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UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

ESTUDIO DE ECUACIONES DIFERENCIALES FRACCIONARIAS Y SUS APLICACIONES

PARA OPTAR POR EL GRADO DE MAESTRO EN CIENCIAS DE LA COMPUTACIÓN CON OPCIÓN A MATEMÁTICAS APLICADAS

PRESENTA

Gerardo Becerra Guzmán

TUTOR

Dr. José Villa Morales

COMITÉ TUTORAL

Dra. Luz Judith Rodríguez Esparza (Asesor)
Dr. Manuel Ramírez Aranda (Asesor)

Aguascalientes, Ags., 9 de junio de 2026

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Dr. Manuel Ramírez Aranda
Asesor de *tesis*

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Erroneous Applications of Fractional Calculus: The Catenary as a Prototype

Gerardo Becerra-Guzmán and José Villa-Morales *

Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad No. 940, Aguascalientes C.P. 20100, Ags, Mexico
* Correspondence: j1v2m32004@gmail.com

Abstract: In this work, we study the equation of the catenary curve in the context of the Caputo derivative. We solve this equation and compare the solution with real physical models. From the experiments, we find that the best approximation is achieved in the classical case. Therefore, introducing a fractional parameter arbitrarily can be detrimental. However, we observe that, when adding a certain weight to the chain, fractional calculus produces better results than classical calculus for modeling the minimum height.

Keywords: fractional catenary curve; Caputo differential equations; fractional models

MSC: 34A08; 34A34; 34A06; 35R11

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The catenary curve, that elegant and resilient shape describing the configuration of a hanging chain, has left an indelible mark on the modern world. In the field of civil engineering, the catenary has played a fundamental role in the design of iconic structures, such as the Sagrada Familia church in Barcelona, Spain. Furthermore, its ability to evenly distribute loads along its curve makes it an ideal solution for bridges, arches, and other structures that must bear heavy weights.

The influence of the catenary extends far beyond civil engineering. In dentistry, its shape describes the natural curvature of healthy dental arches, providing a basis for understanding oral anatomy and designing dental prostheses. In the oil industry, the catenary is used to optimize well drilling, ensuring an efficient and safe trajectory. Even in the medical field, the catenary is employed to measure breast curvature in cancer diagnosis, providing crucial information for the detection and treatment of this disease (see [1,2], or the references mentioned therein).

The fascination with the catenary dates back to the time of Galileo Galilei, who compared the shape of a hanging chain to a parabola. However, it was Joachim Jungius who demonstrated that hanging chains have a distinct shape, laying the groundwork for a deeper study of this curve. In 1691, the catenary was formally defined and named, opening the doors to countless applications (see [3]).

The catenary curve, with its rich history and diverse applications, continues to inspire researchers and professionals in various fields. Its still untapped potential opens up new possibilities for designing stronger, more efficient, and more aesthetic structures, as well as for better a understanding of natural and complex phenomena. The catenary will undoubtedly remain a cornerstone of the modern world and a symbol of human creativity and ingenuity.

In this context, it is natural to consider the differential equation modeling the catenary curve in the fractional context, as we will do next to explore a new facet of the catenary. It should be noted that the equation modeling the catenary curve has already been studied in [4] using the Caputo–Fabrizio derivative. Here, we will use the fractional derivative in

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Optimal control of linear fractional differential equations in the Caputo sense

José Villa-Morales [a],*, Luis Rincón [b], Gerardo Becerra-Guzmán [a]

[a] *Universidad Autónoma de Aguascalientes, Departamento de Matemáticas y Física, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico*

[b] *Universidad Nacional Autónoma de México, Facultad de Ciencias, Departamento de Matemáticas, Circuito Exterior de CU, 04510 CDMX, Mexico*

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ABSTRACT

In this work, we study a time-optimal control problem involving bounded controls to regulate the dynamics of a linear fractional differential equation in the Caputo sense. We prove that the equation governing the dynamics admits optimal controls, which are of the bang–bang type. Additionally, we establish the validity of a maximum principle for determining the optimal control. The paper includes two application examples. The first example demonstrates that the optimal time can be achieved by appropriately selecting the fractional index. The second example illustrates that the choice between the classical and the fractional model may depend on the initial state of the dynamics.

1. Introduction

In recent years, fractional calculus has emerged as a powerful tool in modeling and analyzing complex dynamic systems. Unlike classical calculus, fractional calculus allows the incorporation of memory and hereditary effects into mathematical models, which is particularly useful in various fields such as physics, biology, and engineering. In this context, fractional differential equations have garnered attention due to their ability to more accurately describe real phenomena that cannot be adequately addressed by ordinary differential equations, see for example [1–4]. See also the work of I. Matychyn and V. Onyshchenko [5], which is closely related to ours.

The control of systems described by fractional differential equations presents new challenges and opportunities. In particular, the theory of optimal control for fractional systems is not as well-developed as its classical counterpart is. Nevertheless, a significant amount of work has been done so far. Among other developments, Pontryagin's maximum principle has been proved under very general assumptions (though with a fixed terminal time), and Bellman's dynamic programming method has been developed, see for example [6–8]. See also [9], where particular time-optimal control problems for fractional-order systems are considered. The present work aims to contribute to this field by studying bang–bang controls for linear fractional differential equations in the sense of Caputo.

Let us introduce some notation before presenting and discussing our results. Let $m, n \in \mathbb{N}$. We will say that a control u is admissible if $u : [0, \infty) \to [-1, 1]^m$ is a measurable function. We denote by $\mathcal{A}$ the set of all admissible controls. In what follows, we will study the control problem for the linear fractional differential equation

$$({}^{C}D^{\alpha}_{0+} y)(t) = Ay(t) + Bu(t), \quad \text{for all } t > 0, \tag{1.1}$$

* Corresponding author.
E-mail addresses: jvilla@correo.uaa.mx (J. Villa-Morales), lars@ciencias.unam.mx (L. Rincón), gb81202@gmail.com (G. Becerra-Guzmán).

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Resumen

En este trabajo de tesis se aborda el estudio de ecuaciones diferenciales fraccionarias en el sentido de Caputo desde dos enfoques: el primero orientado al modelado de un fenómeno físico y el otro orientado al análisis teórico de sistemas de control.

En el primer capítulo, los datos mostraron que, para una cadena homogénea sin perturbaciones, la ecuación que mejor modela la curva de la catenaria es la clásica. Esto evidencia que al introducir un parámetro fraccionario arbitrario no se garantiza una mejora respecto al modelo clásico. Sin embargo, al añadir diferentes pesos en el centro de la cadena, se observó que ciertos órdenes fraccionarios describen mejor este nuevo fenómeno y además permiten obtener una mejor aproximación para la altura mínima de la curva.

Por otra parte, en el segundo capítulo se mostró que la conveniencia de emplear un modelo fraccionario frente al clásico depende del índice fraccionario y de las condiciones iniciales del sistema, reforzando la idea de que ningún enfoque debe privilegiarse de manera automática. En conjunto, los resultados de esta tesis evidencian que la aplicación del cálculo fraccionario debe ser cuidadosa, ya que debe estar respaldada por evidencia experimental o por las condiciones particulares del sistema estudiado, evitando su uso indiscriminado en problemas donde los modelos clásicos ya son adecuados.

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Abstract

This thesis studies fractional differential equations in the Caputo sense from two different perspectives: the modeling of a physical phenomenon and the theoretical analysis of control systems.

In the first chapter, experimental data showed that, for a homogeneous hanging chain without perturbations, the classical catenary equation provides the best description of the curve. This result indicates that the arbitrary introduction of a fractional parameter does not necessarily improve the classical model. However, when additional weights are placed at the center of the chain, certain fractional orders provide a more accurate description of the resulting phenomenon and yield a better approximation of the minimum height of the curve.

In the second chapter, it is shown that the convenience of using a fractional model instead of a classical one depends on the fractional order as well as on the initial conditions of the system, reinforcing the idea that neither approach should be preferred automatically. Overall, the results of this thesis show that the application of fractional calculus must be carried out with caution and supported by either experimental evidence or the particular characteristics of the system under study. Consequently, the indiscriminate use of fractional models should be avoided in situations where classical models already provide an adequate description.

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Introducción

Objetivos y alcance

El cálculo fraccionario no es una herramienta tan empleada como lo es el cálculo clásico. Cuando se quiere resolver un fenómeno físico o un problema de aplicación y encontrar su solución se suele emplear el cálculo clásico antes que el fraccionario.
El cálculo fraccionario nació casi al mismo tiempo que el cálculo clásico desarrollado por Newton y Leibniz, porque el interés por órdenes no enteros no se hizo esperar. La primera mención histórica del cálculo fraccionario fue en el año 1695 donde L'Hôpital le mandó una carta a Leibniz sobre una pregunta que se le había generado con la notación de derivada $\dfrac{d^n y}{dx^n}$, donde mencionaba que aplicaciones había si n, no era entero por ejemplo si $n = 1/2$, es decir $\dfrac{d^{1/2} y}{dx^{1/2}}$, Leibniz ante esa pregunta escribió lo siguiente "esta es una aparente paradoja de la cual, algún día, se obtendrán consecuencias útiles" [26].

En muchos trabajos se introducen derivadas fraccionarias en modelos clásicos con la expectativa de obtener mejores resultados, sin embargo, no siempre existe una justificación física o experimental que respalde dicha elección. El presente trabajo busca aportar evidencia teórica y experimental sobre el uso del cálculo fraccionario, mostrando tanto sus alcances como sus limitaciones en fenómenos físicos y problemas de control.

Organización del trabajo

La tesis está organizada de la siguiente forma:

- **Capítulo 1:** En este capítulo se obtiene la ecuación diferencial fraccionaria de la catenaria con el uso de las derivadas de Caputo, respetando las propiedades físicas de la curva catenaria, que sería su simetría. Con el fin de realizar la comparación con la ecuación clásica, se analizan los datos obtenidos mediante digitalización de imágenes de una cadena suspendida de diferentes longitudes, considerando tanto el caso en que la única fuerza externa que actúa sobre ella es su peso, como el caso en que se añade un peso adicional en su punto central.

- **Capítulo 2:** Se aborda un problema de control óptimo en tiempo para ecuaciones diferenciales fraccionarias lineales en el sentido de Caputo, donde se busca un control acotado que lleve la dinámica del sistema al origen en el menor tiempo posible. Se establece la existencia de un control óptimo de tipo bang-bang, así como un principio del máximo análogo al de Pontryagin para este contexto fraccionario. Además, se presentan dos ejemplos de aplicación, uno relacionado con un problema

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de equilibrio y otro con el desplazamiento de un vagón, en los cuales se compara el comportamiento de los modelos clásico y fraccionario.

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Aplicaciones erróneas del cálculo fraccionario: la catenaria como prototipo

1.1 Artículo

La curva de la catenaria es la forma que se genera al tener una cuerda suspendida a dos puntos fijos a una misma altura sometida únicamente a su propio peso, este fenómeno lo podemos ver día tras día como una cadena colgante. Esta curva tiene aplicaciones en la arquitectura con el nombre de arco catenario. Para la obtención de la curva de la catenaria de forma matemática se plantea una ecuación diferencial con condiciones de frontera dadas, en esta primera parte se obtendrá la curva de la catenaria usando el cálculo fraccionario mediante las derivadas fraccionarias de Caputo [16].

La formulación propuesta preserva las propiedades físicas fundamentales del fenómeno, particularmente la simetría característica de la curva. A partir de esta ecuación se obtuvo una solución analítica que posteriormente fue comparada con la solución clásica de la catenaria.

Con el propósito de evaluar ambos modelos, se realizaron experimentos utilizando cadenas suspendidas de diferentes longitudes. Las curvas obtenidas fueron digitalizadas y comparadas con las predicciones teóricas de las versiones clásica y fraccionaria. Los resultados mostraron que, para una cadena homogénea cuya única fuerza externa es su propio peso, el modelo clásico proporciona la mejor aproximación experimental. En particular, se observó que la solución fraccionaria reproduce adecuadamente el fenómeno únicamente cuando el orden fraccionario se aproxima al valor ($\alpha = 1$), correspondiente al caso clásico.

Posteriormente, se modificó el sistema físico añadiendo diferentes pesos en el centro de la cadena. Esta perturbación altera la forma natural de la curva y genera un comportamiento más complejo. En este escenario se observó que ciertos órdenes fraccionarios permiten describir con mayor precisión la deformación producida, obteniendo errores menores que los proporcionados por el modelo clásico. Además, la estimación de la altura mínima de la curva presentó una mejora significativa cuando se empleó el modelo fraccionario.

Los resultados obtenidos permiten concluir que la introducción de un parámetro fraccionario no garantiza por sí misma una mejora en la modelación de un fenómeno físico. Su utilización debe estar respaldada por evidencia experimental y por una jus-

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tificación física adecuada. Sin embargo, los resultados también sugieren que el cálculo fraccionario puede constituir una herramienta útil para modelar fenómenos sometidos a perturbaciones que modifican su estado natural, donde los modelos clásicos presentan ciertas limitaciones.

Los resultados descritos en este capítulo fueron publicados en el artículo titulado *Erroneous Applications of Fractional Calculus: The Catenary as a Prototype*, publicado en la revista *Mathematics* en el año 2024. A continuación, se presenta el artículo en su versión original.

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Erroneous Applications of Fractional Calculus: The Catenary as a Prototype

Gerardo Becerra-Guzmán and José Villa-Morales *

Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad No. 940, Aguascalientes C.P. 20100, Ags, Mexico
* Correspondence: j1v2m32004@gmail.com

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MSC: 34A08; 34A34; 34A06; 35R11

1. Introduction and Statement of the Results

The catenary curve, that elegant and resilient shape describing the configuration of a hanging chain, has left an indelible mark on the modern world. In the field of civil engineering, the catenary has played a fundamental role in the design of iconic structures, such as the Sagrada Familia church in Barcelona, Spain. Furthermore, its ability to evenly distribute loads along its curve makes it an ideal solution for bridges, arches, and other structures that must bear heavy weights.

The influence of the catenary extends far beyond civil engineering. In dentistry, its shape describes the natural curvature of healthy dental arches, providing a basis for understanding oral anatomy and designing dental prostheses. In the oil industry, the catenary is used to optimize well drilling, ensuring an efficient and safe trajectory. Even in the medical field, the catenary is employed to measure breast curvature in cancer diagnosis, providing crucial information for the detection and treatment of this disease (see [1,2], or the references mentioned therein).

The fascination with the catenary dates back to the time of Galileo Galilei, who compared the shape of a hanging chain to a parabola. However, it was Joachim Jungius who demonstrated that hanging chains have a distinct shape, laying the groundwork for a deeper study of this curve. In 1691, the catenary was formally defined and named, opening the doors to countless applications (see [3]).

The catenary curve, with its rich history and diverse applications, continues to inspire researchers and professionals in various fields. Its still untapped potential opens up new possibilities for designing stronger, more efficient, and more aesthetic structures, as well as for better a understanding of natural and complex phenomena. The catenary will undoubtedly remain a cornerstone of the modern world and a symbol of human creativity and ingenuity.

In this context, it is natural to consider the differential equation modeling the catenary curve in the fractional context, as we will do next to explore a new facet of the catenary. It should be noted that the equation modeling the catenary curve has already been studied in [4] using the Caputo–Fabrizio derivative. Here, we will use the fractional derivative in

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the sense of Caputo. The Caputo–Fabrizio derivative involves a non-singular kernel, and in modeling, it often yields different results from the Caputo derivative (see [5]).

Nowadays, it unfortunately happens all too often that a classical differential equation is considered, and instead of the integer-order derivative operator, it is replaced with a fractional derivative operator. In the vast majority of these cases, it is tacitly assumed that the fractional dynamics improve the model obtained using classical differential calculus. Under this premise, that is, assuming there is an improvement using fractional calculus, various authors engage in a series of theoretical developments or numerical manipulations that later make no sense (see [6]).

To illustrate this point, we have chosen the catenary curve because, as we mentioned before, it is an important phenomenon and relatively easy to reproduce. Using a hanging chain and a scanner, one can obtain the shape of the catenary curve (see Section 3). Furthermore, the associated differential equation is a second-order nonlinear differential equation (see (3) and (4)). By replacing the integer-order derivative with the fractional-order derivative (see (2) and (8)), a fractional differential equation is obtained (see (10)). This equation is solved, and the key point here is that the fractional parameters do not model the physical curve of the hanging chain. In other words, it is generally incorrect to think that the fractional model improves the classical model. In some cases, such as in the study of control systems, nonlinear and nonlocal models are often modeled by nonlinear fractional differential equations (see [6]). In our case, we see that this is not the case; indeed, the modeling of the catenary is a nonlinear model, and any (local) perturbation of the chain affects the chain globally.

On the other hand, when a certain weight is applied to the center of the chain, relatively minor compared to the total weight of the chain, it is observed that there are fractional indices that better model the deformed curve than the classical case. This is consistent with what is known (that it is a nonlocal phenomenon), and it opens a possible opportunity for fractional calculus to model this phenomenon, at least improving the classical case. In this case, as we mentioned, by modifying the weight of the chain at its center, the overall structure of the curve is altered; changes are not limited to a neighborhood of the center of the catenary. Since modeling the catenary curve with weight classically is a complicated problem (see [7,8]), and we achieve it with little effort when using fractional calculus, we can say that, for modeling this phenomenon, fractional calculus proves to be useful.

In summary, the fractional index should not be introduced into a differential equation without real evidence that it improves the model (see [6]). At best, the fractional analytical model represents a perturbation of the classical model. This observation should not be taken lightly, as there is a wide variety of fractional derivatives, and theoretical aspects are sometimes studied without any basis other than the aesthetic aspect of mathematics (see [9,10]).

The article is organized as follows. In Section 2, we recall the derivation of the classical catenary, which serves to introduce some concepts, and we propose the relevant modifications in the fractional case. In Section 3, we define some concepts of fractional calculus and state some basic properties. In Section 4, we present the fractional differential equation in the sense of Caputo and solve it. In Section 5, we describe the physical experiment and present some images of catenaries. Finally, in Section 6, we present some conclusions.

2. Classical Catenary Curve

In this section, we will briefly review the classical derivation of the catenary curve. Consider a homogeneous rope or string with linear density ρ and a length greater than a, where a is a fixed positive number. Let T represent the tension at the midpoint $a/2$. The downward displacement of the hanging chain at the point x will be denoted as $y(x)$. With this information, we can create a diagram similar to the one shown in Figure 1.

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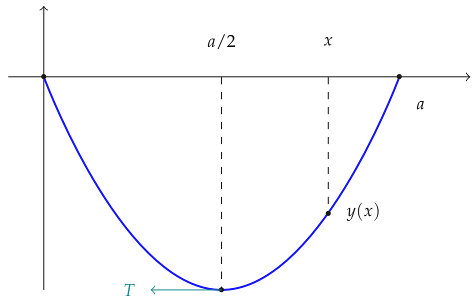

Figure 1. The catenary curve, $y(x)$.

Carrying out an analysis of the forces (see, for example, [11]), we arrive at the following equation,

$$\frac{1}{c}\int_{a/2}^{x}\sqrt{1+(u(t))^2}dt = u(x), \ \ x \in (0,a), \tag{1}$$

where

$$u(x) = y'(x), \tag{2}$$

$c = T/(g\rho)$, and g is the gravitational constant. Taking derivatives in (1), we deduce the second-order nonlinear differential equation

$$\frac{1}{c}\sqrt{1+(u(x))^2} = u'(x), \tag{3}$$

with the initial condition

$$u(a/2) = y'(a/2) = 0. \tag{4}$$

The general solution of (3) has the form

$$u(x) = \sinh\left(\frac{1}{c}x + c_0\right), \ \ x \in [0,a], \tag{5}$$

where c_0 is a constant. Using the initial conditions (4), we obtain

$$c_0 = -\frac{a}{2c}. \tag{6}$$

Now, from (2) and using $y(0) = 0$, we get the classical catenary curve,

$$y(x) = c\left\{\cosh\left(\frac{x}{c} - \frac{a}{2c}\right) - \cosh\left(\frac{a}{2c}\right)\right\}, \ \ x \in [0,a]. \tag{7}$$

The minimum of the curve will be

$$y\left(\frac{a}{2}\right) = c\left\{1 - \cosh\left(\frac{a}{2c}\right)\right\}, \ \ x \in [0,a].$$

3. Some Preliminaries on Fractional Calculus

To introduce the fractional model of the catenary curve, it is helpful to revisit some key concepts of fractional calculus. Numerous excellent texts are available on this subject; for our purposes, we adopted the notation and refer to the results presented in [12,13].

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The left Riemann–Liouville integral, $I^{\alpha}_{a+}f$, of f of order $\alpha \in \mathbb{R}$, is defined as*

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \ \ x \in (a,b).$$

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Now, let us suppose that $\alpha > 0$ and set $n := [\alpha] + 1$, where $[\cdot]$ is the largest integer less than or equal to α. If $f^{(n)}$ exists and is continuous, then the fractional derivative of Caputo, by the left, ${}^{C}\mathcal{D}^{\alpha}_{a+}f$, is defined as

$$\left({}^{C}\mathcal{D}^{\alpha}_{a+}f\right)(x) = \left(I^{n-\alpha}_{a+}f^{(n)}\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{1+\alpha-n}}dt, \;\; x \in (a,b).$$

In the above definition, $\Gamma(\alpha)$, with $\alpha > 0$, represents the usual gamma function. Since we will not be utilizing any other fractional derivative, we will omit the C in the definition of a Caputo derivative, denoting it simply as $D^{\alpha}f$.

Proposition 1. *Let $\alpha > 0$ and $n := [\alpha] + 1$. If f has continuous derivatives up to order $n-1$, and $f^{(n)}$ is absolutely continuous, then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Proof. See Lemma 2.22 in [12]. □

The linearity of the classical integral implies the linearity of the fractional integral. Furthermore, we have the following result.

Proposition 2. *Let $\alpha \geq 0$ and $\beta > 0$; then,*

$$\left(I^{\alpha}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}.$$

Proof. See formulas (2.1.16) of [12]. □

4. Fractional Catenary Curve

Let $\alpha \in (\mathbb{R}_+ \setminus \mathbb{N})$ and $n := [\alpha] + 1$. A common method to extend the classical model to the fractional context involves introducing the fractional derivative operator into expression (2), thereby replacing the classical derivative with the fractional one; see [4,6],

$$u = D^{\alpha}_{\frac{a}{2}+}y. \tag{8}$$

On the other hand, physical conditions require that the resulting curve be symmetric. Keeping this in mind, we consider the following equation (see Equation (1)):

$$\frac{1}{c}\int_{a/2}^{x}\sqrt{1+(u(t))^2}dt = u(x), \;\; \frac{a}{2} < x < a. \tag{9}$$

If we take the derivative, we obtain the fractional differential equation

$$\frac{1}{c}\sqrt{1+\left(D^{\alpha}_{\frac{a}{2}+}y(x)\right)^2} = D^{1+\alpha}_{\frac{a}{2}+}y(x), \tag{10}$$

with boundary conditions

$$y(a) = 0, \; y'\left(\frac{a}{2}\right) = 0, \; \cdots, \; y^{(n)}\left(\frac{a}{2}\right) = 0. \tag{11}$$

Equation (9) is the same as Equation (1); thus, the solution is given by (5),

$$u(x) = \sinh\left(\frac{1}{c}x + c_1\right), \;\; \frac{a}{2} < x < a, \tag{12}$$

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for some constant, c_1. From Proposition 1, we get

$$y(x) - \sum_{k=0}^{n} \frac{y_+^{(k)}\left(\frac{a}{2}\right)}{k!}\left(x - \frac{a}{2}\right)^k = \left(I^{\alpha}_{\frac{a}{2}+} D^{\alpha}_{\frac{a}{2}+} y\right)(x) = I^{\alpha}_{\frac{a}{2}+} u(x). \quad (13)$$

On the other hand, using (12) and the linearity of fractional integral, we obtain

$$I^{\alpha}_{\frac{a}{2}+} u(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} I^{\alpha}_{\frac{a}{2}+}\left(\frac{1}{c}x + c_1\right)^{2k+1}. \quad (14)$$

Since

$$\left(\frac{1}{c}x + c_1\right)^{2k+1} = \left(\frac{1}{c}\right)^{2k+1} \sum_{j=0}^{2k+1} \binom{2k+1}{j} \left(\frac{a}{2} + c \cdot c_1\right)^{2k+1-j} \left(x - \frac{a}{2}\right)^j,$$

then Propositions 2 and (11) yield

$$\begin{aligned} y(x) &= \sum_{k=0}^{n} \frac{y^{(k)}\left(\frac{a}{2}\right)}{k!}\left(x - \frac{a}{2}\right)^k + I^{\alpha}_{\frac{a}{2}+} u(x) \\ &= y\left(\frac{a}{2}\right) + \sum_{k=0}^{\infty}\left(\frac{1}{c}\right)^{2k+1} \sum_{j=0}^{2k+1} \frac{1}{(2k+1-j)!\,\Gamma(j+1+\alpha)}\left(\frac{a}{2} + c \cdot c_1\right)^{2k+1-j}\left(x - \frac{a}{2}\right)^{j+\alpha}. \end{aligned}$$

By hypothesis, the n-th derivative of y exists at $a/2$. Since $\alpha - n < 0$, then

$$\frac{a}{2} + c \cdot c_1 = 0,$$

otherwise, $y^{(n)}(a/2)$ would not exist. This particular value of c_0 corresponds to the classical case, as depicted in (6). Therefore, (11) suggests

$$y(x) = \sum_{k=0}^{\infty}\left(\frac{1}{c}\right)^{2k+1} \frac{1}{\Gamma(2k+2+\alpha)}\left[\left(x - \frac{a}{2}\right)^{2k+1+\alpha} - \left(\frac{a}{2}\right)^{2k+1+\alpha}\right], \quad \frac{a}{2} \le x \le a.$$

The fractional catenary curve y_α is given by

$$y_\alpha(x) = \begin{cases} y(a-x), & 0 \le x \le \frac{a}{2}, \\ y(x), & \frac{a}{2} \le x \le a. \end{cases} \quad (15)$$

When $\alpha = 1$, employing the Taylor series of the hyperbolic cosine yields the classical catenary (7). In the fractional scenario, the minimum is given by

$$y_\alpha\left(\frac{a}{2}\right) = -c^{\alpha} \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2+\alpha)}\left(\frac{a}{2c}\right)^{2k+1+\alpha}. \quad (16)$$

5. Physical Experiments

The experiment consisted of capturing several images of a hanging chain to determine the shape of the curve formed by the chain. A scanner was placed vertically with the chain positioned in front of it. When a photograph of the hanging chain is taken with a camera, the resulting curve depends on the angle of the photograph. However, this method avoids any dependence on the angle in the image.

Note that the function $y(x)$, given in (15), depends on the parameter c, which in turn depends on the linear density of the chain. The criterion we used to determine the value of c is the one that minimizes the error in approximating $y(x)$ to the curve determined via the chain, taking a fixed value of the index α. In Figure 2a,c, the classical catenary curve is

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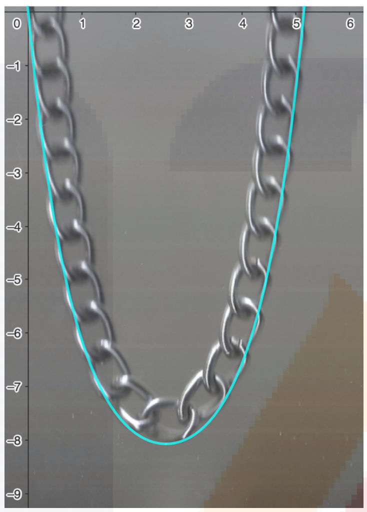
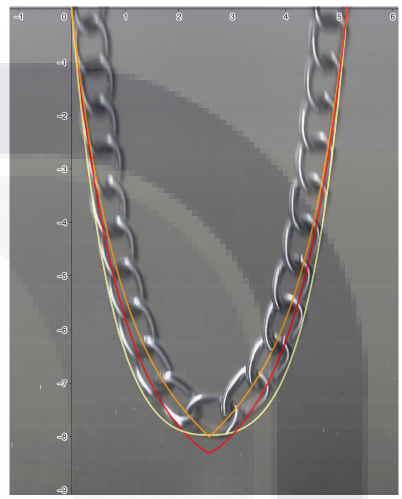
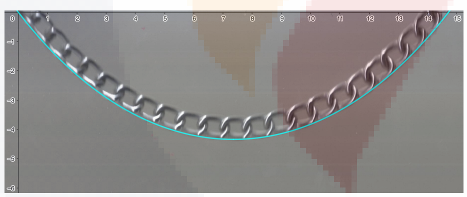
shown in blue. As we can observe from Table 1, when α is 1, the errors are smaller compared to the fractional case, with curves in red ($\alpha = 0.28$, $c = 0.9$), yellow ($\alpha = 1.5$, $c = 0.78$), and orange ($\alpha = 0.05$, $c = 0.93$) when $a = 5.1$ and in red ($\alpha = 0.1$, $c = 3.46$), yellow ($\alpha = 1.5$, $c = 11$), and orange ($\alpha = 0.05$, $c = 3.42$) when $a = 14.71$; see Figure 2b,d. Indeed, in the classical case, the error is 0.042 cm when $a = 5.15$ cm and 0.028 cm when $a = 14.71$ cm, considerably lower than in the fractional case; see Table 1. The errors in the figures were measured using Digimizer 2024, an image analysis software package that allows precise manual measurements (see [14]). In this way, in Figure 2b,d, we see that the fractional curves (i.e., $\alpha \neq 1$) deviate from the curve determined according to the chain. In other words, the fractional solution (15) does not model the curve produced via the chain.



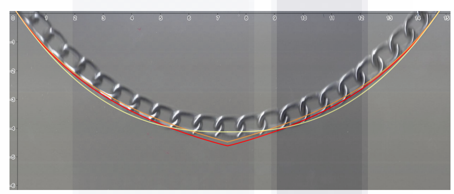
(**a**) $a = 5.15$, $c = 0.845$.



(**b**) $a = 5.15$, $\alpha = 0.28, 0.05, 1.5$.



(**c**) $a = 14.71$, $c = 6.85$.



(**d**) $a = 14.71$, $\alpha = 0.1, 0.05, 1.5$.

Figure 2. Weightless pendant chain.

Table 1. Different values for fractional catenary parameters; see (15).

a (cm)	α	c	Error (cm)
5.15	0.05	0.925	0.361
	0.28	0.9	0.273
	1	0.845	0.017
	1.5	0.78	0.079
14.71	0.05	3.42	0.264
	0.1	3.46	0.241
	1	6.85	0.009
	1.5	11	0.206

Next, we will modify the system by placing a weight of 4.344 grams at the center of the chain in both cases, i.e., for $a = 5.15$ and $a = 14.71$. In Figure 3a,c, the classical curve, shown in blue, corresponds to $\alpha = 1$. In this case, the errors are 0.258 cm when $a = 5.15$ and 0.307 cm when $a = 14.71$. On the other hand, by selecting the fractional index α that

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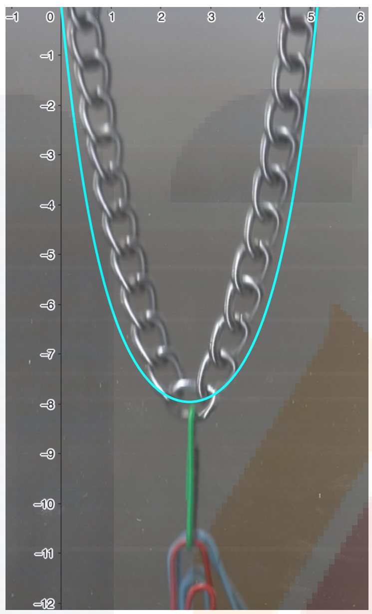
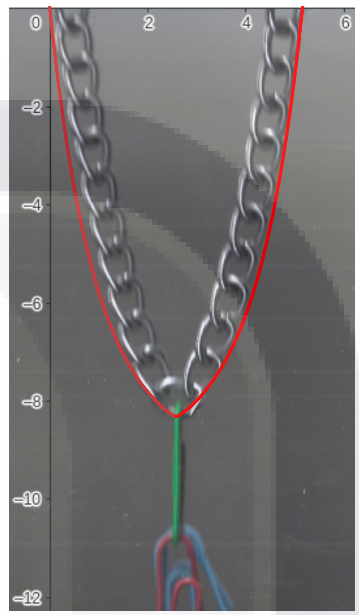
best fits the curve, it turns out that, for $a = 5.15$, the error is 0.19 cm, and when $a = 14.71$, the error is 0.16 cm; see Figure 3b,d. This means that the perturbation is better modeled with a fractional index; see (16).

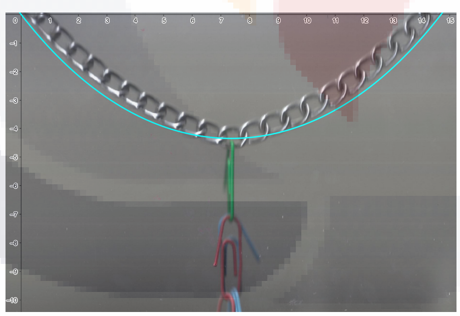
It may be of particular interest to model the minimum height of the chain with the weight, as this can occur, for example, in a cable car cable. In this case, we see that the actual minimum height is -8.324 when $a = 5.15$. In the classical case, the error is 0.252, and in the fractional case with $\alpha = 0.28$, the error is 0.012. On the other hand, when $a = 14.71$ cm, the minimum height is -4.668. In the classical case, the error is 0.325 cm, and in the fractional case with $\alpha = 0.1$, the error is 0.053 cm.



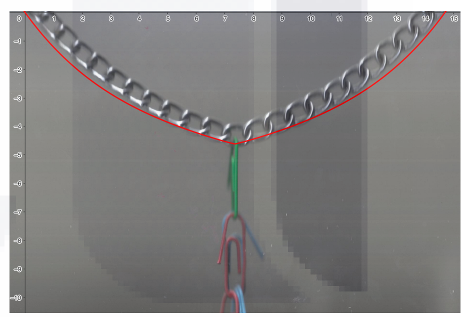
(**a**) $\alpha = 1$, $c = 0.85$.



(**b**) $\alpha = 0.28$, $c = 0.9$.



(**c**) $\alpha = 1$, $c = 6.85$.



(**d**) $\alpha = 0.1$, $c = 3.46$.

Figure 3. Weighted pendant chain.

6. Conclusions

It is well known that fractional calculus is useful for modeling certain phenomena, such as temperature controllers [15] or battery charges [16]. However, we can conclude that this is not always the case. Introducing a fractional index arbitrarily into a model based on ordinary differential equations can lead to results that deviate significantly from expectations. Some authors argue that modeling with fractional calculus provides an advantage by adding an additional degree of freedom, the fractional index α. While this can be true, our study shows that the best results may be obtained by not modifying this parameter, i.e., by setting $\alpha = 1$.

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In summary, the main contribution of our work lies in advocating for the rational use of fractional calculus. The simplicity and replicability of the model studied here underline its value, demonstrating that this approach is not merely a matter of substituting classical derivatives with fractional ones and performing analytical or numerical manipulations without real value. Our study, conducted with the Caputo derivative, can serve as a starting point for analyzing the real behavior of a chain with other fractional derivatives as well; see, for example, the excellent paper [17].

On the other hand, we have observed that fractional calculus can be useful for modeling a perturbation of the original phenomenon. In our case, the minimum height of the weighted chain is better modeled using fractional calculus. Specifically, the approximation error at the minimum point decreases by approximately twenty percent.

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2

Control óptimo de ecuaciones diferenciales fraccionarias lineales en el sentido de Caputo

2.1 Artículo

El contenido de este capítulo está basado en el artículo titulado Optimal control of linear fractional differential equations in the Caputo sense, publicado en la revista Journal of Computational and Applied Mathematics. En dicho trabajo se estudian ecuaciones diferenciales fraccionarias lineales en el sentido de Caputo desde la perspectiva del control óptimo.

En este capítulo se estudian ecuaciones diferenciales fraccionarias lineales en el sentido de Caputo desde la perspectiva del control óptimo. El problema considerado consiste en determinar un control acotado que permita conducir la dinámica del sistema hacia el origen en el menor tiempo posible.

Para abordar este problema, primero se analizan las propiedades básicas de las ecuaciones diferenciales fraccionarias involucradas y se establece la existencia y unicidad de sus soluciones. Posteriormente, se demuestra la existencia de controles óptimos de tipo bang-bang, es decir, controles que alcanzan únicamente los valores extremos permitidos. Además, se obtiene un principio del máximo análogo al de Pontryagin para este contexto fraccionario, el cual proporciona una condición necesaria para la optimalidad de los controles.

Finalmente, se presentan dos ejemplos de aplicación. El primero corresponde a un problema de equilibrio, mientras que el segundo estudia el movimiento de un vagón controlado mediante una fuerza externa. Estos ejemplos permiten comparar el comportamiento de los modelos clásicos y fraccionarios, mostrando que la elección entre ambos depende de las características particulares del sistema y de las condiciones iniciales consideradas.

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Optimal control of linear fractional differential equations in the Caputo sense

José Villa-Morales [a],*, Luis Rincón [b], Gerardo Becerra-Guzmán [a]

[a] Universidad Autónoma de Aguascalientes, Departamento de Matemáticas y Física, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico
[b] Universidad Nacional Autónoma de México, Facultad de Ciencias, Departamento de Matemáticas, Circuito Exterior de CU, 04510 CDMX, Mexico

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ABSTRACT

In this work, we study a time-optimal control problem involving bounded controls to regulate the dynamics of a linear fractional differential equation in the Caputo sense. We prove that the equation governing the dynamics admits optimal controls, which are of the bang–bang type. Additionally, we establish the validity of a maximum principle for determining the optimal control. The paper includes two application examples. The first example demonstrates that the optimal time can be achieved by appropriately selecting the fractional index. The second example illustrates that the choice between the classical and the fractional model may depend on the initial state of the dynamics.

1. Introduction

In recent years, fractional calculus has emerged as a powerful tool in modeling and analyzing complex dynamic systems. Unlike classical calculus, fractional calculus allows the incorporation of memory and hereditary effects into mathematical models, which is particularly useful in various fields such as physics, biology, and engineering. In this context, fractional differential equations have garnered attention due to their ability to more accurately describe real phenomena that cannot be adequately addressed by ordinary differential equations, see for example [1–4]. See also the work of I. Matychyn and V. Onyshchenko [5], which is closely related to ours.

The control of systems described by fractional differential equations presents new challenges and opportunities. In particular, the theory of optimal control for fractional systems is not as well-developed as its classical counterpart is. Nevertheless, a significant amount of work has been done so far. Among other developments, Pontryagin's maximum principle has been proved under very general assumptions (though with a fixed terminal time), and Bellman's dynamic programming method has been developed, see for example [6–8]. See also [9], where particular time-optimal control problems for fractional-order systems are considered. The present work aims to contribute to this field by studying bang–bang controls for linear fractional differential equations in the sense of Caputo.

Let us introduce some notation before presenting and discussing our results. Let $m, n \in \mathbb{N}$. We will say that a control u is admissible if $u : [0, \infty) \to [-1, 1]^m$ is a measurable function. We denote by $\mathcal{A}$ the set of all admissible controls. In what follows, we will study the control problem for the linear fractional differential equation

$$({}^{C}D^{\alpha}_{0+} y)(t) = Ay(t) + Bu(t), \quad \text{for all } t > 0, \tag{1.1}$$

* Corresponding author.
E-mail addresses: jvilla@correo.uaa.mx (J. Villa-Morales), lars@ciencias.unam.mx (L. Rincón), gb81202@gmail.com (G. Becerra-Guzmán).

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$$y(0) = x^0,$$

where ${}^C D^{\alpha}_{0+}$ denotes the left-sided Caputo fractional derivative of order $\alpha \in (0,1]$, $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$ are matrices, $u \in \mathcal{A}$ and $x^0 \in \mathbb{R}^n$ is the initial value. We denote the solution of the problem (1.1) by $y(t; x^0, u)$ or, if there is no confusion, simply by $y(t)$.

Let us denote by G the set of initial values x^0 for which there exists a control $u \in \mathcal{A}$ such that $y(t; x^0, u) = 0$ for some $t \geq 0$. In this case, we will say that 0 is reachable from x^0. Thus, the set G is the set of reachable points. In this paper, we will demonstrate that if $x^0 \in G$, then there exists an admissible control u^* such that the dynamics $y(t; x^0, u^*)$ bring x^0 to the origin 0 in the shortest possible time, τ^* (see Lemma 3.1). Moreover, we will show that this optimal control u^* is of the bang–bang type (see Theorem 3.2). Furthermore, in order to find an optimal control u^* a maximum principle is established. Such maximum principle is analogous to the classical maximum principle of Pontryagin (see Theorem 3.3). The optimal control problem addressed in this work is a specific type of optimal control problem, see for example [10]. However, we believe that it serves as a first step towards understanding and addressing, for example, the existence of bang–bang controls in the more general case.

Next, we will discuss the context and contribution of our work. The literature on the existence and uniqueness of solutions to fractional differential equations is extensive; in fact, many results include linear equations as a particular case (see [11–15], and the references therein). However, most of these works assume that the control u must satisfy certain regularity conditions. For example, if u satisfies that $t^{1-\alpha}u(t)$ is a continuous function on $[0,\infty)$, then the unique solution to (1.1) is given by (see Theorem 7.13 in [13])

$$y(t) = E_\alpha[t^\alpha A]x^0 + \int_0^t e_\alpha^{(t-s)A} Bu(s)ds, \quad t > 0, \tag{1.2}$$

where E_α is the Mittag-Leffler function (or operator) and e_α is the fractional exponential function (or operator), see [13]. On the other hand, few works consider control problems in a more general context. For example, assuming that the control u is integrable in a fractional sense of order $1-\alpha$, it is shown in [16] that the problem (1.1) has a unique solution (see also [17,18]). In our case, we consider essentially bounded controls. In this context, a function y is a solution of Eq. (1.1) if the condition for all $t > 0$, in (1.1), is replaced by for almost all $t > 0$. In this way, as we will see, the existence of a solution to (1.1) follows as a consequence of Theorem 3.1 in [19]. On the other hand, some works study linear fractional differential equations in the sense of Caputo for unbounded controls (see [20] or [21]). In the case of unbounded controls, the techniques used to address the solution of these problems are quite different and, in some cases, easier to apply (see [10]).

On the other hand, the bang–bang property is of great importance in optimal control theory, as noted in [22]. For certain fractional partial differential equations, assuming integrability constraints on the control (see [23]) or restricting the value of the fractional parameter (see [24]), the existence of bang–bang type controls has been demonstrated. However, to the best of our knowledge, there are no results in the context we address here (see Lemma 3.1 and Theorem 3.2). When attempting to follow classical proofs to address problems in the fractional case, we encounter the difficulty that now the reachable set G is not convex. This implies that certain controllability criteria are no longer met (see [25,26], and [27]). In many practical control applications, relying solely on a criterion that guarantees the existence of optimal bang–bang controls is often insufficient. To address this limitation for the applications, we introduce a maximum Pontryagin principle. As we will see, this principle is a consequence of more general recent theoretical developments.

To illustrate the results, we study two application examples. The first example shows that for certain dynamics, it is always possible to choose a fractional parameter that minimizes the optimal time to reach the origin. The second example demonstrates that the choice between a classical model and a fractional one depends on the initial conditions of the system's dynamics. These observations are crucial since we cannot dismiss classical models outright, a fact that is unfortunately occurring without a thorough study of the phenomenon in question.

The organization of the paper is as follows. In Section 2, we state the existence and uniqueness of the solution to system (1.1) with essentially bounded controls. In Section 3, we prove the existence of bang–bang controls, and we present a fractional Pontryagin maximum principle. In Section 4, we provide two application examples of the bang–bang principle and the maximum principle. Finally, in Section 5, we conclude with a brief discussion on some potential future work.

2. Solution to the linear differential equation with essentially bounded control

Let us suppose that $0 < \alpha < 1$ and $a \in \mathbb{R}$. According to formula (2.4.1) in [13], the left-sided Caputo derivative on (a,∞) of order α is defined as

$$({}^C D_{a+} y)(x) = (D^{\alpha}_{a+} y)(x) - \frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}, \tag{2.1}$$

where

$$(D^{\alpha}_{a+} y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{y(t)}{(x-t)^\alpha} dt$$

is the left-sided Riemann–Liouville fractional derivative. In the sequel, the essential supremum of a function v is denoted by $\|v\|_\infty$. We say that v is essentially bounded if $\|v\|_\infty < \infty$.

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Theorem 2.1. *Let $A \in M_{n\times n}(\mathbb{R})$ and $x^0 \in \mathbb{R}^n$. If v is essentially bounded, then the initial value problem*

$$\begin{aligned}({}^{C}D^{\alpha}_{a+}y)(t) &= Ay(t) + v(t), \quad \text{for almost all } t > a,\\ y(a) &= x^0,\end{aligned} \tag{2.2}$$

has a unique global solution given by

$$y(t) = E_\alpha[(t-a)^\alpha A]x^0 + \int_a^t e_\alpha^{(t-s)A} v(s)ds, \quad t \geq a. \tag{2.3}$$

A proof of Theorem 2.1 can be found in [19] (see Theorem 6 and Remark 2) and in [28] (see Theorem 5.1).

3. Existence of a bang–bang optimal control and a Pontryagin's maximum principle

For each $t \geq 0$, let us define the set

$$G(t) = \{x^0 \in \mathbb{R}^n : \text{there exists } u \in \mathcal{A} \text{ such that } y(t; x^0, u) = 0\}.$$

In this way, by Theorem 2.1, $x^0 \in G(t)$ if and only if

$$E_\alpha\left[t^\alpha A\right] x^0 = -\int_0^t e_\alpha^{(t-s)A} Bu(s)ds, \quad \text{for some } u \in \mathcal{A}. \tag{3.1}$$

Let $x^0 \in \mathbb{R}^n$. Over all the admissible controls u that drive the system from x^0 to the origin 0, we choose that control that minimizes the time to achieve the goal. The minimal time is denoted by τ^*, in symbols,

$$\tau^* = \inf\{t \geq 0 : x^0 \in G(t)\}.$$

Let us set $G := \bigcup_{t\geq 0} G(t)$.

Lemma 3.1. *If $x^0 \in G$, then $x^0 \in G(\tau^*)$. In other words, there exists $u^* \in \mathcal{A}$ such that $y(\tau^*; x^0, u^*) = 0$.*

Proof. There exists one $t_1 > 0$ such that $x^0 \in G(t_1)$. Let $\{t_n\}$ be a decreasing sequence such that $x^0 \in G(t_n)$ and $t_n \downarrow \tau^*$. Since $x^0 \in G(t_n)$, there exists $u_n \in \mathcal{A}$ such that

$$E_\alpha\left[(t_n)^\alpha A\right] x^0 = -\int_0^{t_n} e_\alpha^{(t_n-s)A} Bu_n(s)ds. \tag{3.2}$$

Without loss of generality we will suppose that $u_n = 0$ in $[t_n, t_1]$. Since $\{u_n\} \subset L^\infty([0,t_1])$, by the Banach–Alaoglu theorem (see [29]), there exists $u^* \in L^\infty([0,t_1])$ such that

$$\lim_{k\to\infty} \int_0^{t_1} v(s)u_{n_k}(s)\,ds = \int_0^{t_1} v(s)u^*(s)ds,$$

for some subsequence $\{u_{n_k}\}$ and for each $v \in L^1([0,t_1])$. Let us take

$$v(s) = \text{sign}(u^*(s))\, 1_{[\tau^*,t_1]}(s), \quad s \in [0,t_1].$$

Then, the Dominated Convergence Theorem implies

$$0 = \lim_{k\to\infty} \int_0^{t_1} \text{sign}(u^*(s))\, 1_{[\tau^*,t_1]}(s)\, u_{n_k}(s)\,ds = \int_{\tau^*}^{t_1} |u^*(s)|ds.$$

Hence,

$$u^* = 0, \quad \text{a.e. on} \quad [\tau^*, t_1]. \tag{3.3}$$

Furthermore, using (3.3), we note that

$$\begin{aligned}\left|\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu^*(s)ds + E_\alpha\left[(\tau^*)^\alpha A\right] x^0\right| \leq{}& \left|\int_0^{t_1} e_\alpha^{(\tau^*-s)A} Bu^*(s)ds - \int_0^{t_1} e_\alpha^{(\tau^*-s)A} Bu_{n_k}(s)ds\right|\\ &+ \left|\int_0^{t_1} e_\alpha^{(\tau^*-s)A} Bu_{n_k}(s)ds - \int_0^{t_1} e_\alpha^{(t_{n_k}-s)A} Bu_{n_k}(s)ds\right|\\ &+ \left|E_\alpha\left[(t_{n_k})^\alpha A\right] x^0 - E_\alpha\left[(\tau^*)^\alpha A\right] x^0\right|.\end{aligned}$$

Given that

$$\lim_{n\to\infty}\left(E_\alpha\left[(\tau^*)^\alpha A\right] - E_\alpha\left[t_n^\alpha A\right]\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(e_\alpha^{A(\tau^*-s)} - e_\alpha^{A(t_n-s)}\right) = 0,$$

and noticing that the function $s \mapsto e_\alpha^{(\tau^*-s)A}B$ is integrable, we can use again the Dominated Convergence Theorem to deduce that

$$E_\alpha\left[(\tau^*)^\alpha A\right] x^0 = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu^*(s)ds, \tag{3.4}$$

which means that $x^0 \in G(\tau^*)$. □

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In particular, from (3.4) we observe that $\tau^* > 0$ when $x^0 \neq 0$. For practical applications, the following result is crucial. Recall that an admissible control $u(t) = (u^1(t), \dots, u^m(t))^T$ is called bang–bang if for almost all $t \geq 0$ and each $i \in \{1, \dots, m\}$, $|u^i(t)| = 1$.

Theorem 3.2 (*Bang–bang control principle*). *If $x^0 \in G$ then there exists an optimal bang–bang control.*

Proof. The proof will be divided into several steps to enhance clarity in the presentation.

1. Let K be the set of admissible controls $u \in \mathcal{A}$ such that u leads x^0 to 0 at time τ^*, that is, $K = \{u \in \mathcal{A} : y(\tau^*; x^0, u) = 0\}$. By Lemma 3.1, $u^* \in K$, i.e. $K \neq \emptyset$.
2. Now let us see that K is convex. Let $u, v \in K$. Then
$$E_\alpha\left[(\tau^*)^\alpha A\right] x^0 = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu(s)ds \quad \text{and} \quad E_\alpha\left[(\tau^*)^\alpha A\right] x^0 = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bv(s)ds.$$
Therefore, for $0 \leq \lambda \leq 1$,
$$E_\alpha\left[(\tau^*)^\alpha A\right] x^0 = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} B\left(\lambda u(s) + (1-\lambda)v(s)\right) ds.$$
Since $\lambda u + (1-\lambda)v \in \mathcal{A}$, we have $\lambda u + (1-\lambda)v \in K$.
3. It is now shown that K is a closed set in the weak-* topology. Since $(L^\infty([0,t_1]))^* = L^1([0,t_1])$ is separable, it is enough to prove that K is sequentially closed (see Theorem 3.2.8 in [29]). Let $\{u_n\}$ be a sequence in K. By the Banach–Alaoglu theorem, there exists $u \in \mathcal{A}$ and a subsequence $\{u_{n_k}\}$ such that
$$\lim_{k\to\infty} \int_0^{\tau^*} v(s)u_{n_k}(s)\,ds = \int_0^{\tau^*} v(s)u(s)ds,$$
for all $v \in L^1([0,\tau^*])$. Proceeding as in the proof of Lemma 3.1 we can see that
$$E_\alpha\left[(\tau^*)^\alpha A\right] x^0 = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu(s)ds.$$
This means $u \in K$.
4. By the Krein-Milman theorem (see [30]), the set K has at least one extreme point. Let $u = (u^1, \dots, u^m)^T \in K$ be such an extreme point. It is shown next that u is a bang–bang control. For a contradiction, suppose it is not. Then there exists an index $i_0 \in \{1, \dots, m\}$ and a positive Lebesgue measurable set $H \subset [0,\tau^*]$ such that $|u^{i_0}(s)| < 1$, for all $s \in H$. Observe that
$$H = \bigcup_{n=1}^\infty \left(\left\{s : |u^{i_0}(s)| \leq 1 - \frac{1}{n}\right\} \cap H\right) := \bigcup_{n=1}^\infty H_n.$$
Therefore, there exists $n_0 \in \mathbb{N}$ such that $(1/2)\lambda(H) < \lambda(H_{n_0})$. Thus, $\lambda(H_{n_0}) > 0$ and
$$|u^{i_0}(s)| \leq 1 - \frac{1}{n_0}, \quad \text{for all} \quad s \in H_{n_0}. \tag{3.5}$$
5. Let us define the linear mapping $I : L^\infty(H_{n_0}) \to \mathbb{R}^n$ as
$$I(f) := \int_{H_{n_0}} e_\alpha^{(\tau^*-s)A} Bv_f(s)ds,$$
where $v_f(s) := (0, \dots, 0, f(s), 0, \dots, 0)^T$, i.e. $f(\cdot)$ is the i_0th entry of the vector v_f. If $\ker(I) = \{0\}$, then the dimension of $L^\infty(H_{n_0})$ would be less than or equal to n. But this is impossible since $L^\infty(H_{n_0})$ has infinite dimension. Therefore, there exists $f \in L^\infty([0,t_1])$ such that $|f| > 0$ a.s. on H_{n_0} and $I(f) = 0$. Thus, $v := v_{f/\|f\|_\infty}$ is an admissible control.
6. Consider the measurable functions u_1 and u_2 given by
$$u_1 = u + \frac{1}{n_0}v \quad \text{and} \quad u_2 = u - \frac{1}{n_0}v.$$
It is shown next that $u_1 \in K$. Observe that
$$u_1(s) = \begin{cases} u(s) & \text{if } s \in [0,\tau^*]\backslash H_{n_0}, \\ u(s) + \frac{1}{n_0}v(s) & \text{if } s \in H_{n_0}. \end{cases}$$
Then, inequality (3.5) implies
$$|u_1^{i_0}(s)| \leq |u^{i_0}(s)| + \frac{1}{n_0}|v^{i_0}(s)| \leq 1 - \frac{1}{n_0} + \frac{1}{n_0} = 1.$$
This means u_1 is an admissible control. On the other hand, since $u \in K$, see (3.4),
$$-\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu_1(s)ds = -\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bu(s)ds - \frac{1}{n_0}\int_0^{\tau^*} e_\alpha^{(\tau^*-s)A} Bv(s)ds$$

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$$= E\left[(\tau^*)^\alpha A\right] x^0 - \frac{1}{n_0}\int_{H_{n_0}} e_\alpha^{(\tau^*-s)A} Bv(s)ds$$
$$= E\left[(\tau^*)^\alpha A\right] x^0 - I(f)$$
$$= E\left[(\tau^*)^\alpha A\right] x^0.$$

Therefore $u_1 \in K$. A similar argument can be used to show that $u_2 \in K$.

7. Due to the fact that H_{n_0} has positive Lebesgue measure, $u_1 \neq u$ and $u_2 \neq u$. However, $u_1/2 + u_2/2 = u$, which means u is a convex combination of points in K. This contradicts the fact that u is an extreme point in K.

Therefore, u is an optimal bang–bang control. □

From a practical point of view, the mere existence of optimal controls is not sufficient; a criterion is needed to find them. This motivates us to present the following special case of Pontryagin's Theorem, whose proof follows from Theorem 3.2 in [6], considering the Hamiltonian defined as $H(x, u, p) = (Ax + Bu) \cdot p$.

Theorem 3.3 (*A Maximum Principle*). *Let $x^0 \in G$. There exists a vector $h \in \mathbb{R}^n \setminus \{0\}$ such that for almost all $s \in [0, \tau^*]$,*

$$h^T e_\alpha^{(\tau^*-s)A} Bu^*(s) = \max_{a\in[-1,1]^m} \left\{ h^T e_\alpha^{(\tau^*-s)A} Ba \right\}. \tag{3.6}$$

4. Examples

In this section, we present two examples of the results obtained in the preceding sections. In these applications, we will highlight some features of fractional calculus that are important to keep in mind when applying fractional calculus in the modeling of real-world phenomena.

Example 1 (*A Basic Balancing Problem*). A walker on a hanging cable must balance to maintain equilibrium. The study to control the walker's balance on the hanging rope is multidimensional and complicated (see [26]). Next, we will study a simpler one-dimensional version, which nevertheless exhibits the main characteristics of the phenomenon. Specifically, we will consider the initial value problem

$$\begin{aligned} ({}^C D_{0+}^\alpha y)(t) &= y(t) + u(t), \quad t > 0, \\ y(0) &= x^0, \end{aligned} \tag{4.1}$$

where the control $u : [0, \infty] \to [-1, 1]$ is a measurable function. The solution of the above equation is, see the general solution (1.2),

$$y(t) = E_\alpha[t^\alpha] x^0 + \int_0^t e_\alpha^{t-s} u(s)ds. \tag{4.2}$$

Let us set the reachable set from x^0 as

$$F(t) = \{x^1 \in \mathbb{R} : \text{there exists } u \in \mathcal{A} \text{ such that } y(x^0; t, u) = x^1\}.$$

Therefore, if $x^1 \in F(t)$, then

$$E_\alpha[t^\alpha]x^0 - \int_0^t e_\alpha^{t-s} ds \le x^1 \le E_\alpha[t^\alpha]x^0 + \int_0^t e_\alpha^{t-s} ds.$$

Using the formula (see Proposition 1.12 (ii) in [31])

$$\lambda \int_a^t e_\alpha^{(t-s)\lambda} ds = E_\alpha[(t-a)^\alpha \lambda] - 1, \quad \lambda \in \mathbb{R}, \tag{4.3}$$

we conclude that $F(t)$ is the closed interval

$$F(t) = [\, E_\alpha[t^\alpha](x^0 - 1) + 1,\ E_\alpha[t^\alpha](x^0 + 1) - 1\,], \quad t > 0.$$

If $x^0 \in G$, then

$$x^0 = \frac{1}{E_\alpha[t^\alpha]} \int_0^t e_\alpha^{t-s} u(s)ds$$

for some $t > 0$ and a control u. Since $|u| \le 1$, we have

$$-\frac{1}{E_\alpha[t^\alpha]} \int_0^t e_\alpha^{t-s} ds \le x^0 \le \frac{1}{E_\alpha[t^\alpha]} \int_0^t e_\alpha^{t-s} ds.$$

Applying (4.2), we deduce

$$-1 + \frac{1}{E_\alpha[t^\alpha]} \le x^0 \le 1 - \frac{1}{E_\alpha[t^\alpha]}.$$

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Table 1
Optimal time τ^* for different values of the parameters y_0 and α, for the dynamics of Example 1.

$y_0 \setminus \alpha$	0.125	0.25	0.5	0.75	1
2	0.0028	0.053	0.265	0.490	0.693
5	0.163	0.495	0.998	1.345	1.609
10	0.590	1.052	1.645	2.025	2.302
20	1.390	1.735	2.320	2.712	2.995
50	3.291	2.833	3.238	3.626	3.912
100	5.714	3.843	3.955	4.319	4.605

Since $E_\alpha[t^\alpha] > 0$, we conclude that $x^0 \in (-1,1)$, that is, $G \subset (-1,1)$. On the other hand, if $x^0 \in (-1,1)$, we seek a time $t > 0$ and a control u such that $y(x^0; t, u) = 0$. We achieve this, for example, by choosing the control $u = -1$ in (4.2),

$$0 = E_\alpha[t^\alpha]x^0 - \int_0^t e_\alpha^{t-s} ds.$$

Thus, from (4.3), we obtain

$$E_\alpha[t^\alpha] = \frac{1}{1-x^0}, \quad x^0 \in (-1,1). \tag{4.4}$$

Since the Mittag-Leffler function E_α is continuous and increasing on $(0,\infty)$, it follows that there exists a unique $t > 0$ such that $y(t; x^0, -1) = 0$, that is, $x^0 \in G(t)$. In this way, $G = (-1,1)$.

Now let us apply the maximum principle. Let $x^0 \in G$. Then there exists $h \neq 0$ such that

$$he_\alpha^{(\tau^*-s)}u^*(s) = \max_{a\in[-1,1]} \left\{ he_\alpha^{(\tau^*-s)}a \right\},$$

thus

$$u^*(s) = \operatorname{sign}(he_\alpha^{(\tau^*-s)}), \quad 0 \le s \le \tau^*.$$

Since $e_\alpha^{(\tau^*-s)} > 0$,

$$u^*(s) = \operatorname{sign}(h).$$

This means that the optimal control u^* is constant. From the previous comments, we have $u^* = -1$ if $x^0 \in (-1,1)$. Moreover, the optimal time τ^* will be given by (4.4), taking the inverse of the Mittag-Leffler function.

Next, we want to study numerically the optimal time in terms of the fractional parameter α. Given $x^0 \in (-1,1)$, we seek the minimum time that drives the system to the origin, which is given by the expression (4.4), in other words,

$$1 < \sum_{k=0}^{n} \frac{(\tau^*)^{\alpha k}}{\Gamma(\alpha k + 1)} \approx (1-x^0)^{-1} =: y_0, \tag{4.5}$$

where $n \in \mathbb{N}$. Let us consider 15 terms in the sum (4.5), that is $n = 14$. Moreover, by varying the parameters y_0 and α, we obtain the values of τ^* given in Table 1. It can be observed that for small values of y_0, the optimal time is obtained at the smallest fractional index α. However, this behavior no longer occurs for large values of y_0. Nonetheless, we observe that for both large and small values of y_0, it is always possible to choose a fractional index that provides the best arrival time to equilibrium. That is, in this case, it is observed that using fractional dynamics, with the control $u^* = -1$, yields the smallest optimal arrival time to equilibrium.

Example 2 (*A Railroad Car with One Control*). In the manufacturing of electrical microcircuits, it is necessary to move an object in a straight line to a specific position. Thus, the state of the system is determined by the position and velocity of the object. What is controlled in this system is the force acting on the object. In our example, we will consider a railway car that has a motor at each end. We denote the motor on the right side by -1 and the one on the left side by 1. We assume that the control is the thrust applied to each motor (thrust is not applied to both motors simultaneously). Additionally, we introduce the following notation:

- Let $q(t)$ be the position of the railroad car at time $t \ge 0$.
- Let ${}^C\!D_{0+}^\alpha q(t)$ be the fractional velocity at time $t \ge 0$.
- Let $u(t)$ be the thrust of the motors, $-1 \le u(t) \le 1$, for $t \ge 0$.

Assuming that a fractional second law of motion holds and supposing that the mass of the railroad car is 1, it follows that

$${}^C\!D_{0+}^{2\alpha} q(t) = u(t),$$

where ${}^C\!D_{0+}^{2\alpha} = {}^C\!D_{0+}^{\alpha}\,{}^C\!D_{0+}^{\alpha}$ is the composition operator. Let us set $x(t) := (q(t), v(t))^T$ with $v(t) := {}^C\!D_{0+}^\alpha q(t)$. So we will get

$${}^C\!D_{0+}^\alpha x(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

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and $x(0) = (q_0, v_0)^T$. Let us note that

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^k = 0, \quad \text{for} \quad k \geq 2.$$

Then

$$e_\alpha^{A(\tau^*-s)} = \begin{pmatrix} (\tau^* - s)^{\alpha-1}/\Gamma(\alpha) & (\tau^* - s)^{2\alpha-1}/\Gamma(2\alpha) \\ 0 & (\tau^* - s)^{\alpha-1}/\Gamma(\alpha) \end{pmatrix}.$$

This implies that

$$h^T e_\alpha^{A(\tau^*-s)} B = \frac{h_1(\tau^* - s)^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{h_2(\tau^* - s)^{\alpha-1}}{\Gamma(\alpha)}.$$

The maximum principle yields

$$\left(\frac{h_1(\tau^* - s)^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{h_2(\tau^* - s)^{\alpha-1}}{\Gamma(\alpha)} \right) u^*(s) = \max_{|a|\leq 1} \left\{ \left(\frac{h_1(\tau^* - s)^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{h_2(\tau^* - s)^{\alpha-1}}{\Gamma(\alpha)} \right) a \right\},$$

then

$$u^*(s) = \operatorname{sign} \left(\frac{h_1(\tau^* - s)^{\alpha}}{\Gamma(2\alpha)} + \frac{h_2}{\Gamma(\alpha)} \right).$$

This is a bang–bang control. Moreover, if $\alpha = 1$, then

$$u(s) = \operatorname{sign}(-sg_1 + g_2),$$

where $(g_1, g_2) = (h_1, h_1\tau^* + h_2)$, which coincides with the classical control (see [32]).

Moreover, we will see that only one switch is required to reach the origin. Let us suppose that we start with the controllability of the railroad car when we turn on the motor -1 (it is assumed that shortly before we traveled using the motor 1). In this way, our system (or initial dynamics) is of the form

$$\begin{aligned} v_0(t) &= ({}^C D_{0+}^\alpha q_0)(t), \\ ({}^C D_{0+}^\alpha v_0)(t) &= -1, \end{aligned}$$

with $q_0(0) = q_0$, $v_0(0) = v_0$. Using formula (1.2) we find that

$$v_0(t) = v_0 - \frac{1}{\Gamma(1+\alpha)} t^\alpha, \quad q_0(t) = q_0 + \frac{v_0}{\Gamma(1+\alpha)} t^\alpha - \frac{1}{\Gamma(1+2\alpha)} t^{2\alpha}, \quad t \geq 0.$$

If we set $c_\alpha := \Gamma(1+2\alpha)/(\Gamma(1+\alpha))^2$, then

$$c_\alpha q_0(t) = c_\alpha(q_0 + v_0^2) - v_0^2 + (2 - c_\alpha)v_0\, v_0(t) - (v_0(t))^2. \tag{4.6}$$

This produces a family of parabolas shown in Fig. 1(a).

The motor -1 is used during the time period $[0, t_s]$, then we change to motor 1. In that period of time, the dynamics are determined by the equation

$$\begin{aligned} v_1(t) &= ({}^C D_{t_s+}^\alpha q_1)(t), \\ ({}^C D_{t_s+}^\alpha v_1)(t) &= 1, \end{aligned}$$

with $q_1(t_s) = q_1$, $v_1(t_s) = v_1$. Proceeding as before we obtain

$$v_1(t) = v_1 + \frac{1}{\Gamma(1+\alpha)} t^\alpha, \quad q_1(t) = q_1 + \frac{v_1}{\Gamma(1+\alpha)} t^\alpha + \frac{1}{\Gamma(1+2\alpha)} t^{2\alpha}, \quad t \geq 0.$$

In this case, we obtain the following formula

$$c_\alpha q_1(t) = c_\alpha(q_1 - v_1^2) + v_1^2 - (2 - c_\alpha)v_1\, v_1(t) + (v_1(t))^2, \tag{4.7}$$

which produces the family of parabolas shown in Fig. 1(b).

From the families of parabolas in Fig. 1, we mark in red a trajectory that drives the fractional dynamics $\alpha = 0.541$, with initial value at the point $(q_0, v_0) = (1, 0.5)$, to the origin in the shortest time. In Fig. 2, we observe that there is a switching time, t_s. Let us calculate this switching time as well as the arrival time at the origin, τ^*.

We look for a time τ^* such that

$$q_1(\tau^*) = 0, \quad v_1(\tau^*) = 0,$$

and

$$q_0(t_s) = q_1(0), \quad v_0(t_s) = v_1(0).$$

Solving this system of equations we have

$$(\tau^*)^\alpha = (t_s)^\alpha - \Gamma(1+\alpha)v_0,$$

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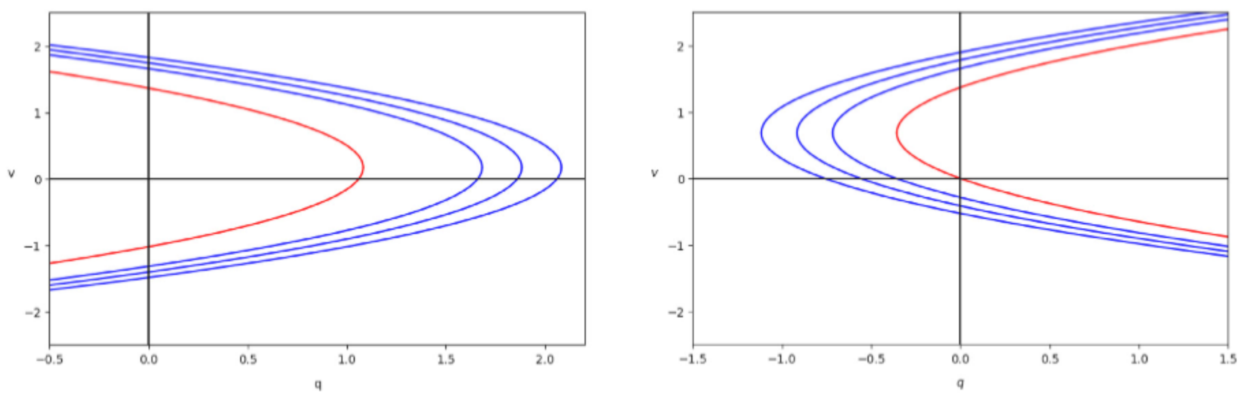

(a) Family of parabolas determined by (4.6).   (b) Family of parabolas determined by (4.7).

Fig. 1. Families of curves that produce the optimal trajectory for the dynamics of Example 2, with the initial value $(q_0, v_0) = (1, 0.5)$, and for parameter values of α equal to 0.541, 0.6, 0.65, and 0.7, respectively.

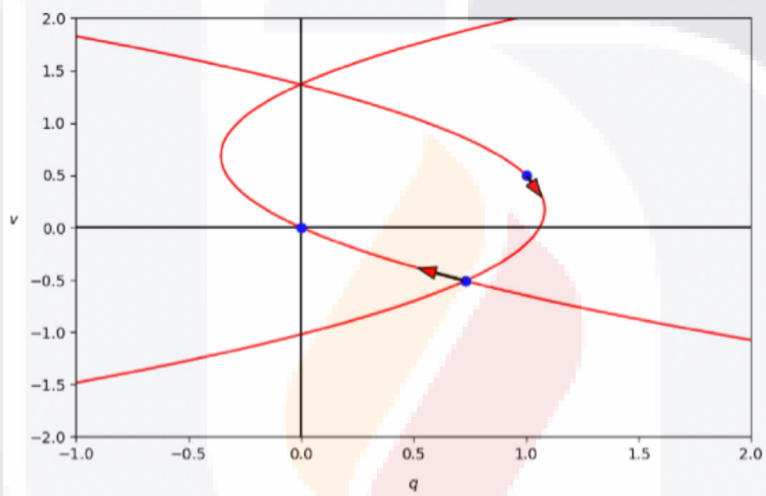

Fig. 2. Optimal trajectory when $(q_0, v_0) = (1, 0.5)$ and $\alpha = 0.541$ for the dynamics of Example 2.

where

$$(t_s)^\alpha = \frac{\left(\frac{3}{\Gamma(1+\alpha)} - \frac{2\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right) v_0 + \sqrt{\left(\frac{3}{\Gamma(1+\alpha)} - \frac{2\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right) v_0^2 - \frac{4}{(\Gamma(1+\alpha))^2}\left(v_0^2 - q_0 - \frac{(\Gamma(1+\alpha))^2 v_0^2}{\Gamma(1+2\alpha)}\right)}}{2(\Gamma(1+\alpha))^{-2}}.$$

In this way, the optimal arrival time will be

$$\tau^* := t_s + ((t_s)^\alpha - \Gamma(1+\alpha) v_0)^{1/\alpha}. \tag{4.8}$$

When $(q_0, v_0) = (1, 0.5)$ we observe that $\tau^* = 2.486$, when $\alpha = 0.541$, and that, for this same initial condition, $\tau = 2.736$, when $\alpha = 1$. Moreover, from graph (a) of Fig. 3 we see that $\alpha = 0.541$ is the fractional parameter that provides us with the optimal arrival time at the origin. On the other hand, when the initial conditions are $(q_0, v_0) = (2, 1.599)$ the minimum time is $\tau^* = 4.945$ and is reached in $\alpha = 1$. In fact, from graph (b) of Fig. 3 we see that for these initial conditions the minimum value of the time τ^* is obtained in the classical case. That is, if the objective is to minimize the arrival time at the origin, the choice between a classical and a fractional model depends on where the control of the system's dynamics begins.

5. Conclusions

In this paper, we prove the bang–bang principle and state Pontryagin's maximum principle for the linear case of fractional dynamics in the sense of Caputo, specifically for a time-optimal control problem. We examine two applications of these results, demonstrating that, in some cases, an appropriately chosen fractional parameter can enhance performance compared to classical dynamics. More importantly, these examples illustrate that the choice between a fractional or classical model may depend on the

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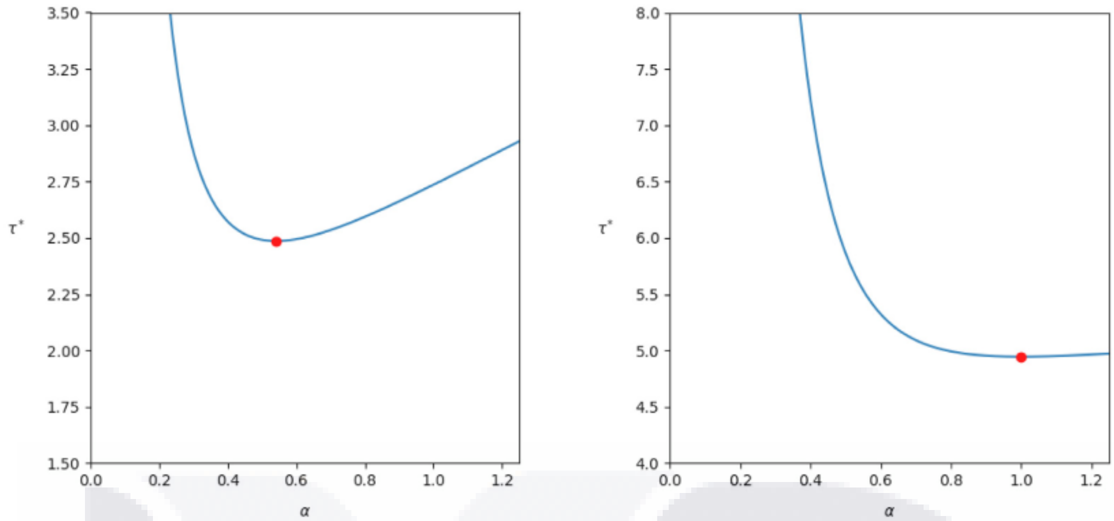

(a) $(q_0, v_0) = (1, 0.5)$. (b) $(q_0, v_0) = (2, 1.599)$.

Fig. 3. Graph of τ^* with respect to the parameter α, see Eq. (4.8), for the dynamics of Example 2.

initial conditions of the system. This highlights the importance of setting aside biases when modeling and considering both classical and fractional approaches on equal footing.

A wide variety of topics can be further addressed. One of particular interest is for example the study of controllability criteria in terms of the controllability matrix or the study of observability (see [10]).

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Data availability

No data was used for the research described in the article.

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3
Conclusión

- **Capítulo 1:** En este capítulo se obtuvo la ecuación de la catenaria fraccionaria con el uso de las derivadas de Caputo, de tal forma que esta ecuación mantuviera la simetría que presenta el fenómeno físico. Una vez obtenida esta solución, se comparó con la solución clásica de la catenaria usando los datos que se obtuvieron al analizar la curva catenaria formada por diferentes tamaños de la misma cadena. Los datos mostraron que el mejor modelo que aproxima la forma de la curva es cuando α es igual a 1, es decir, el modelo clásico lo describe mejor que el fraccionario. La curva de la cadena se modificó añadiendo pesos en su centro, de tal manera que se generó una perturbación que la afectara globalmente, donde la curva de la catenaria tenía una forma diferente a su estado natural. Este cambio fue modelado mediante las versiones clásica y fraccionaria. Se pudo apreciar que la ecuación fraccionaria describió mejor la curva con peso añadido que la expresión clásica, además, mejoró la predicción de la altura mínima comparada con la ecuación clásica. La introducción de derivadas fraccionarias en un modelo clásico debe estar justificada por la evidencia física o experimental, ya que la presencia de un parámetro fraccionario no garantiza una mejora en la modelación.

- **Capítulo 2:** Se abordó un problema de control óptimo en tiempo para ecuaciones diferenciales fraccionarias lineales en el sentido de Caputo, en el cual se busca un control acotado que lleve la dinámica del sistema al origen en el menor tiempo posible. Se demostró la existencia de un control óptimo de tipo bang-bang, así como un principio del máximo análogo al de Pontryagin para este contexto fraccionario, extendiendo así estos resultados clásicos de la teoría de control al caso fraccionario. Para ilustrar estos resultados, se analizaron dos ejemplos: un problema de equilibrio y el desplazamiento de un vagón con dos motores. En el primero, se observó que siempre es posible elegir un índice fraccionario que minimice el tiempo óptimo de llegada al origen. En el segundo, se mostró que la conveniencia de emplear el modelo clásico o el fraccionario depende de las condiciones iniciales del sistema, de manera que ninguno de los dos enfoques resulta preferible en todos los casos. Esto confirma que la elección entre un modelo clásico y uno fraccionario debe sustentarse en las características particulares del sistema que se quiere modelar, y no asumirse de manera automática.

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