mathbf{C}$, usually written $r = \underrightarrow{Lim}F$ or $r = ColimF$, together with a natural transformation $u : F \to \Delta r$ which is universal among natural transformations $\tau : F \to \Delta c$.

Given categories $\mathbf{C}$, $\mathbf{J}$, and the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$, a **limit** or **inverse limit** for a functor $F : \mathbf{J} \to \mathbf{C}$ is a universal arrow $\langle r, v\rangle$ from Δ to F. It consists of an object r of $\mathbf{C}$, usually written $r = \underleftarrow{Lim}F$ or $LimF$ and called the limit object of the functor F, together with a natural transformation $v : \Delta r \to F$ which is universal among natural transformations $\tau : \Delta c \to F$, for objects c of $\mathbf{C}$

A category $\mathbf{C}$ is called **complete** if for every category $\mathbf{J}$ and functor $F : \mathbf{J} \to \mathbf{C}$ then $\underleftarrow{Lim}F$ exists. Analogous, a category $\mathbf{C}$ is called **cocomplete** if for every category $\mathbf{J}$ and functor $F : \mathbf{J} \to \mathbf{C}$ then $\underrightarrow{Lim}F$ exists.

4.4 Grothendieck Category

In 1957 was published the article “Sur quelques points d'algÃ¨bre homologique” by Alexander Grothendieck, now often referred to as the TÃ´hoku paper, in which it was introduced the concept of **Grothendieck Category**, it is a certain kind of cocomplete abelian category $\mathbf{C}$ which satisfies the next condition for any object $C \in \mathbf{C}$

$$\left(\sum_I C_i\right) \cap B = \sum_I (C_i \cap B) \tag{AB5}$$

where B is a subobject of C, and $(C_i)_I$ is a directed family of subobjects of C, the last condition is called "AB5" condition in the article of Grothendieck, is important to mention that for cocomplete abelian categories the last condition is equivalent to direct limits are exact in the category.

Let C a object of a Grothendieck category $\mathbf{C}$. A subobject B of C is called **essential** if $B \cap C' \neq 0$ for every non zero $C' \subset C$. And also a monomorphism $\alpha : B \to C$ is called **essential** if $Im(\alpha)$ is an essential subobject of C.

Lemma 4.4.1. *If $A \subseteq B$ subobjects of C, then A is essential in C if and only if A is essential in B and B is essential in C*

Proof. Suppose A is essential in C thus for every subobject $D \subseteq C$ we have $A \cap D \neq 0$ but remember that $A \subseteq B$ then we obtain let $B \cap D \neq 0$, in other words B is essential in C. Let a subobject $E \subseteq B$, then we have that $E \subseteq C$ thus $A \cap E \neq 0$. In other words A is essential in B.

On the other hand, suppose A is essential in B and B is essential in C, let $D \subseteq C$, since B is essential, we have $B \cap D \neq 0$ but $B \cap D \subseteq B$ and also A is essential in B, thus $A \cap (B \cap D) \neq 0$ but we need to remember that $A \subseteq B$ that implies that $A \cap B = A$. In conclusion $A \cap D \neq 0$ in other words A is essential in C. $\square$

Lemma 4.4.2. *If $\alpha : B \to C$ and $\beta : C \to D$ are monomorphisms, then $\beta \circ \alpha$ is essential if and only if both α and β are essential.*

Proof. Clearly $Im(\beta \circ \alpha) \subseteq Im(\beta)$ are subobjects of D. Suppose that $\beta \circ \alpha$ is essential, thus $Im(\beta \circ \alpha)$ is essential in D, by the Lemma 1 we have $Im(\beta)$ is essential in D in other words β is a essential monomorphism, and also $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$.
Let $N \subseteq C$ then we know that $\beta(N) \subseteq Im(\beta)$ thus $Im(\beta \circ \alpha) \cap \beta(N) \neq 0$ then there is $m \neq 0$ such that $m \in Im(\beta \circ \alpha)$ and $m \in \beta(N)$, then there is $x \in B$ such that $m = (\beta \circ \alpha)(x)$ and also there is $y \in N$ such that $m = \beta(y)$ then since β is a monomorphism then $0 \neq (\beta \circ \alpha)(x) = \beta(y)$ implies $\alpha(x) = y \neq 0$. In conclusion $0 \neq y \in Im(\alpha) \cap N$, in other words α is essential.
Now, suppose α and β are essential, let $E \subseteq Im(\beta)$ then there is $F \subseteq C$ such that $\beta(F) = E$ but $Im(\alpha) \cap F \neq 0$ thus there exists $m \neq 0$ such that $0 \neq \beta(m) \in Im(\beta \circ \alpha)$ and $0 \neq \beta(m) \in \beta(F) = E$, in other words $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$.
In conclusion we have $Im(\beta \circ \alpha)$ is essential in $Im(\beta)$ and $Im(\beta)$ is essential in D by Lemma 4.4.1 $Im(\beta \circ \alpha)$ is essential in D, thus $\beta \circ \alpha$ is a essential monomorphism. $\square$

Lemma 4.4.3. *If $\alpha : C \to E$ is a monomorphism and E is an injective object, then for every essential monomorphism $\beta : C \to C'$ there exists a monomorphism $\gamma : C' \to E$ such that $\gamma \circ \beta = \alpha$.*

Proof. Since E is injective, there is a $\gamma : C' \to E$ such that $\gamma \circ \beta = \alpha$. Since α is a monomorphism we have $0 = Ker(\alpha) = Ker(\gamma \circ \beta) = Ker(\gamma) \cap Im(\beta)$, but β is essential, that implies $Ker(\gamma) = 0$. In conclusion γ is a monomorphism. $\square$

Let C be a object of $\mathbf{C}$, an **injective envelope** of C is an essential monomorphism $C \to E$, where E is an injective object and we will denote $E(C)$.

In the next proposition we have that the injective envelope is unique up to isomorphism.

Proposition 4.4.4. *If $\alpha : C \to E$ and $\alpha' : C \to E'$ are injective envelopes of C. Then there is an isomorphism $\gamma : E \to E'$ such that $\gamma \circ \alpha = \alpha'$.*

Proof. By Lemma 4.4.3 there is a monomorphism $\gamma : E \to E'$ such that $\gamma \circ \alpha = \alpha'$, since α' is a essential monomorphism then γ is a essential monomorphism, since E is injective γ splits thus $E' = Im(\gamma) \oplus M$ for some subobject M, and also $Im(\gamma \circ \alpha) = Im(\alpha') \subseteq Im(\gamma)$. Suppose that $M \neq 0$, since $Im(\alpha') \cap M \neq 0$ then $Im(\gamma) \cap M \neq 0$ that implies $M = 0$. In conclusion $Im(\gamma) = E'$ in other words γ is onto. □

Proposition 4.4.5. *An object E is injective if and only if every essential monomorphism $\alpha : E \to C$ is an isomorphism.*

Proof. Suppose E is injective, and $\alpha : E \to C$ an essential monomorphism, thus α splits, in conclusion α must be onto. Now suppose every essential monomophism is an isomorphism. Let $M \supseteq E$, by Zorn's lemma there is a subobject $K \subseteq M$ maximal such that $E \cap K = 0$. Let the canonical projection $\beta : M \to M/K$ and consider $\mu : E \to M/K$ given by $\mu(x) = \beta(x)$ for every $x \in E$. Let $0 \neq S \subseteq M/K$ then there is $L \subseteq M$ such that $S = L/K$.
We have two cases, the first one, suppose $E \cap L = 0$ that implies $L \subseteq K$ and also $L/K \subseteq K$ then $L/K \cap E/K \subseteq K \cap Im(\mu) \neq 0$ in conclusion μ is an essential monomorphism. On the other hand, suppose $E \cap L \neq 0$ then there is $m \neq 0$ such that $m \in E \cap L$ thus $m + K \in E/K \cap L/K = Im(\mu) \cap L/K$, in conclusion μ is an essential monomorphism.
Since μ is essential monomorphism, then it is an isomorphism, hence $M = E + K$, since $E \cap K = 0$ thus $M = E \oplus K$, in other words E is injective. □

Proposition 4.4.6. *If C is a subobject of some injective object E then C has an injective envelope.*

Proof. Suppose C is a subobject of an injective object E, since the category $\mathbf{C}$ is cocomplete we can choose a maximal essential extension C' of C within E, with $\epsilon : C \to C'$. Then for every essential monomorphism $\beta : C' \to C''$ there exists a monomorphism $\gamma : C' \to E$ by Lemma 4.4.3.

An also by Lemma 4.4.2, $\beta \circ \epsilon$ is a essential monomorphism and also an isomorphism, since the maximality of C'. Then C' is injective by the Proposition 4.4.5 and so it's an injective envelope of C. □

4.5 Generalization of Bumby's Theorem

Theorem 4.5.1. *Let $\boldsymbol{C}$ be a Grothendieck category. If $A \supseteq B \in Ob(\boldsymbol{C})$, both injective and if there is a monomorphism $\varphi : A \to B$ then $A \cong B$.*

Proof. Let $A \supseteq B \in \mathbf{C}$ then we can find a subobject C such that $A = C \oplus B$. Now

$$\begin{aligned} A = C \oplus B &\supseteq C \oplus \varphi(A) \\ &= C \oplus \varphi(C \oplus B) \\ &= C \oplus \varphi(C) \oplus \varphi(B) \end{aligned}$$

$$\subseteq C \oplus \varphi(C) \oplus \varphi(\varphi(A))$$
$$\subseteq C \oplus \varphi(C) \oplus \varphi(\varphi(C)) \oplus \dots$$

Call it $P = C \oplus \varphi(C) \oplus \varphi(\varphi(C)) \oplus \dots$ and we have $A \supseteq P$ and also $P \cap B = \varphi(P)$, because $P \cap B = \varphi(C) \oplus \varphi(\varphi(C)) \oplus \dots$, and $P \cap B \subseteq B$ by Proposition 4.4.6, $P \cap B$ has an injective envelope, denoted by $E(P \cap B) \subseteq B$ and also $E(P \cap B)$ is injective then we can find a subobject K such that $B = E(P \cap E) \oplus K$.

And remember $A = C \oplus B = C \oplus (E(P \cap B) \oplus K) = (C \oplus E(P \cap B)) \oplus K$ that implies $C \oplus E(P \cap B)$ is injective:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "\varphi"'] & C \oplus E(P \cap B) \arrow[dl, dashed, "\overline{\varphi}"] \\
 & E(P \cap B) &
\end{tikzcd}$$

with the diagram we define $\overline{\varphi}$, by Proposition 5 $\overline{\varphi}$ is an isomorphism. Finally $\overline{\varphi} \oplus 1_K : (C \oplus E(P \cap E)) \oplus K \to E(P \cap E) \oplus K$ is also an isomorphism but $\overline{\varphi} \oplus 1_K : A \to B$ then is the required isomorphism.

□

Theorem 4.5.2. *Let $\boldsymbol{C}$ be a Grothendieck category. If A and B are objects of $\boldsymbol{C}$ which are isomorphic to subobjects of each others, their injective envelopes are are isomorphic.*

Proof. Let $E(A)$,$E(B)$ be the injective envelopes of A,B, respectively. Then if $\varphi : A \to B$ is a monomorphism so much be any map $\bar{\varphi} : E(A) \to E(B)$ which extends it (the injectivity of $E(B)$ guarantees the existence of maps $\bar{\varphi}$).

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"'] & E(A) \arrow[ddl, dashed, "\overline{\varphi}"] \\
 & B \arrow[d] & \\
 & E(B) &
\end{tikzcd}$$

Likewise if we have $\psi : B \to A$, a monomorphism, we obtain a monomorphism $\bar{\psi} : E(B) \to E(A)$ extending it. Now apply the last theorem with $E(A)$, $\bar{\psi}(E(B))$, $\bar{\psi} \circ \bar{\varphi}$ on the roles of A, B, φ, respectively. Now that we know that $E(A) \cong E(B)$. □

Finally, it iw worth to point out that the last theorems are also valid for pure-injectivity and RD-injectivity of modules. The proofs for those cases are similar to the proofs of our last results.

Conclusions

In this work, we extended the Cantor-Bernstein-Schröder theorem to a generalized form of injectivity in Grothendieck categories. Our main result generalizes Bumby's criterion of injectivity to this type of categories. As particular cases, we extended some criteria under which two RD-injective (respectively, pure-injective) modules are isomorphic when they are isomorphic to submodules of each other. These and other corollaries are in agreement with propositions previously reported in the literature. Several results that extend some known properties of injective modules and hulls to arbitrary categories are derived in the way.

After the conclusion of this work, various avenues of research still remain open. Indeed, many of the definitions and results presented in this thesis are valid not only for Grothendieck, but for categories in general. In fact, all the concepts, lemmas and propositions in injective objects hold for arbitrary categories. Thus, a natural direction of further investigation would be to elucidate categorical conditions on a class of morphisms for which Bumby's criterion of isomorphism holds. In other words, given a category $\mathcal{C}$ and the class of monic morphisms therein, one wishes to determine conditions under which two injective objects are isomorphic when they are subobjects of each other. To the best of the author's knowledge, the problem is still open.

Bibliography

[1] R. T. Bumby. Modules which are isomorphic to submodules of each other. *Archiv der Math.*, 16(1):184–185, 1965.

[2] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, New Jersey, 1st edition, 1999.

[3] W. Chacholski, W. Pitsch, and J. Scherer. Injective classes of modules. *J. Algebra Appl.*, 12(04):1250188, 2013.

[4] A. Facchini and M. H. Bien. Maximal ideals of the endomorphism ring of an injective module. *J. Algebra Appl.*, 13(04):1350131, 2014.

[5] L. Fuchs and L. Salce. *Modules over Valuation Domains*, volume 91 of *Lecture Notes in Pure and Applied Mathematics.* Marcel-Dekker, New York, 1st edition, 1985.

[6] L. Fuchs and L. Salce. *Modules over non-Noetherian Domains*, volume 84 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, Rhode Island, 1st edition, 2001.

[7] T. Jech. *Set Theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 3th edition, 2003.

[8] I. Kaplansky. Projective modules. *Annals of Math.*, 68(2):372–377, 1958.

[9] T. Y. Lam. *Lectures on Modules and Rings*, volume 189 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1st edition, 1999.

[10] L. Mao and N. Ding. Gorenstein FP-injective and Gorenstein flat modules. *J. Algebra Appl.*, 7(4):491–506, 2008.

[11] B. Mitchell. *Theory of categories*, volume 17. Academic Press, London, 1965.

[12] S. S. Page and Y. Zhou. Direct sums of quasi-injective modules, injective covers, and natural classes. *Communications in Algebra*, 22(8):2911–2923, 1994.

[13] J. Rosický. Injectivity and accessible categories. *Cubo Matem. Educ.*, 4:201–211, 2002.

[14] R. B. Warfield. Purity and algebraic compactness for modules. *Pac. J. Math.*, 28(3):699–719, 1969.