UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

MODELACIÓN CON ECUACIONES DIFERENCIALES ESTOCÁSTICAS

PRESENTA

Gabriela de Jesús Cabral García

PARA OPTAR POR EL GRADO DE DOCTOR EN CIENCIAS APLICADAS Y TECNOLOGÍAS

TUTOR

Dr. José Villa Morales

COMITÉ TUTORAL

Dra. Mariana Alfaro Gómez (Asesor)
Dr. Luis Antonio Rincón Solís (Asesor)
Dra. Luz Judith Rodríguez Esparza(Asesor)

Aguascalientes, Ags., 17 de febrero de 2025

CARTA DE VOTO APROBATORIO
INDIVIDUAL

M. en C. Jorge Martín Alférez Chávez
DECANO DEL CENTRO **DE CIENCIAS BÁSICAS**

P R E S E N T E

Por medio del presente como **TUTOR** designado de la estudiante **GABRIELA DE JESÚS CABRAL GARCÍA** con ID 496 quien realizó la tesis titulada: **MODELACIÓN CON ECUACIONES DIFERENCIALES ESTOCÁSTICAS** un trabajo propio, innovador, relevante e inédito y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia doy mi consentimiento de que la versión final del documento ha sido revisada y las correcciones se han incorporado apropiadamente, por lo que me permito emitir el **VOTO APROBATORIO**, para que ella pueda proceder a imprimirla así como continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y sin otro particular por el momento, me permito enviarle un cordial saludo.

A T E N T A M E N T E
"Se Lumen Proferre"
Aguascalientes, Ags., a 17 de febrero de 2025

Dr. José Villa Morales
Tutor de tesis

c.c.p.- Interesado
c.c.p.- Secretaría Técnica del Programa de Posgrado

M. en C. Jorge Martín Alférez Chávez
DECANO DEL CENTRO **DE CIENCIAS BÁSICAS**

P R E S E N T E

Por medio del presente como **ASESORA** designado de la estudiante **GABRIELA DE JESÚS CABRAL GARCÍA** con ID 496 quien realizó la tesis titulada: **MODELACIÓN CON ECUACIONES DIFERENCIALES ESTOCÁSTICAS** un trabajo propio, innovador, relevante e inédito y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia doy mi consentimiento de que la versión final del documento ha sido revisada y las correcciones se han incorporado apropiadamente, por lo que me permito emitir el **VOTO APROBATORIO**, para que ella pueda proceder a imprimirla así como continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y sin otro particular por el momento, me permito enviarle un cordial saludo.

A T E N T A M E N T E
"Se Lumen Proferre"
Aguascalientes, Ags., a 28 de marzo del 2025

Dra. Mariana Alfaro Gómez
Asesor de tesis.

c.c.p.- Interesado
c.c.p.- Secretaría Técnica del Programa de Posgrado

CARTA DE VOTO APROBATORIO
INDIVIDUAL

M. en C. Jorge Martín Alférez Chávez
DECANO DEL CENTRO **DE CIENCIAS BÁSICAS**

P R E S E N T E

Por medio del presente como **ASESOR** designado de la estudiante **GABRIELA DE JESÚS CABRAL GARCÍA** con ID 496 quien realizó la tesis titulada: **MODELACIÓN CON ECUACIONES DIFERENCIALES ESTOCÁSTICAS** un trabajo propio, innovador, relevante e inédito y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia doy mi consentimiento de que la versión final del documento ha sido revisada y las correcciones se han incorporado apropiadamente, por lo que me permito emitir el **VOTO APROBATORIO**, para que ella pueda proceder a imprimirla así como continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y sin otro particular por el momento, me permito enviarle un cordial saludo.

A T E N T A M E N T E
"Se Lumen Proferre"
Aguascalientes, Ags., a 27 de marzo de 2025

Dr. Luis Antonio Rincón Solís
Asesor de tesis.

c.c.p.- Interesado
c.c.p.- Secretaría Técnica del Programa de Posgrado

CARTA DE VOTO APROBATORIO
INDIVIDUAL

M. en C. Jorge Martín Alférez Chávez
DECANO DEL CENTRO **DE CIENCIAS BÁSICAS**

____
P R E S E N T E

Por medio del presente como **ASESORA** designado de la estudiante **GABRIELA DE JESÚS CABRAL GARCÍA** con ID 496 quien realizó la tesis titulada: **MODELACIÓN CON ECUACIONES DIFERENCIALES ESTOCÁSTICAS** un trabajo propio, innovador, relevante e inédito y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia doy mi consentimiento de que la versión final del documento ha sido revisada y las correcciones se han incorporado apropiadamente, por lo que me permito emitir el **VOTO APROBATORIO**, para que ella pueda proceder a imprimirla así como continuar con el procedimiento administrativo para la obtención del grado.

Pongo lo anterior a su digna consideración y sin otro particular por el momento, me permito enviarle un cordial saludo.

A T E N T A M E N T E
"Se Lumen Proferre"
Aguascalientes, Ags., a 27 de marzo de 2025

Luz Judith R.E.

Dra. Luz Judith Rodríguez Esparza
Asesor de tesis.

c.c.p.- Interesado
c.c.p.- Secretaría Técnica del Programa de Posgrado

UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES

DICTAMEN DE LIBERACIÓN ACADÉMICA PARA INICIAR LOS TRÁMITES DEL EXAMEN DE GRADO

POSGRADOS UAA

Fecha de dictaminación dd/mm/aaaa: 21/07/2025

NOMBRE: Gabriela de Jesús Cabral García **ID** 496

PROGRAMA: Doctorado en Ciencias Aplicadas y Tecnología **LGAC (del posgrado):** Matemáticas Aplicadas

TIPO DE TRABAJO: (X) Tesis ( ) Trabajo Práctico

TITULO: Modelación con Ecuaciones Diferenciales Estocásticas

IMPACTO SOCIAL (señalar el impacto logrado): La tesis presenta diferentes modelos estocásticos, de los cuales se calcula el tiempo promedio para que dicho proceso pase de un nivel de interes a otro. Este tiempo de calcula para modelos de crecimiento de células cancerígenas, así como de infectados por alguna enfermedad. Por último se presenta un teorema de punto fijo aplicable al espacio de funciones uniformemente integrables.

INDICAR SI NO N.A. (NO APLICA) SEGÚN CORRESPONDA:

	Elementos para la revisión académica del trabajo de tesis o trabajo práctico:
SI	El trabajo es congruente con las LGAC del programa de posgrado
SI	La problemática fue abordada desde un enfoque multidisciplinario
SI	Existe coherencia, continuidad y orden lógico del tema central con cada apartado
SI	Los resultados del trabajo dan respuesta a las preguntas de investigación o a la problemática que aborda
SI	Los resultados presentados en el trabajo son de gran relevancia científica, tecnologíca o profesional según el área
SI	El trabajo demuestra más de una aportación original al conocimiento de su área
SI	Las aportaciones responden a los problemas prioritarios del país
NO	Generó transferecia del conocimiento o tecnológica
SI	Cumple con la ética para la investigación (reporte de la herramienta antiplagio)
	El egresado cumple con lo siguiente:
SI	Cumple con lo señalado por el Reglamento General de Docencia
SI	Cumple con los requisitos señalados en el plan de estudios (créditos curriculares, optativos, actividades complementarias,estancia, predoctoral, etc)
SI	Cuenta con los votos aprobatorios del comité tutoral, en caso de los posgrados profesionales si tiene solo tutor podrá liberar solo el tutor
N.A.	Cuenta con la carta de satisfacción del Usuario
SI	Coincide con el título y objetivo registrado
SI	Tiene congruencia con cuerpos académicos
SI	Tiene el CVU del Conacyt actualizado
SI	Tiene el artículo aceptado o publicado y cumple con los requisitos institucionales (en caso que proceda)
	En caso de Tesis por artículos científicos publicados
SI	Aceptación o Publicación de los articulos según el nivel del programa
SI	El estudiante es el primer autor
SI	El autor de correspondencia es el Tutor del Núcleo Académico Básico
SI	En los artículos se ven reflejados los objetivos de la tesis, ya que son producto de este trabajo de investigación.
SI	Los artículos integran los capítulos de la tesis y se presentan en el idioma en que fueron publicados
SI	La aceptación o publicación de los artículos en revistas indexadas de alto impacto

Con base a estos criterios, se autoriza se continúen con los trámites de titulación y programación del examen de grado: Sí X No ____

FIRMAS

Elaboró:

* NOMBRE Y FIRMA DEL CONSEJERO SEGÚN LA LGAC DE ADSCRIPCION: Dr. José Antonio Guerrero Díaz

NOMBRE Y FIRMA DEL SECRETARIO TÉCNICO: Dr. Francisco Javier Alvarez Rodríguez

*** En caso de conflicto de intereses, firmará un revisor miembro del NAB de la LGAC correspondiente distinto al tutor o miembro del comité tutoral, asignado por el Decano**

Revisó:

NOMBRE Y FIRMA DEL SECRETARIO DE INVESTIGACIÓN Y POSGRADO: Dr. Alejandro Padilla Díaz

Autorizó:

NOMBRE Y FIRMA DEL DECANO: M.C. Jorge Martín Alferez Chávez

Nota: procede el trámite para el Depto. de Apoyo al Posgrado

En cumplimiento con el Art. 105C del Reglamento General de Docencia que a la letra señala entre las funciones del Consejo Académico: .... Cuidar la eficiencia terminal del programa de posgrado y el Art. 105F las funciones del Secretario Técnico, llevar el seguimiento de los alumnos.

Rendiconti del Circolo Matematico di Palermo Series 2
https://doi.org/10.1007/s12215-021-00714-7

A fixed point theorem in the space of integrable functions and applications

G. J. de Cabral-García[1] · K. Baquero-Mariaca[1] · J. Villa-Morales[1]

Received: 29 October 2021 / Accepted: 16 December 2021

Abstract
We give sufficient conditions to ensure when a mapping $T : \boldsymbol{E} \to \boldsymbol{E}$ has a unique fixed point, $\boldsymbol{E}$ is a set of measurable functions that is uniformly continuous, closed, and convex. The proof of the existence of the fixed point depends on a certain type of sequential compactness for uniformly integrable functions that is also studied. The fixed point theorem is applied in the study of the uniqueness and existence of some Fredholm and Caputo equations.

Keywords Fixed point theorem · Uniform integrability · Fredholm equations · Caputo fractional equations · Sequential compactness

Mathematics Subject Classification Primary 47H10 · Secondary 54C05

1 Introduction and statement of the results

Let $\boldsymbol{E}$ be a non-empty set and $T : \boldsymbol{E} \to \boldsymbol{E}$ be a mapping. We say that T has a fixed point if there exists an $x \in \boldsymbol{E}$ such that $T(x) = x$. The literature on fixed point theorems is very abundant. This is due to its elegant theoretical aspect and its many applications in differential equations, game theory, and functional analysis, to name just a few. Broadly speaking, there are two results that helped drive the study of fixed point theorems. Namely, Banach's contraction principle and Browder's fixed point theorem. In the first case, it is assumed that $\boldsymbol{E}$ is a complete metric space and T is a contraction. On the other hand, in the second case it is assumed that $\boldsymbol{E}$ is a compact set, and the mapping T is non-expansive. Based on them, the study of fixed point theorems has diversified into different aspects, which are active branches of non linear analysis.

In the case where $\boldsymbol{E}$ is a complete metric space, one aspect in the study of fixed point theorems is to weaken the definition of metric, see for example [1, 7, 10, 15, 18, 19] and

✉ J. Villa-Morales
jvilla@correo.uaa.mx

G. J. de Cabral-García
gabriela.cabral@edu.uaa.mx

1 Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad No. 940, Cd. Universitaria, C.P. 20131 Aguascalientes, AGS, Mexico

Applied Mathematical Modelling 128 (2024) 272–286

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Certain aspects of the SIS stochastic epidemic model

Gabriela de Jesús Cabral-García, José Villa-Morales *

Universidad Autónoma de Aguascalientes, Departamento de Matemáticas y Física, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico

A R T I C L E I N F O

MSC:
65L12
65L05
65L70

Keywords:
First moment
Asymptotic behavior
Some real-world SIS epidemic models

A B S T R A C T

The SIS model is a fundamental tool in epidemiology for understanding the spread of infectious diseases. This article focuses on the stochastic SIS model, introducing randomness through the disease transmission parameter. The study investigates the behavior of the model, revealing the conditions for the recurrence or extinction of the disease. In particular, it addresses the calculation of the conditional expected time for the disease to exceed a certain threshold, using both Laplace transforms and numerical techniques for its specific application. Real-world phenomena are discussed, and a method for determining the most suitable stochastic parameter is proposed, with examples such as gonorrhea and pneumococcus.

1. Introduction

The SIS (Susceptible-Infectious-Susceptible) model is a classic mathematical epidemiology model used to describe and understand the spread of infectious diseases within a population. In this model, individuals can be categorized into two states: susceptible (S), representing those who can contract the disease, and infectious (I), representing those who are infected and can transmit the disease to others. The SIS model serves as a foundational framework for comprehending the dynamics of infectious diseases within a population. It stands as a crucial tool in epidemiology for predicting disease dissemination and assessing public health interventions, such as vaccination and social distancing (for instance, refer to Mollison [1] and the related references cited therein).

Let us denote by S_t and I_t the numbers of susceptible and infected individuals at time t, respectively. We assume that the birth and death rates in the population are the same, leading to a constant population size denoted as N over time. Symbolically, the dynamics of the SIS epidemic model are described by the following system of differential equations:

$$\frac{dS_t}{dt} = (\tilde{b}+\gamma)I_t - \frac{\beta}{N} S_t \cdot I_t, \tag{1.1}$$

$$\frac{dI_t}{dt} = \frac{\beta}{N} S_t \cdot I_t - (\tilde{b}+\gamma)I_t. \tag{1.2}$$

Here, $\tilde{b} \geq 0$ represents the rate at which an infected person recovers, and $\gamma > 0$ is the death rate. We will introduce randomness to the crucial term $\beta \in \mathbb{R}$, which is associated with disease transmission. This method of introducing randomness into a system is widely recognized and referred to as stochastic parameter perturbation (refer to [2] for more information).

Using the assumption of a constant population over time, instead of considering a system of differential equations, we focus on a single differential equation. In this approach, by introducing stochastic parameter perturbation, we address a single stochastic differential equation corresponding to the number of infected individuals (referred to as equation (2.2)).

* Corresponding author.
E-mail addresses: gabriela.cabral@edu.uaa.mx (G. de Jesús Cabral-García), jvilla@correo.uaa.mx (J. Villa-Morales).

https://doi.org/10.1016/j.apm.2024.01.027
Received 23 August 2023; Received in revised form 10 January 2024; Accepted 16 January 2024

Available online 22 January 2024

Applied Mathematics and Computation 470 (2024) 128569

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Conditional moments of the first-passage time of a crowed population

Gabriela de Jesús Cabral-García, José Villa-Morales *

Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico

A R T I C L E I N F O

MSC:
92D25
60G53
60G17
92B15

Keywords:
Stochastic logistic differential equation
Conditional moments
First-passage time
Tumor
World population

A B S T R A C T

Using the method of stochastic variation of parameters in the logistic differential equation a stochastic logistic differential equation is obtained. For this stochastic differential equation, the first two conditional moments of the first-passage time are found. These results are applied in the study of the growth of cancerous tumors and in the study of the growth of the world population when a random perturbation is incorporated into the deterministic models.

1. Introduction

The study of population growth models dates back to 1798 when Thomas Robert Malthus wrote an essay on the exponential growth of population and the linear growth of resources. Modeling the growth of a system is a classic topic in applied mathematics; additionally, due to its importance, its applications in the real world, and the complexity of the problem, this is a current issue (see, for example, [25], [4], [27], [21], [20], [14], [11], [10] and the references contained therein).

In our case, we will consider the growth of a population in which the environment (or ecosystem) has the property of maintaining at most a population of size $K > 0$ indefinitely, we will say that the environment has a carrying capacity K. Furthermore, we will assume that the quality of the environment is determined by a parameter $\lambda \in \mathbb{R}$. In this case, if X_t is the size of the population at time $t > 0$, then it is well known that the dynamics or growth of the population is determined by the logistic differential equation

$$\frac{d}{dt} X_t = \lambda X_t (K - X_t), \quad t > 0, \tag{1.1}$$

where $X_0 = x_0 > 0$ is the initial population. In this case, the solution is given by the sigmoidal function

$$X_t = \frac{x_0 K}{x_0 + (K - x_0) e^{-\lambda K t}}, \quad t \geq 0. \tag{1.2}$$

* Corresponding author.
E-mail addresses: gabriela.cabral@edu.uaa.mx (G. de Jesús Cabral-García), jvilla@correo.uaa.mx (J. Villa-Morales).

https://doi.org/10.1016/j.amc.2024.128569
Received 11 April 2023; Received in revised form 14 December 2023; Accepted 19 January 2024

Agradecimientos

Agradezco a la Universidad Autónoma de Aguascalientes por haberme brindado la oportunidad de formar parte de este programa. Mi afecto hacia esta institución es inmenso. Además, quiero reconocer el valioso apoyo financiero otorgado por CONAHCYT, sin el cual este trabajo no habría sido posible.

Quiero expresar mi más profundo agradecimiento a mi compañero de estudio y de vida, mi esposo Luis. Tu constante apoyo y aliento han sido el pilar fundamental que me ha impulsado en este viaje. Sin ti, nada de esto sería posible, amo el equipo que somos. También deseo extender mi más sincero agradecimiento a mis padres, Gabriela y Guillermo, así como a mi hermano Memo. Gracias por creer en mí desde el principio, por estar a mi lado en cada paso de este camino, y por ser mi fuente inagotable de amor y apoyo.

Este trabajo lo llevo a cabo con el objetivo de asegurar un futuro mejor para mis dos hijos, Ana Lía y Luis Noé. Son mi mayor inspiración y motor para seguir adelante. Los amo profundamente y cada logro alcanzado en este proceso está dedicado a ustedes.

Asimismo, deseo destacar el invaluable apoyo brindado por mi asesor, el Dr. José Villa Morales. Le agradezco sinceramente por darme la oportunidad de trabajar bajo su dirección. Estoy enormemente agradecida por haber contado con su orientación y mentoría durante este proceso.

Índice general

Resumen

Esta tesis aborda dos temas principales. En primer lugar, se explora un teorema de punto fijo, el cual entre muchas aplicaciones es una herramienta fundamental en la demostración de soluciones para una amplia gama de ecuaciones diferenciales, tanto ordinarias como parciales. Este teorema se formula en el contexto del espacio de funciones uniformemente integrables.

En la segunda parte, se adopta un enfoque estocástico al trabajar con dos modelos matemáticos, los cuales se muestran en los capítulos dos y tres. El objetivo es calcular el primer y segundo momento de primera pasada, es decir, el tiempo promedio que un proceso estocástico requiere para pasar de un estado a otro. El primer modelo que se examina, usualmente conocido como SIS, divide la población en dos clases: susceptibles e infectados. Este modelo se utiliza para analizar el comportamiento de enfermedades infecciosas y cómo se produce el intercambio entre ambas clases. El segundo modelo es el crecimiento logístico, ampliamente conocido por modelar el crecimiento de poblaciones.

Para ambos modelos, se aplican los resultados teóricos en datos reales. En efecto, se evalúa si estos datos se ajustan a los modelos propuestos y, finalmente, se calculan valores esperados de manera numérica. Es importante destacar que los datos reales utilizados son de relevancia clínica o poblacional, lo que añade un componente práctico significativo al análisis realizado.

Abstract

This thesis addresses two main topics. Firstly, it explores a fixed-point theorem, which, among many applications, is a fundamental tool in proving the existence of solutions for a wide range of differential equations, both ordinary and partial. This theorem is formulated within the context of the space of uniformly integrable functions.

In the second part, a stochastic approach is adopted by working with two mathematical models, which are presented in chapters two and three. The objective is to calculate the first and second moments of the first passage time, that is, the average time a stochastic process requires to transition from one state to another. The first model examined, commonly known as the SIS model, divides the population into two classes: susceptible and infected. This model is used to analyze the behavior of infectious diseases and how the exchange between these two classes occurs. The second model is the logistic growth model, widely recognized for modeling population growth.

For both models, the theoretical results are applied to real-world data. Specifically, it is evaluated whether these data fit the proposed models, and finally, some expected values are calculated numerically. It is important to note that the real-world data used are of clinical or population relevance, adding a significant practical component to the analysis conducted.

Introducción

Objetivos y alcance

Las ecuaciones diferenciales son una herramienta fundamental en el estudio y comprensión de una amplia gama de fenómenos, abarcando desde aspectos sociales, como el comportamiento de enfermedades contagiosas, hasta fenómenos físicos, económicos y financieros [8]. El enfoque de este trabajo es en dos aspectos centrales.

En primer lugar, se establece un nuevo teorema de punto fijo, el cual proporciona condiciones suficientes para garantizar la existencia de un valor fijo único en el contexto de una transformación T que opera en un espacio de funciones medibles uniformemente integrables, cerradas y convexas a él mismo.

En segundo lugar, el estudio se centra en ecuaciones diferenciales estocásticas. Estas emergen a partir de ecuaciones diferenciales ordinarias al introducir ruido aleatorio en uno de sus parámetros, método conocido como perturbación estocástica de parámetros [3], lo que permite que el modelo se ajuste mejor a la realidad. Esta modificación da lugar a una variedad infinita de posibles procesos para un mismo modelo. El objetivo es considerar todas las trayectorias posibles que puede seguir el proceso y calcular el tiempo promedio que le toma alcanzar un nivel de interés particular por primera vez, ya sea superior o inferior al nivel inicial. Este tiempo promedio es crucial, ya que proporciona información valiosa para la implementación de políticas de prevención o intervención en diferentes escenarios. Para calcular este valor esperado se utilizan herramientas de análisis numérico, para posteriormente hacer el cálculo de dichos valores esperados en escenarios con datos reales de diversos fenómenos sociales, algunos de estos datos se retoman de [25], [32]. Los escenarios que se consideran son el crecimiento poblacional, la proliferación de células cancerosas, la propagación de enfermedades infecciosas como la gonorrea y el neumococo, y el crecimiento en tamaño de poblaciones de peces guppy.

Esta investigación no solo amplía nuestro entendimiento teórico de los sistemas dinámicos subyacentes a estos fenómenos, sino que también proporciona herramientas prácticas para abordar y gestionar situaciones concretas en la realidad.

Organización del trabajo

La tesis está organizada de la siguiente forma:

- **Capítulo 1:** El objetivo principal del trabajo presente en este capítulo es el de establecer un teorema de punto fijo sobre un espacio métrico completo, cerrado, convexo y uniformemente integrable y una transformación que no es necesariamente una contracción. Como introducción al artículo, presentado en la segunda sección del capítulo, se recuerdan algunos teoremas de punto fijo (por ejemplo [6]) y las distintas condiciones que estos requieren. Posteriormente se presentan los conceptos sobre los que se sustenta el teorema de punto fijo para finalmente dar los enunciados importantes del trabajo realizado.

- **Capítulo 2:** En este capítulo se presenta el primer trabajo con ecuaciones diferenciales estocásticas, se trabaja con el modelo matemático SIS (Susceptible-Infeccioso-Susceptible), el cual es un modelo clásico en epidemiología. Este modelo se utiliza para describir y comprender la propagación de enfermedades infecciosas en una población, analizando el intercambio de individuos entre las clases de susceptible e infectado (ver [9]).

  En la introducción del capítulo se da a conocer el sistema dinámico de ecuaciones diferenciales ordinarias que definen el modelo. Se establece lo que representa cada uno de los paramétros utilizados y a cual de éstos se le agrega el ruido aleatorio obteniendo así un modelo estocástico.

- **Capítulo 3:** Aquí se presenta otra ecuación diferencial estocástica, la cual tiene como base una ecuación diferencial ordinaria conocida, la ecuación que modela el crecimiento logístico. Se estudia el comportamiento del modelo al agregar ruido aleatorio a uno de los parámetros que rigen dicho modelo. Esto con el mismo fin que en el capítulo anterior, calcular el valor esperado que le toma al proceso estócastico pasar por un nivel de interés, dado que inicio en un nivel superior o inferior. Además, aquí se expone el hecho de que este modelo es una generalización del modelo estudiado en el capítulo anterior. Por último, al igual que en el modelo SIS se hace el cálculo numérico del valor esperado de interés aplicado a casos con datos reales.

1 Un teorema de punto fijo en el espacio de funciones integrales y aplicaciones

Sea E un conjunto no vacío y $T : E \to E$ una transformación. Se dice que T tiene un punto fijo si existe un $x \in E$ tal que $T(x) = x$. La literatura sobre teoremas de punto fijo es muy abundante. Esto se debe a su elegante aspecto teórico y a sus muchas aplicaciones en ecuaciones diferenciales, teoría de juegos y análisis funcional, por nombrar sólo algunas. En términos generales, hay dos resultados que ayudan a impulsar el estudio de los teoremas de punto fijo. A saber, el principio de contracción de Banach [16] y el teorema del punto fijo de Browder [5]. En el primer caso, se supone que E es un espacio métrico completo y T es una contracción. En cambio, en el segundo caso se supone que E es un conjunto compacto, y la transformación T es no-expansiva. A partir de ellos el estudio de los teoremas de punto fijo se ha diversificado en diferentes aspectos, que son ramas activas del análisis no lineal, ver [22].

En el caso en que E sea un espacio métrico completo, un aspecto importante en el estudio de los teoremas de punto fijo es debilitar la definición de métrica, véase por ejemplo [30], [34], [15], [31], [42], [43] y las referencias dadas allí. Cuando E es un conjunto compacto, el resultado clásico es el teorema de Edelstein, que tiene muchas generalizaciones importantes, por ejemplo ver [10], [14], [40], [39], [38] o las referencias en ellos.

Se denota con (X, A, μ) a un espacio de medida y por $L(\mu) = L(X, A, \mu)$ al espacio de funciones integrables, además si $x \in L(\mu)$, se denota con $\|x\| = \int |x| d\mu$, como la norma de x.

Teorema 1.1 Sea $E \subset L(\mu)$ un conjunto cerrado, convexo y uniformemente integrable. Sea $T : E \to E$ una función. Si el mapeo $x \mapsto \|x - Tx\|$ es cuasiconvexo, semicontinuo inferior y si

$$\frac{1}{2}\|x - Tx\| < \|x - y\|$$

siempre implica que

$$\|Tx - Ty\| < \text{máx}\left\{\|x - y\|, \frac{1}{2}\left(\|y - Tx\| + \|x - Ty\|\right), \frac{1}{2}\left(\|x - Tx\| + \|y - Ty\|\right)\right\}$$

para todo $x, y \in E$ con $x \neq y$, entonces T tiene un único punto fijo.

La importancia de este resultado radica en que no se asume la condición de compacidad sobre E. Del mismo modo, E es efectivamente un espacio métrico completo, sin embargo, la aplicación T no es necesariamente una contracción. En el caso de no expansividad existen algunos resultados en este sentido, ver por ejemplo [13], sin embargo en ellos se asume la compacidad. De esta manera el teorema anterior no cae en el contexto clásico. Además, nótese que E tiene un orden parcial natural, sin embargo, también en este caso la revisión de la literatura sobre teoremas de punto fijo en espacios métricos ordenados indica que no existen resultados similares.

Como se sabe, cuando se estudia la existencia de un punto fijo una forma de proceder es considerar una función continua de la forma $x \mapsto d(x, Tx)$ y verificar que dicha función alcanza su ínfimo en 0. Esto funciona bien cuando el espacio métrico E es compacto. Aquí la falta de esta propiedad está cubierta por el siguiente resultado, que es un tipo de compacidad secuencial.

Teorema 1.2 Sea $(f_i)_{i\in I}$ una red en $L(\mu)$. Para cada $i \in I$ se denota por $C_i = \mathrm{co}\{f_j : j \geq i\}$ la envolvente convexa (casco) de todas las f_j, $j \geq i$. Si $(f_i)_{i\in I}$ es uniformemente integrable, entonces para cada $i \in I$ existe $g_i \in C_i$ tal que $(g_i)_{i\in I}$ converge en $L(\mu)$ a una función medible de valor real g.

Todas las sucesiones uniformemente integrables están acotadas en $L_p(\mu)$, $1 \leq p < +\infty$. Si $1 < p < +\infty$ entonces el teorema de Mazur implica que existe una sucesión convergente en el casco convexo de la sucesión completa. Este resultado no necesariamente es cierto cuando $p = 1$. Además, un punto clave es que en el Teorema 1.2 la nueva red se construye en la cola de la envolvente convexa de la primera red. Estas consideraciones son fundamentales, como se expone en la demostración del Teorema 1.1 (véase en particular la demostración del Lemma 2.1). La importancia de este trabajo radica en el Teorema 1.2, el cual puede tener una gran variedad de aplicaciones en diferentes ramas de las matemáticas, véase por ejemplo [41], [29], [44], [45].

1.2. Artículo

Rendiconti del Circolo Matematico di Palermo Series 2
https://doi.org/10.1007/s12215-021-00714-7

A fixed point theorem in the space of integrable functions and applications

G. J. de Cabral-García[1] · K. Baquero-Mariaca[1] · J. Villa-Morales[1]

Received: 29 October 2021 / Accepted: 16 December 2021

Abstract
We give sufficient conditions to ensure when a mapping $T : \boldsymbol{E} \to \boldsymbol{E}$ has a unique fixed point, $\boldsymbol{E}$ is a set of measurable functions that is uniformly continuous, closed, and convex. The proof of the existence of the fixed point depends on a certain type of sequential compactness for uniformly integrable functions that is also studied. The fixed point theorem is applied in the study of the uniqueness and existence of some Fredholm and Caputo equations.

Keywords Fixed point theorem · Uniform integrability · Fredholm equations · Caputo fractional equations · Sequential compactness

Mathematics Subject Classification Primary 47H10 · Secondary 54C05

1 Introduction and statement of the results

Let $\boldsymbol{E}$ be a non-empty set and $T : \boldsymbol{E} \to \boldsymbol{E}$ be a mapping. We say that T has a fixed point if there exists an $x \in \boldsymbol{E}$ such that $T(x) = x$. The literature on fixed point theorems is very abundant. This is due to its elegant theoretical aspect and its many applications in differential equations, game theory, and functional analysis, to name just a few. Broadly speaking, there are two results that helped drive the study of fixed point theorems. Namely, Banach's contraction principle and Browder's fixed point theorem. In the first case, it is assumed that $\boldsymbol{E}$ is a complete metric space and T is a contraction. On the other hand, in the second case it is assumed that $\boldsymbol{E}$ is a compact set, and the mapping T is non-expansive. Based on them, the study of fixed point theorems has diversified into different aspects, which are active branches of non linear analysis.

In the case where $\boldsymbol{E}$ is a complete metric space, one aspect in the study of fixed point theorems is to weaken the definition of metric, see for example [1, 7, 10, 15, 18, 19] and

✉ J. Villa-Morales
jvilla@correo.uaa.mx

G. J. de Cabral-García
gabriela.cabral@edu.uaa.mx

1 Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad No. 940, Cd. Universitaria, C.P. 20131 Aguascalientes, AGS, Mexico

Published online: 07 January 2022

the references given there. When $\boldsymbol{E}$ is a compact set the classical result is Edelstein's theorem which has many important generalizations, for a recent account of the theory see for instance [5, 8, 11, 14, 17] or the references in them.

To state the fixed point theorem let us introduce some notations. Let $(X, \mathcal{A}, \mu)$ be a measure space. We will denote by $L(\mu) = L(X, \mathcal{A}, \mu)$ the space of integrable functions. If $x \in L(\mu)$, we will write $||x|| = \int |x| d\mu$.

Theorem 1.1 *Let $\boldsymbol{E} \subset L(\mu)$ be a set which is closed, convex, and uniformly integrable. Let $T : \boldsymbol{E} \to \boldsymbol{E}$ be a function. If the mapping $x \mapsto \|x - Tx\|$ is quasi-convex, lower semi-continuous and if*

$$\frac{1}{2}\|x - Tx\| < \|x - y\|$$

always implies

$$\|Tx - Ty\| < \max\left\{ \|x - y\|, \frac{1}{2}(\|y - Tx\| + \|x - Ty\|), \frac{1}{2}(\|x - Tx\| + \|y - Ty\|) \right\}, \tag{1.1}$$

for all $x, y \in \boldsymbol{E}$ with $x \neq y$. Then T has a unique fixed point.

The importance of this result lies in the fact that the compactness condition is not assumed on $\boldsymbol{E}$. Similarly, $\boldsymbol{E}$ is indeed a complete metric space, however, the application T is not necessarily a contraction. In the case of non-expansiveness there are some results in this sense, see for example [9], however compactness is assumed in them. In this way the previous fixed pint theorem does not fall into the classical context. Furthermore, note that $\boldsymbol{E}$ has a natural partial order, however also in this case our literature review on fixed point theorems on metric spaces indicates that there are no similar results.

As it is known, when studying the existence of a fixed point one way to proceed is to consider a continuous function of the form $x \mapsto d(x, Tx)$ and verify that such function reaches its infimum at 0. This works very well when the metric space $\boldsymbol{E}$ is compact. Here the lack of this property is covered by the following result, which is a type of sequential compactness, that we consider being important in itself.

Theorem 1.2 *Let $(f_i)_{i \in I}$ be a net in $L(\mu)$. For each $i \in I$ we denote by $C_i = \mathrm{co}\{f_j : j \geq i\}$ the convex envelope (hull) of all f_j, $j \geq i$. If $(f_i)_{i \in I}$ is uniformly integrable, then for each $i \in I$ there exists $g_i \in C_i$ such that $(g_i)_{i \in I}$ converges in $L(\mu)$ to a real-valued measurable function g.*

All uniformly integrable sequences are bounded in $L^p(\mu)$, $1 \leq p < +\infty$. If $1 < p < +\infty$ then Mazur's theorem implies that there is a convergent sequence in the convex hull of the whole sequence. This result is not necessarily true when $p = 1$. Moreover one key point is that in Theorem 1.2 the new net is built in the tail of the convex hull of the first net. These kind of considerations are fundamental as we will see in the proof of Theorem 1.1 (see in particular the proof of Lemma 2.1). We think that Theorem 1.2 can have a wide variety of applications in different branches of mathematics, see for example [6, 12, 20, 21].

Fixed point theorems are particularly useful for proving the existence of solutions of differential and integral equations. In our case, we will see that Theorem 1.1 can be applied to demonstrate the uniqueness and existence of certain types of Fredholm equations and some fractional differential equations in the Caputo sense, which is an area of recent interest in fixed point theory.

The article is organized in the following manner. In Sect. 2 we give the proof of Theorem 1.1, in Sect. 3 we study the applications of the fixed point theorem, and Sect. 4 is dedicated to the proof of Theorem 1.2.

2 Proof of the fixed point theorem

The demonstration of the fixed point theorem is based on a minimal principle that we examine next. We recall that a family $\boldsymbol{F}$ of $\mathcal{A}$ measurable functions is uniformly integrable if for all $\varepsilon > 0$ there exists $w_\varepsilon \in L(\mu)$, $w_\varepsilon \geq 0$, such that

$$\sup_{u \in \boldsymbol{F}} \int_{\{|u| > w_\varepsilon\}} |u| d\mu < \varepsilon,$$

where sets like $\{|u| > w_\varepsilon\}$ mean $\{t \in X : |u(t)| > w_\varepsilon(t)\}$.

Lemma 2.1 *Let $\boldsymbol{E} \subset L(\mu)$ be a closed convex set that is uniformly integrable. Let $T : \boldsymbol{E} \to \boldsymbol{E}$ be a function. If the mapping $x \mapsto \|x - Tx\|$ is quasi-convex and lower semi-continuous, then there exists $x^0 \in \boldsymbol{E}$ such that*

$$\|x^0 - Tx^0\| = \inf_{x \in \boldsymbol{E}} \|x - Tx\|.$$

Proof Set $\alpha = \inf_{x \in \boldsymbol{E}} \|x - Tx\|$ and let (x_n) be a minimizing sequence, $\alpha = \lim_{n \to +\infty} \|x_n - Tx_n\|$. For $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\|x_n - Tx_n\| < \alpha + \varepsilon, \quad \text{for all } n \geq n_0. \tag{2.1}$$

By Theorem 1.2 there exists a sequence (y_n) in $L(\mu)$ such that $y_n \in C_n = \text{co}\{x_i : i \geq n\}$, for all $n \in \mathbb{N}$ and let $y^0 \in L(\mu)$ such that (y_n) converges to y^0 in $L(\mu)$. Let us see that the infimum is reached at y^0. Since

$$y_n = \sum_{i=1}^{k_n} \lambda_i^n x_{n_i(n)}, \quad \text{with } n_i(n) \geq n,$$

then the quasi-convexity property of $x \mapsto \|x - Tx\|$ yields

$$\begin{aligned} \|y_n - Ty_n\| &= \left\| \sum_{i=1}^{k_n} \lambda_i^n x_{n_i(n)} - T \sum_{i=1}^{k_n} \lambda_i^n x_{n_i(n)} \right\| \\ &\leq \max\{\|x_{n_i(n)} - Tx_{n_i(n)}\| : i = 1, \ldots, k_n\}. \end{aligned}$$

If $n \geq n_0$ then (2.1) implies $\|y_n - Ty_n\| < \alpha + \varepsilon$. Considering $\lim_{n \to +\infty} y_n = y^0$, in $L(\mu)$, the lower semi-continuity of $x \mapsto \|x - Tx\|$ implies

$$\|y^0 - Ty^0\| \leq \liminf_{n \to +\infty} \|y_n - Ty_n\| \leq \alpha + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, in the previous inequality we deduce $\|y_0 - Ty_0\| \leq \alpha$.

On the other hand, since $\boldsymbol{E}$ is closed, then $y^0 \in \boldsymbol{E}$. The definition of α, implies $\alpha \leq \|y^0 - Ty^0\|$. □

Now we proceed to prove the fixed point theorem.

Proof of Theorem 1.1 By Lemma 2.1 there exists $z^0 \in \boldsymbol{E}$ such that

$$\|z^0 - Tz^0\| = \inf_{z \in \boldsymbol{E}} \|z - Tz\|.$$

Let us see that z^0 is the fixed point that we are looking for.

If $\|z^0 - Tz^0\| > 0$, then $\frac{1}{2}\|z^0 - Tz^0\| < \|z^0 - Tz^0\|$. Thus

$$\begin{aligned}\|Tz^0 - T(Tz^0)\| < \max\Big\{ &\|z^0 - Tz^0\|, \frac{1}{2}\|z^0 - T(Tz^0)\|,\\ &\frac{1}{2}\big(\|z^0 - Tz^0\| + \|Tz^0 - T(Tz^0)\|\big)\Big\}.\end{aligned}$$

Since $\|z^0 - T(Tz^0)\| \leq \|z^0 - Tz^0\| + \|Tz^0 - T(Tz^0)\|$, then

$$\|Tz^0 - T(Tz^0)\| < \max\Big\{\|z^0 - Tz^0\|, \frac{1}{2}\big(\|z^0 - Tz^0\| + \|Tz^0 - T(Tz^0)\|\big)\Big\}.$$

We have two possibilities

$$\|Tz^0 - T(Tz^0)\| < \|z^0 - Tz^0\|$$

or well

$$\|Tz^0 - T(Tz^0)\| < \frac{1}{2}\big(\|z^0 - Tz^0\| + \|Tz^0 - T(Tz^0)\|\big).$$

Any of the two cases turns out

$$\|Tz^0 - T(Tz^0)\| < \|z^0 - Tz^0\|,$$

which is a contradiction to the minimality of z^0. Therefore $Tz^0 = z^0$.

Now let us see the uniqueness. Let x^0 be other fixed point, $Tx^0 = x^0$. Suppose $x^0 \neq z^0$, thus

$$\frac{1}{2}\|x^0 - Tx^0\| = 0 < \|x^0 - z^0\|.$$

The hypothesis implies

$$\begin{aligned}\|x^0 - z^0\| &= \|Tx^0 - Tz^0\|\\ &< \max\Big\{\|x^0 - z^0\|, \frac{1}{2}\big(\|z^0 - x^0\| + \|x^0 - z^0\|\big), 0\Big\}\\ &= \|x^0 - z^0\|.\end{aligned}$$

That is, $\|x^0 - z^0\| < \|x^0 - z^0\|$. Which is absurd, hence $x^0 = z^0$. □

Corollary 2.2 *Let $\boldsymbol{E} \subset L(\mu)$ be a closed convex set that is uniformly integrable. Let $T : \boldsymbol{E} \to \boldsymbol{E}$ be a function. If the application $x \mapsto \|x - Tx\|$ is quasi-convex and lower semi-continuous and*

$$\|Tx - Ty\| < \|x - y\|, \quad x, y \in \boldsymbol{E}, \quad x \neq y, \tag{2.2}$$

then T has a unique fixed point.

Proof *Since*

$$\|x - y\| \leq \max \Big\{ \|x - y\|, \frac{1}{2}(\|y - Tx\| + \|x - Ty\|), \\ \frac{1}{2}(\|x - Tx\| + \|y - Ty\|) \Big\},$$

then (2.2) trivially implies (1.1). The result follows from Theorem 1.1. □

3 Applications

In this part, we are going to study two applications of Theorem 1.1.

Let $U \subset \mathbb{R}^n$ denote a measurable set and let $h_0 : U \to \mathbb{R}$ be an integrable function. In this section we assume that $(X, \mathcal{A}, \mu) = (U, \mathcal{B}(U), dx)$, where $\mathcal{B}(U)$ is the Borel sigma-algebra and dx is the Lebesgue measure. If $h_0 : U \to \mathbb{R}$ is a given non-negative integrable function, let us define the set of dominated functions by h_0 as

$$L(U; h_0) = \{f \in L(dx) \,:\, |f| \leq h_0 \text{ a.e.}\}.$$

It is not difficult to verify that $\boldsymbol{E} = L(U; h_0)$ is a closed convex subset of $L(U, \mathcal{B}(U), dx)$ and it is uniformly integrable.

3.1 Fredholm equations

Let us assume that U has finite Lebesgue measure. Moreover, we have an integrable function $f_0 : U \to \mathbb{R}$ and a non-negative measurable $k : U \times U \to \mathbb{R}$. Let us see that the integral equation (a Fredholm equation of the second kind)

$$g(x) = f_0(x) + \int_U k(x, y) g(y) dy, \quad x \in U, \tag{3.1}$$

has a unique solution in $L(U; h_0)$, for some integrable function h_0.

Proposition 3.1 *Let us suppose that*

$$\int_U k(x, y) dx < 1, \quad y \in U, \tag{3.2}$$

$$\sup_{x \in U} \int_U k(x, y) |f_0(y)| dy < +\infty, \tag{3.3}$$

$$\sup_{y \in U} \int_U k(x, y) dx > \sup_{x \in U} \int_U k(x, y) dy, \tag{3.4}$$

then there exists a constant $c > 0$ such that (3.1) has a unique solution in $L(U; |f_0| + c)$.

Proof From (3.3) and (3.4) we see that we can take $c \in (0, +\infty)$ such that

$$\sup_{x\in U} \int_U k(x,y)|f_0(y)|dy \leq c\left(\sup_{y\in U} \int_U k(x,y)dx - \sup_{x\in U} \int_U k(x,y)dy\right). \tag{3.5}$$

Let us consider the operator $T : \boldsymbol{E} \to \boldsymbol{E}$, defined as

$$(Tg)(x) = f_0(x) + \int_U k(x,y)g(y)dy, \quad x \in U, \tag{3.6}$$

where $\boldsymbol{E} = L(U; h_0)$ and $h_0 := |f_0| + c$ is integrable over U, due to U has finite Lebesgue measure. We have that $Tg \in \boldsymbol{E}$ if $g \in \boldsymbol{E}$. Indeed

$$\begin{aligned}
|(Tg)(x)| &\leq |f_0(x)| + \int_U k(x,y)|g(y)|dy \\
&\leq |f_0(x)| + \int_U k(x,y)\big(|f_0(y)| + c\big)dy \\
&\leq |f_0(x)| + \sup_{x\in U} \int_U k(x,y)|f_0(y)|dy + c \sup_{x\in U} \int_U k(x,y)dy.
\end{aligned}$$

Using (3.5) and (3.2) we can conclude that $Tg \in \boldsymbol{E}$.

We want to apply Theorem 1.1, so we need to see that $g \mapsto \|g - Tg\|$ satisfies the required hypotheses by such result. Let us see that g is convex. If $g_1, g_2 \in \boldsymbol{E}$ and $\lambda_1 + \lambda_2 = 1$, with $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, then

$$\begin{aligned}
&||\big(\lambda_1 g_1 + \lambda_2 g_2\big) - T\big(\lambda_1 g_1 + \lambda_2 g_2\big)|| \\
&= \int \left|(\lambda_1 g_1 + \lambda_2 g_2)(x) - T\big(\lambda_1 g_1 + \lambda_2 g_2\big)(x)\right| dx \\
&= \int \left|\lambda_1 g_1(x) + \lambda_2 g_2(x) - f_0(x) - \int_U k(x,y)\big(\lambda_1 g_1(x) + \lambda_2 g_2(y)\big)dy\right| dx \\
&= \int \Big|\lambda_1 g_1(x) - \lambda_1 f_0(x) - \lambda_1 \int_U k(x,y)g_1(y)dy \\
&\quad + \lambda_2 g_2(x) - \lambda_2 f_0(x) - \lambda_2 \int_U k(x,y)g_2(y)dy\Big| dx \\
&\leq \lambda_1 \int \left|g_1(x) - f_0(x) - \int_U k(x,y)g_1(y)dy\right| dx \\
&\quad + \lambda_2 \int \left|g_2(x) - f_0(x) - \int_U k(x,y)g_2(y)dy\right| dx \\
&= \lambda_1 \|g_1 - Tg_1\| + \lambda_2 \|g_2 - Tg_2\|.
\end{aligned}$$

In particular $g \mapsto \|g - Tg\|$ is quasi-convex.

If $g_1, g_2 \in \boldsymbol{E}$, the inequality (3.2) and Fubini's theorem yields

$$\begin{aligned}\|Tg_1 - Tg_2\| &= \int_U \left| \int_U k(x,y)g_1(y)dy - \int_U k(x,y)g_2(y)dy \right| dx \\ &\le \int_U |g_1(y) - g_2(y)| \int_U k(x,y)dxdy \\ &< \int_U |g_1(y) - g_2(y)|dy = \|g_1 - g_2\|.\end{aligned} \tag{3.7}$$

This implies T is continuous, therefore by Corollary 2.2 there exists a unique $g_0 \in \boldsymbol{E}$ such that $Tg_0 = g_0$. □

In a sense, inequality (3.1) is apparently related to Edelstein's fixed point theorem, however the metric space $\boldsymbol{E}$ is not compact in general. Moreover, if $\sup_{y\in U} \int_U k(x,y)dx < 1$ the inequality (3.7) implies T is contraction and Proposition 3.1 follows from the classical contraction principle. However, in the following example T is not a contraction but Proposition 3.1 still applies.

Example The Fredholm equation

$$g(x) = x^{-1/2} + \int_0^1 ae^{x-2y} g(y)dy, \quad x \in (0,1), \tag{3.8}$$

where $a := 1/(e-1)$, has a unique solution in $L((0,1);h_0)$, for some integrable function h_0.

Proof First let us observe that

$$\int_0^1 ae^{x-2y}dx = e^{-2y} < 1, \quad y \in (0,1), \tag{3.9}$$

$$\sup_{x\in(0,1)} \int_0^1 ae^{x-2y}y^{-1/2}dy = \frac{e}{e-1}\int_0^1 e^{-2y}y^{-1/2}dy \le \frac{2e}{e-1}, \tag{3.10}$$

$$\sup_{y\in(0,1)} \int_0^1 ae^{x-2y}dx = a\int_0^1 e^x dx \sup_{y\in(0,1)} e^{-2y} = 1, \tag{3.11}$$

$$\sup_{x\in(0,1)} \int_0^1 ae^{x-2y}dy = a\int_0^1 e^{-2y}dy \sup_{x\in(0,1)} e^x = \frac{e+1}{2e}. \tag{3.12}$$

Since $e > 1$, then (3.11) and (3.12) imply that the corresponding inequality (3.4) is also true. To determine the function T, defined in (3.6), it is necessary to find a constant $c > 0$ such that (3.5) is valid, in this case

$$\frac{2e}{e-1} \le c\left(1 - \frac{e+1}{2e}\right), \tag{3.13}$$

therefore we can take $c := (2e/(e-1))^2$. The inequalities (3.9), (3.10) and (3.13) are the required by Proposition 3.1, then there is a unique solution $g_0 \in L\big((0,1);c + x^{-1/2}\big)$ of (3.8). From (3.11) it is important to observe that the application T is not a contraction since

$\sup_{y\in U}\int_U k(x,y)dx = 1$ (see (3.7)). In this way, the existence of solutions to (3.8) does not fall into the classical context. □

3.2 Fractional equations

At present there is in the literature a great variety of fractional derivative concepts, even more the number increases every year. However, some of such concepts are used more frequently by the different applications they have. For instance, the Riemann-Liouville derivative is used in real analysis, the derivative in the Caputo sense is used in physics and numerical methods, and the derivative in the Grunwald-Letnikov sense is used among others in signal processing and in automatic control. In order to show one more application of the results previously presented, we develop in this subsection a method to prove the existence and uniqueness of solutions for certain fractional differential equations in the Caputo sense. With appropriate adjustments we think that the same approach can be applied to equations involving a kernel of the type $(t-s)^{\alpha-1}$, $0<s<t$. As will be seen right away (see for example the estimation (3.16)) such kernel plays an important role. We believe that the same strategy can be applied to fractional derivatives of the type Riemann-Liouville, Chen, Davidson-Essex, Coimbra, Canavati, Jumarie, Riesz, Cossar, etc. However, for derivatives of the Caputo-Fabrizio, Atangana-Baleanu, Marchaud, etc. type substantial changes to the method presented here may be required. In other fractional derivative concepts that do not involve a kernel as in the derivatives of Grunwald-Letnikov, Weyl, Hadamard, Yang, etc. it may be preferable to resort to other schemes (see for example [2–4]).

Next, we present one more application of Theorem 1.1, namely, we will show the existence and uniqueness of solutions for certain fractional differential equations in the Caputo sense, as we have previously mentioned, the same scheme can be applied to other fractional derivative concepts. More specifically we study the initial value problem

$$\begin{cases} {}^{c}D^{\alpha}u(t) &= f(t)u(t), \quad t>0 \\ u(0) &= u_0, \end{cases} \tag{3.14}$$

where $0<\alpha\le 1$ and $f:(0,+\infty)\to(0,+\infty)$ is a measurable function. The integral representation of (3.14) is

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)u(s)ds, \quad t>0, \tag{3.15}$$

where Γ is the usual gamma function.

Before we present the uniqueness and existence result we introduce some necessary notation. Set $A=\{t>0 : f(t)>\Gamma(\alpha+1)\}$. We will denote by $t_A=\inf A$, where we introduce the convention that if $A=\emptyset$, then $\inf\emptyset=+\infty$. For each measurable functions $g:(0,+\infty)\to(0,+\infty)$ and $h:(0,+\infty)\to(0,+\infty)$, let us define

$$k(s,t) := \frac{\Gamma(\alpha)}{I(g)}\cdot\frac{g(s)}{f(st)}\cdot\frac{h(t)}{t^{\alpha}} - h(st), \quad s\in(0,1), \quad t\in(0,+\infty),$$

where

$$I(g) := \int_0^1 (1-s)^{\alpha-1} g(s)ds < +\infty.$$

We also set $B = \{t > 0 : k(s,t) < |u_0|, \text{ for all } s \in (0,1)\}$ and $t_B = \inf B$.

Proposition 3.2 *If there exist measurable functions $g : (0,+\infty) \to (0,+\infty)$ and $h : (0,+\infty) \to (0,+\infty)$, as before, such that $0 < \delta := \min\{t_A, t_B\} < +\infty$, then the equation (3.15) has a unique solution in $L((0,\delta);|u_0| + h)$.*

Proof Let us consider the application T defined on $\boldsymbol{E} = L\big((0,\delta);|u_0| + h\big)$ as

$$(Tu)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)u(s)ds, \quad t \in (0,\delta).$$

If $u \in \boldsymbol{E}$, then

$$|(Tu)(t)| \le |u_0| + \int_0^1 (1-s)^{\alpha-1} \frac{f(st)}{\Gamma(\alpha)} \big(|u_0| + h(st)\big) t^\alpha ds.$$

From the definition of t_B we have that $k(s,t) \ge |u_0|$, for each $t \le t_B$ and $s \in (0,1)$. This implies

$$|u_0| + h(st) \le \frac{\Gamma(\alpha)}{I(g)} \cdot \frac{g(s)}{f(st)} \cdot \frac{h(t)}{t^\alpha}, \quad t \in (0,t_A), \quad s \in (0,1),$$

this in turn produce the inequality

$$|(Tu)(t)| \le |u_0| + \frac{h(t)}{I(g)} \int_0^1 (1-s)^{\alpha-1} g(s)ds. \tag{3.16}$$

Thus $Tu \in \boldsymbol{E}$, therefore $T : \boldsymbol{E} \to \boldsymbol{E}$.

Let us see that $u \mapsto \|u - Tu\|$ is convex and continuous. Let $u_1, u_2 \in \boldsymbol{E}$ and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned}
&\|\lambda_1 u_1 + \lambda_2 u_2 - T\big(\lambda_1 u_1 + \lambda_2 u_2\big)\| \\
&\le \lambda_1 \int_0^\delta \left| u_1(t) - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)u_1(s)ds \right| dt \\
&\quad + \lambda_2 \int_0^\delta \left| u_2(t) - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)u_2(s)ds \right| dt \\
&= \lambda_1 \|u_1 - Tu_1\| + \lambda_2 \|u_2 - Tu_2\|.
\end{aligned}$$

To verify the continuity of $u \mapsto \|u - Tu\|$ it is sufficient to note that

$$\begin{aligned}
\|Tu_1 - Tu_2\| &\le \frac{1}{\Gamma(\alpha)} \int_0^\delta \int_0^t (t-s) f(s) \big|u_1(s) - u_2(s)\big| dsdt \\
&= \int_0^t \frac{f(s)}{\Gamma(1+\alpha)} \big|u_1(s) - u_2(s)\big| ds.
\end{aligned}$$

Now, from the definition of t_A, it follows that

$$\frac{1}{\Gamma(1+\alpha)} f(s) < 1, \quad s \in (0, t_B).$$

Thus, if $u_1 \neq u_2$, then

$$\|Tu_1 - Tu_2\| < \sup_{s\in(0,\delta)} \frac{f(s)}{\Gamma(1+\alpha)} \cdot \|u_1 - u_2\|. \tag{3.17}$$

Therefore, by Corollary 2.2 the application T has a unique fixed point, the solution of (3.15). □

If $\sup_{s\in(0,\delta)} f(s) < \Gamma(1+\alpha)$, then (3.17) implies that T is a contraction. The following example tell us that the Proposition 3.2 can be applied in cases where T is not necessarily a contraction, extending in this way the classical results.

Example Let $0 < \alpha \leq 1$. We will see that

$$\begin{cases} {}^cD^\alpha u(t) &= \left(at^\alpha + b + \frac{c}{t}\right)^{-1} u(t), \quad t > 0, \\ u(0) &= 1, \end{cases} \tag{3.18}$$

where

$$a := 2e + \frac{2e}{\Gamma(1+\alpha)}, \quad b := \frac{1}{2\Gamma(1+\alpha)}, \quad c := \frac{1}{a^{1/\alpha}(8\Gamma(1+\alpha))^{1+1/\alpha}},$$

has a unique solution in $L((0,\delta); 1 + at^\alpha + b + c/t)$, for some $\delta > 0$.

Proof Le us consider the functions $g : (0,+\infty) \to (0,+\infty)$ and $h : (0,+\infty) \to (0,+\infty)$ defined as

$$g(t) = e^t, \quad h(t) = at^\alpha + b + \frac{c}{t}.$$

The intermediate value theorem for integrals yields

$$I(g) = \int_0^1 (1-s)^{\alpha-1} e^s ds = \frac{e^{s_0}}{\alpha},$$

for some $s_0 \in (0,1)$. In this way

$$\begin{aligned} k(s,t) &= \alpha\Gamma(\alpha)\frac{e^s}{e^{s_0}} h(st) \frac{h(t)}{t^\alpha} - h(st) \\ &\geq h(st)\left\{\frac{\Gamma(1+\alpha)}{e} \cdot \frac{h(t)}{t^\alpha} - 1\right\}. \end{aligned}$$

Using that $h(t) \geq b$ and $h(t) \geq at^\alpha$, when $t > 0$, in the previous inequality

$$k(s,t) \geq b\left\{\frac{\Gamma(1+\alpha)}{e} a - 1\right\} = 1, \quad s \in (0,1), \quad t \in (0,+\infty).$$

That is to say, the set B is empty, $t_B = +\infty$. Thus

$$|(Tu)(t)| \leq 1 + at^\alpha + b + \frac{c}{t}, \quad t \in (0, +\infty).$$

On the other hand, let us note that

$$\lim_{t \downarrow 0} \frac{f(t)}{\Gamma(1+\alpha)} = 0,$$

where we remember that $f(t) = (at^\alpha + b + c/t)^{-1}$. Since

$$\frac{f(t_0)}{\Gamma(1+\alpha)} = \frac{4}{3} > 1, \quad t_0 := (8a\Gamma(1+\alpha))^{-1/\alpha},$$

then there exist a $t_1 > 0$ such that

$$\sup_{s \in (0,t_1)} \frac{f(s)}{\Gamma(1+\alpha)} = 1. \tag{3.19}$$

By the definition of A we have that $t_A = t_1$. Therefore $0 < \delta = t_1 < +\infty$, then by Proposition 3.2 the fractional equation (3.18) has a unique solution in $L((0, \delta); 1 + at^\alpha + b + c/t)$. From (3.17) and (3.19) we conclude that the corresponding application T is not a contraction, but we have a unique solution to (3.18). □

4 Proof of the type of sequential compactness

This section will be devoted to the proof of Theorem 1.2. We recall that a net (or sequence) $(f_i)_{I \in I}$ of measurable functions is uniformly integrable if

(I1) $\sup_{i \in I} \int |f_i| d\mu < +\infty$.
(I2) For all $\varepsilon > 0$ there exists $w_\varepsilon \in L(\mu)$ and $\delta > 0$, such that $\int_B w_\varepsilon < \delta$, with $B \in \mathcal{A}$, implies

$$\sup_{i \in I} \int_B |f_i| d\mu < \varepsilon.$$

If $\boldsymbol{F} = \{f_i : i \in I\}$, then this concept of uniform integrability coincides with the given previously in Sect. 2 (see for example the excellent book, [16]).

If $(X, \mathcal{A}, \mu)$ is σ-finite measure space, Vitali's Convergence Theorem state that if the sequence $(f_j)_{j \in \mathbb{N}} \subset L(\mu)$ converges in measure to some measurable function f, then the following assertions are equivalent (see [16]):

(a). $\lim_{j \to +\infty} ||f_j - f|| = 0$.
(b). $(|u_j|)_{j \in \mathbb{N}}$ is uniformly integrable.
(c). $\lim_{j \to +\infty} \int |f_j| d\mu = \int |f| d\mu$.

It is important to observe that (c) does not imply (b) for nets in general. Indeed let us see this in the following example.

Example Write $(X, \mathcal{A}) = (\mathbb{Z}, 2^{\mathbb{Z}})$ and define the measure μ as follows, $\mu(\emptyset) = 0$ and $\mu(\{j\}) = 1_{\mathbb{Z}_{<0}}(j)$, where $\mathbb{Z}_{<0} = \{n \in \mathbb{Z} : n < 0\}$ and $1_{\mathbb{Z}_{<0}}$ is the indicator function. The net $(f_i)_{i\in\mathbb{Z}}$, where $f_i(j) = i1_{\{i\}}(j)$, converges in measure to 0, $\lim_{j\to+\infty} \int |f_j| d\mu = 0$ and $\{\int |f_i| d\mu : i \in \mathbb{Z}\}$ is not bounded.

The previous example tells us that Vitaili's theorem does not hold for nets in general, fortunately the implication (b) implies (a) is true in the nest case. Since we did not find a specific reference for the proof, we include it.

Proposition 4.1 *Let $(f_i)_{i\in I}$ be a net of measurable functions that is uniformly integrable. If $(f_i)_{i\in I}$ is Cauchy in measure, then it converges in $L(\mu)$ to a real-valued measurable function, f.*

Proof We begin by proving $(f_i)_{i\in I}$ is Cauchy in $L(\mu)$. Let $\varepsilon > 0$ be arbitrary and fixed. Since $(f_i)_{i\in I}$ is uniformly integrable, there exists $w_\varepsilon \in L(\mu)$ such that

$$\int_{\{|f_i|>w_\varepsilon\}} |f_i| d\mu < \varepsilon, \quad \text{for all } i \in I.$$

The above inequality and the fact that $\{|f_j - f_k| > 2w_\varepsilon\} \subset \{|f_j| > w_\varepsilon\} \cup \{|f_k| > w_\varepsilon\}$ implies that for each $j, k \in I$,

$$\begin{aligned}
&\int_{\{|f_j-f_k|>2w_\varepsilon\}} |f_j - f_k| d\mu \\
&\leq \int_{\{|f_j-f_k|>2w_\varepsilon\}} 2\max\{|f_j|, |f_k|\} d\mu \\
&\leq 2\left(\int_{\{|f_j|>w_\varepsilon\}\cap\{|f_k|>w_\varepsilon\}} + \int_{\{|f_j|>w_\varepsilon\geq|f_k|\}} + \int_{\{|f_k|>w_\varepsilon\geq|f_j|\}} \right) \max\{|f_j|, |f_k|\} d\mu \\
&\leq 2\left(\int_{\{|f_j|>w_\varepsilon\}} |f_j| d\mu + \int_{\{|f_k|>w_\varepsilon\}} |f_k| d\mu + \int_{\{|f_j|>w_\varepsilon\}} |f_j| d\mu + \int_{\{|f_k|>w_\varepsilon\}} |f_k| d\mu \right) \\
&< 8\varepsilon.
\end{aligned}$$

On the other hand, using the Dominated Convergence Theorem we can find $r_0 > 0$ and $R_0 > 0$ such that

$$\int \min\{r, 2w_\varepsilon\} d\mu < \varepsilon, \quad \text{for all } r \leq r_0,$$

$$\int_{\{2w_\varepsilon>R\}} 2w_\varepsilon d\mu < \varepsilon, \quad \text{for all } R \geq R_0.$$

Furthermore, since $(f_i)_{i\in I}$ is Cauchy in measure, there exists $i_1 \in I$ such that

$$\mu\{|f_j - f_k| > r_0\} < \frac{\varepsilon}{R_0}, \quad \text{for all } j, k \geq i_1.$$

Accordingly, if $j, k \geq i_1$ we have

$$\begin{aligned}
&\int |f_j - f_k| d\mu \\
&= \int_{\{|f_j - f_k| > 2w_\varepsilon\}} |f_j - f_k| d\mu + \int_{\{|f_j - f_k| \leq 2w_\varepsilon\}} |f_j - f_k| d\mu \\
&\leq 8\varepsilon + \int_{\{|f_j - f_k| \leq \min\{r_0, 2w_\varepsilon\}\}} |f_j - f_k| d\mu + \int_{\{r_0 < |f_j - f_k| < 2w_\varepsilon\}} |f_j - f_k| d\mu \\
&\leq 8\varepsilon + \int \min\{r_0, 2w_\varepsilon\} d\mu + \int_{\{r_0 < |f_j - f_k|\} \cap \{R_0 < 2w_\varepsilon\}} 2w_\varepsilon d\mu \\
&\quad + \int_{\{r_0 < |f_j - f_k|\} \cap \{r_0 < 2w_\varepsilon \leq R_0\}} 2w_\varepsilon d\mu \\
&\leq 8\varepsilon + \int \min\{r_0, 2w_\varepsilon\} d\mu + \int_{\{R_0 < 2w_\varepsilon\}} 2w_\varepsilon d\mu + R_0 \mu\{|f_j - f_k| > r_0\} \\
&< 11\varepsilon.
\end{aligned}$$

In this way we have proved that $(f_i)_{i \in I}$ is Cauchy in $L(\mu)$.

We now proceed by induction. Suppose we have chosen $j_1 \leq i_1 \leq j_2 \leq i_2 \leq \cdots \leq j_n \leq i_n$ such that

$$\|f_i - f_j\| < \frac{1}{2}, \text{ for all } i, j \geq j_1, \; \cdots, \; \|f_i - f_j\| < \frac{1}{2^n}, \text{ for all } i, j \geq j_n,$$

and

$$\|f_{i_2} - f_{i_1}\| < \frac{1}{2}, \; \cdots, \; \|f_{i_n} - f_{i_{n-1}}\| < \frac{1}{2^{n-1}}.$$

Since $(f_i)_{i \in I}$ is Cauchy in $L(\mu)$, there exists $\tilde{j}_{n+1} \in I$ for which

$$\|f_i - f_j\| < \frac{1}{2^{n+1}}, \text{ for all } i, j \geq \tilde{j}_{n+1}.$$

Since I is a directed set there exists $j_{n+1} \in I$, for which $j_{n+1} \geq i_n$ and $j_{n+1} \geq \tilde{j}_{n+1}$. Let us take an $i_{n+1} \in I$, with $i_{n+1} \geq i_n$ and $i_{n+1} \geq j_{n+1}$. Inasmuch as $i_{n+1} \geq i_n \geq j_n$, we conclude

$$\|f_{i_{n+1}} - f_{i_n}\| < \frac{1}{2^n}.$$

By induction we have constructed a sequence $(f_{i_n})_{n \in \mathbb{N}}$ that is Cauchy in $L(\mu)$. Therefore $(f_{i_n})_{n \in \mathbb{N}}$ converges in $L(\mu)$ to a real-vakued measurable function f. We can now proceed to prove that the net $(f_i)_{i \in I}$ converges in $L(\mu)$ to f. For $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_{i_n} - f\| < \frac{\varepsilon}{3}, \quad \text{for all } n \geq n_0.$$

Let $m_0 \geq n_0$ for which $2^{-m_0} < \varepsilon/3$. Therefore if $i \geq j_{m_0}$, then

$$\begin{aligned}\|f_i - f\| \leq& \|f_i - f_{j_{m_0}}\| + \|f_{j_{m_0}} - f_{i_{m_0}}\| + \|f_{i_{m_0}} - f\| \\ <& \frac{1}{2^{m_0}} + \frac{1}{2^{m_0}} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned} \tag{4.1}$$

To estimate the second term in the right hand side of (4.1) we have used that $i_{m_0} \geq j_{m_0}$, which completes the proof. □

Before continuing, let us recall one more uniform integrability equivalence that we will use later (see [16]): A net (or sequence) $(f_i)_{I \in I}$ of measurable functions is uniformly integrable if

(II1) $\lim_{R \to +\infty} \sup_{i \in I} \int_{\{|f_i| > R\}} |f_i| d\mu = 0.$

(II2) For all $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{A}$, $\mu(A_\varepsilon) < +\infty$, such that

$$\sup_{i \in I} \int_{A_\varepsilon^c} |f_i| d\mu < \varepsilon.$$

We also will need the following result.

Lemma 4.2 *Let* $u : [0, +\infty) \to [0, 1)$ *be defined as*

$$u(x) = 1 - e^{-x}. \tag{4.2}$$

Let α *and* β *be positive real numbers, then for all* $x, y \geq 0$,

$$1_{[|x-y| \geq \alpha, \min\{x,y\} \leq \beta]}(x, y) \leq \frac{1}{\gamma(\alpha, \beta)} \left(u\left(\frac{x+y}{2} \right) - \frac{1}{2} \left[u(x) + u(y) \right] \right), \tag{4.3}$$

where

$$\gamma(\alpha, \beta) := \frac{1}{2} e^{-\beta} \left(1 - e^{-\frac{\alpha}{2}} \right)^2. \tag{4.4}$$

Proof See Lemma 4 in [20]. □

Lemma 4.3 *Let* $(f_i)_{i \in I}$ *be a net of non-negative measurable functions that is uniform integrable. For each* $i \in I$ *there exists* $g_i \in C_i = \text{co}\{f_j : j \geq i\}$ *such that* $(g_i)_{i \in I}$ *converges in* $L(\mu)$ *to a real-valued measurable function* g.

Proof For each $i \in I$ let us set

$$\begin{aligned} t_i &= \sup \left\{ \int u(f) d\mu : f \in C_i \right\}, \\ t_0 &= \sup \left\{ \int u(f) d\mu : f \in C_0 \right\}, \end{aligned}$$

where $C_0 = \text{co}\{f_j : j \in I\}$ and u is given in (4.2). Notice that $(t_i)_{i \in I}$ is decreasing and $0 \leq t_i \leq t_0$, this implies $t_{+\infty} = \inf_{i \in I} t_i = \lim_{i \in I} t_i$.

If $f \in C_0$ then $f = \sum_{j=1}^{m} \lambda_j f_{i_j}$, where $i_j \in I$ and $\sum_{j=1}^{m} \lambda_j = 1$, with $\lambda_j \in [0,1]$. Since $u(x) \le x$ we deduce

$$\int u(f)d\mu \le \sum_{j=1}^{m} \lambda_j \int |f_{i_j}|d\mu \le \sup_{i\in I} \int |f_i|d\mu,$$

thus $t_0 < +\infty$, to conclude this we have used the uniform integrability of $(f_i)_{i\in I}$, see **(*I*.1)**.

Let $(t_{i_n})_{n\in\mathbb{N}}$ be a sequence such that $i_{n+1} \ge i_n$, for all $n \in \mathbb{N}$, and

$$|t_{+\infty} - t_{i_n}| < \frac{1}{n}, \quad \text{for all } n \in \mathbb{N}. \tag{4.5}$$

For each $j \in I$ we have the following possibilities: $t_j > t_{i_1}$ or $t_j \le t_{i_1}$. In the first case we set $n(j) = 1$ and in the second case we take an $n(j) \in \mathbb{N}$ such that $t_{i_{n(j)+1}} \le t_j \le t_{i_{n(j)}}$. From the definition of supreme in t_j, we deduce that there exists $g_j \in C_j$ for which

$$t_j - \frac{1}{n(j)} < \int u(g_j)d\mu. \tag{4.6}$$

We will see that the net $(g_j)_{j\in I}$ has the desired properties, at the moment we just known that $g_j \in C_j$, for all $j \in I$.

First let us see that $(g_j)_{j\in I}$ is uniformly integrable. For each $j \in I$, g_j can be written as $g_j = \sum_{k=1}^{m} \lambda_k f_{i_k}$, where $i_k \ge j$ and $\sum_{k=1}^{m} \lambda_k = 1$, with $\lambda_k \in [0,1]$. Thus

$$\int_B |g_j|d\mu \le \sum_{k=1}^{m} \lambda_k \int_B |f_{i_k}|d\mu \le \sup_{i\in I} \int_B |f_i|d\mu, \tag{4.7}$$

for each $B \in \mathcal{A}$. Using (4.7) conditions **(*I*1)** and **(*I*2)** are easily verified.

We proceed to prove that $(g_j)_{j\in I}$ is Cauchy in measure. We take $\alpha > 0$ and $\varepsilon > 0$. Since $(g_j)_{j\in I}$ is uniformly integrable, we have by **(*II*1)** that there exists $R > 1$ such that

$$\sup_{j\in I} \int_{\{|g_j|>r\}} |g_j|d\mu < \frac{\varepsilon}{4}, \quad \text{for all } r \ge R. \tag{4.8}$$

On the other hand, let $n_1 \in \mathbb{N}$ such that

$$\frac{1}{n_1} < \frac{\varepsilon\gamma(\alpha, R)}{6},$$

where γ is defined in (4.4).

If $j, k \ge i_{n_1}$, since $\frac{1}{2}(g_j + g_k) \in C_{i_{n_1}}$ then

$$\int u\left(\frac{g_j + g_k}{2}\right)d\mu \le t_{i_{n_1}}. \tag{4.9}$$

From (4.3) we obtain

$$\begin{aligned}&\mu(|g_k-g_j|\geq\alpha)\\&\leq\mu(|g_k-g_j|\geq\alpha,\min(g_k,g_j)\leq R)+\mu(\min(g_k,g_j)>R)\\&\leq\frac{1}{\gamma(\alpha,R)}\left[\int u\left(\frac{g_k+g_j}{2}\right)d\mu-\frac{1}{2}\int\left(u(g_k)+u(g_j)\right)d\mu\right]\\&\quad+\mu(g_k>R)+\mu(g_j>R).\end{aligned}$$

Then, from (4.6), (4.8) and (4.9) it follows that

$$\begin{aligned}&\mu(|g_k-g_j|\geq\alpha)\\&\leq\frac{1}{\gamma(\alpha,R)}\left[\left|t_{i_{n_1}}-t_{+\infty}\right|+\left|\frac{1}{2}t_{+\infty}-\frac{1}{2}t_{i_{n(j)+1}}\right|+\left|\frac{1}{2}t_{+\infty}-\frac{1}{2}t_{i_{n(k)+1}}\right|+\frac{1}{n_1}\right]\\&\quad+\int_{\{g_k>R\}}d\mu+\int_{\{g_j>R\}}d\mu\\&\leq\frac{1}{\gamma(\alpha,R)}\left[\frac{1}{n_1}+\frac{1}{2(n(j)+1)}+\frac{1}{2(n(k)+1)}+\frac{1}{n_1}\right]\\&\quad+\int_{\{g_k>R\}}Rd\mu+\int_{\{g_j>R\}}Rd\mu\\&\leq\frac{1}{\gamma(\alpha,R)}\cdot\frac{3}{n_1}+\int_{\{|g_k|>R\}}|g_k|d\mu+\int_{\{|g_j|>R\}}|g_j|d\mu\\&<\varepsilon.\end{aligned}$$

Therefore $(g_i)_{i\in I}$ is Cauchy in measure that is uniformly integrable. The result follows from Proposition 4.1. □

We are now in position to address the net result in its full generality.

Proof of Theorem 1.2 Let us first consider the net $(f_i^+)_{i\in I}$, where $f_i^+=\sup\{f_i,0\}$. It is easily seen that $(f_i^+)_{i\in I}$ is uniformly integrable. Applying Lemma 4.3 to $(f_i^+)_{i\in I}$ we have that for each $j\in I$ there exists $\hat{g}_j\in\mathrm{co}\{f_i^+ : i\geq j\}$ such that $(\hat{g}_j)_{j\in I}$ converges in $L(\mu)$ to a real-valued measurable function g^+. Thus for each $j\in I$,

$$\hat{g}_j=\sum_{k=1}^{n_j}\lambda_k^j f_{i_{k(j)}}^+,$$

where $\lambda_k^j\in[0,1]$, $\sum_{k=1}^{n_j}\lambda_k^j=1$ and $i_{k(j)}\geq j$.

We consider the function

$$\tilde{g}_j:=\sum_{k=1}^{n_j}\lambda_k^j f_{i_{k(j)}}^-,$$

where $f_i^-=\sup\{-f_i,0\}$. It is easy to check that $(f_i^-)_{i\in I}$ is uniformly integrable. Using a similar argument as in (4.7) we can prove that $(\tilde{g}_j)_{j\in I}$ is uniformly integrable. Then, applying Lemma 4.3 to the net $(\tilde{g}_j)_{j\in I}$, we have that for each $r\in I$ there exists $g_r^-\in\mathrm{co}\{\tilde{g}_j : j\geq r\}$ such that $(g_r^-)_{r\in I}$ converges in $L(\mu)$ to a real-valued measurable function g^-. Thus for each $r\in I$, g_r^- can be written as

$$g_r^- = \sum_{m=1}^{n_r} \gamma_m^r \tilde{g}_{j_{m(r)}} = \sum_{m=1}^{n_r} \gamma_m^r \sum_{k=1}^{n_{j_{m(r)}}} \lambda_k^{j_{m(r)}} f^-_{i_{k(j_{m(r)})}},$$

where $\gamma_m^r \in [0,1]$, $\sum_{k=1}^{n_j} \gamma_m^r = 1$ and $j_{m(r)} \geq r$.

Let us consider the function

$$g_r^+ := \sum_{m=1}^{n_r} \gamma_m^r \hat{g}_{j_{m(r)}} = \sum_{m=1}^{n_r} \gamma_m^r \sum_{k=1}^{n_{j_{m(r)}}} \lambda_k^{j_{m(r)}} f^+_{i_{k(j_{m(r)})}}.$$

We now proceed to verify that $(g_r^+)_{r\in I}$ converges to g^+ in $L(\mu)$. Given $\varepsilon > 0$, since $(\hat{g}_j)_{j\in I}$ converges to g^+ in $L(\mu)$, there exists $i_0 \in I$ such that

$$\|\hat{g}_j - g^+\| < \varepsilon, \quad \text{for all } j \geq i_0.$$

If $r \geq i_0$, then $j_{m(r)} \geq r$ implies

$$\|g_r^+ - g^+\| \leq \sum_{m=1}^{n_r} \gamma_m^r \|\hat{g}_{j_{m(r)}} - g^+\| < \varepsilon.$$

Finally, let us set

$$\begin{aligned} g_r &:= g_r^+ - g_r^- \\ &= \sum_{m=1}^{n_r} \gamma_m^r \sum_{k=1}^{n_{j_{m(r)}}} \lambda_k^{j_{m(r)}} \left(f^+_{i_{k(j_{m(r)})}} - f^-_{i_{k(j_{m(r)})}} \right) \\ &= \sum_{m=1}^{n_r} \gamma_m^r \sum_{k=1}^{n_{j_{m(r)}}} \lambda_k^{j_{m(r)}} f_{i_{k(j_{m(r)})}}. \end{aligned}$$

From this we see that $g_r \in C_r = \text{co}\{f_i : i \geq r\}$ and it is clear that $(g_r)_{r\in I}$ converges in $L(\mu)$ to $g^+ - g^- =: g$. □

Acknowledgements We would like to thank the reviewers for their careful reading of the manuscript, their comments greatly improved the paper. The author JVM was partially supported by the grant PIM22-1 of Universidad Autónoma de Aguascalientes.

References

1. Abtahil, M., Kadelburg, Z., Radenović, S.: Fixed points of Ćirić-Matkowski-type contractions in v -generalized metric spaces. RACSAM (2017). https://doi.org/10.1007/s13398-016-0275-5
2. Adigüzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Meth. Appl. Sci. (2020). https://doi.org/10.1002/mma.665
3. Adigüzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. RACSAM (2021). https://doi.org/10.1007/s13398-021-01095-3
4. Adigüzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via geraghty type hybrid contractions. Appl. Comput. Math. **20**, 313–333 (2021)

5. Carl, S., Heikkilä, S.: Fixed point theory in ordered sets and applications: From differential and integral equations to game theory. Springer, Berlin (2011)
6. Hernández-Hernández, D., Villa-Morales, J.: On the minimax theorem for the space of probability measures on metric spaces. Banach Publications (2020)
7. Jleli, M., Samet, B.: A generalized metric space and related fixed point theorems. J. Fixed Point Theory Appl. (2015). https://doi.org/10.1186/s13663-015-0312-7
8. Karapinar, E.: Edelstein type fixed point theorems. Fixed Point Theory Appl. (2012). https://doi.org/10.1186/1687-1812-2012-107
9. Karapinar, E., Taş, K.: Generalized (C) -conditions and related fixed point theorems. Comp. Math. App. **61**, 3370–3380 (2011)
10. Khamsi, M.A.: Generalized metrics paces: Asurvey. J. Fixed Point Theory Appl. **17**, 455–475 (2015)
11. Paesano, D., Vetro, P.: Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces. Topol. Appl. **159**, 911–920 (2012)
12. Pratelli, M.: A minimax theorem without compactness hypothesis. Mediterr. J. Math. **2**, 103–112 (2005)
13. Samet, B.: Discussion on A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari. Publ. Math. Debrecen **76**(4), 493–494 (2010)
14. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for $\alpha - \psi$ -contractive type mappings. Nonlinear Anal. Theory Methods Appl. **7**, 2154–2165 (2012)
15. Sarma, I.R., Rao, J.M., Rao, S.S.: Contractions over generalized metric spaces. J. Nonlinear Sci. Appl. **2**, 180–182 (2009)
16. Schilling, R.L.: Measures, integrals, and martingales. Cambridge University Press, Cambridge (2005)
17. Shahzad, N., Valero, O.: Nemytskii-Edelstein type fixed point theorem for partial metric spaces. Fixed Point Theory Appl. (2015). https://doi.org/10.1186/s13663-015-0266-9
18. Villa-Morales, J.: A fixed point theorem and some properties of v-generalized metric spaces. J. Fixed Point Theory Appl. (2018). https://doi.org/10.1007/s11784-018-0485-x
19. Villa-Morales, J.: Subordinate Semimetric Spaces and Fixed Point Theorems. J. Math. (2018). https://doi.org/10.1155/2018/7856594
20. Villa-Morales, J.: An extreme value theorem on closed convex subsets of L^p spaces and two applications. J. Math. Anal. Appl. (2020). https://doi.org/10.1016/j.jmaa.2020.124250
21. Villa-Morales, J.: *Dominated weak solutions for a second order ordinary differential equation with boundary conditions in*$(0, +\infty)$, to appear in Mediterr. J. Math. (2022)

2 Aspectos del modelo de epidemia estocástica SIS

El modelo SIS (Susceptible-Infeccioso-Susceptible) es un modelo matemático clásico de epidemiología utilizado para describir y comprender la propagación de enfermedades infecciosas en una población. En este modelo, los individuos pueden clasificarse en dos clases: susceptibles (S), que representa a los que pueden contraer la enfermedad, e infecciosos (I), que representa a los que están infectados y pueden transmitir la enfermedad a otros. El modelo SIS sirve de marco fundamental para comprender la dinámica de las enfermedades infecciosas en una población. Constituye una herramienta crucial en epidemiología para predecir la propagación de enfermedades y evaluar las intervenciones de salud pública, como la vacunación y el distanciamiento social (por ejemplo, véase Mollison [21] y las referencias citadas en él).

Se denota por S_t y I_t el número de individuos susceptibles e infectados en el tiempo t, respectivamente. Un hecho importante que se supone es que las tasas de natalidad y de mortalidad de la población son las mismas, lo que conlleva a un tamaño constante de la población N. Simbólicamente, la dinámica del modelo epidémico SIS se describe mediante el siguiente sistema de ecuaciones diferenciales:

$$\begin{aligned}\frac{dS_t}{dt} &= \left(\tilde{b}+\gamma\right) I_t - \frac{\beta}{N} S_t I_t,\\ \frac{dI_t}{dt} &= \frac{\beta}{N} S_t I_t - \left(\tilde{b}+\gamma\right) I_t.\end{aligned}$$

En donde $\tilde{b} \geq 0$ representa la tasa a la que se recupera una persona infectada y $\gamma > 0$ es la tasa de mortalidad. Se introduce ruido blanco al término crucial $\beta \in \mathbb{R}$, el cual está asociado a la transmisión de la enfermedad. Este método de introducir aleatoriedad en un sistema es ampliamente reconocido y se denomina perturbación estocástica de parámetros (para más información, consulte [3]).

Partiendo del supuesto de una población constante en el tiempo, en lugar de considerar un sistema de ecuaciones diferenciales, es posible trabajar solo una ecuación diferencial. Con este enfoque, al introducir la perturbación estocástica a uno de los parámetros, se aborda una ecuación diferencial estocástica correspondiente al número de individuos infectados.

Existen numerosas contribuciones significativas sobre el modelo SIS estocástico (véanse [12] y [9]). La mayoría de estas contribuciones se centran en investigar la persistencia o extinción de la enfermedad. Otras exploran temas como la existencia de distribuciones estacionarias de la infección. En el caso presentado en este trabajo, se profundiza en el comportamiento asintótico de la enfermedad. Específicamente, se demuestra que si el número de reproducción estocástica es mayor o igual que uno, la enfermedad es recurrente. Por el contrario, si es menor que uno, la enfermedad se desvanece en el infinito. Este resultado responde a una conjetura planteada en [9].

En situaciones en las que el número de reproducción estocástica es superior a uno, la duración del primer tiempo de llegada a un nivel superior al inicial resulta ser infinita. Sin embargo, condicionando el análisis a la ocurrencia de un tiempo finito, es posible calcular el tiempo esperado para este primer paso. El reto para alcanzar este resultado reside en derivar una solución explícita para la transformada de Laplace del primer momento, que se expresa en términos de funciones monótonas convenientes. Además, durante el cálculo numérico del tiempo condicional esperado, se presentan múltiples integrales con integrandos singulares. Estas complejidades dificultan el uso de técnicas de integración convencionales o de software numérico estándar para este fin. En consecuencia, se presenta un método numérico que emplea series de Taylor para abordar estos desafíos.

Por el contrario, en el campo de los modelos estocásticos, no se suelen presentar criterios específicos para determinar el valor más conveniente del parámetro estocástico (véase, por ejemplo, [2], [37], [51]). El enfoque en el que se basa este trabajo es considerar tanto el valor esperado del primer momento de pasada como el valor correspondiente derivado del modelo determinista. Se lleva a cabo este esquema bajo el supuesto de que el modelo determinista encapsula el comportamiento medio del fenómeno en estudio. Se muestran tres modelos basados en datos obtenidos en circunstancias reales, en concreto, la infección por gonorrea, la infección por neumococo y el peso de los peces guppy. Una observación digna de mención es que el modelo resultante tiene una mayor generalidad que el modelo logístico, lo que hace que los resultados presentados en este trabajo puedan aplicarse en escenarios en los que dicho modelo no es aplicable. Esta afirmación se ve respaldada por la aplicación del modelo al peso de los peces guppy.

2.2. Artículo

Applied Mathematical Modelling 128 (2024) 272–286

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Certain aspects of the SIS stochastic epidemic model

Gabriela de Jesús Cabral-García, José Villa-Morales *

Universidad Autónoma de Aguascalientes, Departamento de Matemáticas y Física, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico

ARTICLE INFO

MSC:
65L12
65L05
65L70

Keywords:
First moment
Asymptotic behavior
Some real-world SIS epidemic models

ABSTRACT

The SIS model is a fundamental tool in epidemiology for understanding the spread of infectious diseases. This article focuses on the stochastic SIS model, introducing randomness through the disease transmission parameter. The study investigates the behavior of the model, revealing the conditions for the recurrence or extinction of the disease. In particular, it addresses the calculation of the conditional expected time for the disease to exceed a certain threshold, using both Laplace transforms and numerical techniques for its specific application. Real-world phenomena are discussed, and a method for determining the most suitable stochastic parameter is proposed, with examples such as gonorrhea and pneumococcus.

1. Introduction

The SIS (Susceptible-Infectious-Susceptible) model is a classic mathematical epidemiology model used to describe and understand the spread of infectious diseases within a population. In this model, individuals can be categorized into two states: susceptible (S), representing those who can contract the disease, and infectious (I), representing those who are infected and can transmit the disease to others. The SIS model serves as a foundational framework for comprehending the dynamics of infectious diseases within a population. It stands as a crucial tool in epidemiology for predicting disease dissemination and assessing public health interventions, such as vaccination and social distancing (for instance, refer to Mollison [1] and the related references cited therein).

Let us denote by S_t and I_t the numbers of susceptible and infected individuals at time t, respectively. We assume that the birth and death rates in the population are the same, leading to a constant population size denoted as N over time. Symbolically, the dynamics of the SIS epidemic model are described by the following system of differential equations:

$$\frac{dS_t}{dt} = (\tilde{b}+\gamma)I_t - \frac{\beta}{N}S_t \cdot I_t, \tag{1.1}$$

$$\frac{dI_t}{dt} = \frac{\beta}{N}S_t \cdot I_t - (\tilde{b}+\gamma)I_t. \tag{1.2}$$

Here, $\tilde{b} \geq 0$ represents the rate at which an infected person recovers, and $\gamma > 0$ is the death rate. We will introduce randomness to the crucial term $\beta \in \mathbb{R}$, which is associated with disease transmission. This method of introducing randomness into a system is widely recognized and referred to as stochastic parameter perturbation (refer to [2] for more information).

Using the assumption of a constant population over time, instead of considering a system of differential equations, we focus on a single differential equation. In this approach, by introducing stochastic parameter perturbation, we address a single stochastic differential equation corresponding to the number of infected individuals (referred to as equation (2.2)).

* Corresponding author.
E-mail addresses: gabriela.cabral@edu.uaa.mx (G. de Jesús Cabral-García), jvilla@correo.uaa.mx (J. Villa-Morales).

https://doi.org/10.1016/j.apm.2024.01.027
Received 23 August 2023; Received in revised form 10 January 2024; Accepted 16 January 2024

Available online 22 January 2024

Numerous significant contributions exist regarding the stochastic SIS model (refer to [3] and [4]). Most of these contributions center around investigating the persistence or extinction of the disease. Others explore topics such as the existence of stationary infection distributions. In our case, we delve into the asymptotic behavior of the disease. Specifically, we demonstrate that if the stochastic reproduction number is greater than or equal to one, the disease is recurrent. Conversely, if it is less than one, the disease fades out in the infinite horizon (refer to Theorem 5.2 and Theorem 5.3). This result addresses a conjecture posed in [4], as elaborated in the following section.

In situations where the stochastic reproduction number exceeds one, the duration of the first arrival time at a level higher than the initial level turns out to be infinite. However, by conditioning our analysis to the occurrence of a finite time, we can calculate the expected time for this first step. The challenge in achieving this result lies in deriving an explicit solution for the first arrival time Laplace transform, which is expressed in terms of convenient monotone functions (as presented in the formula (4.9)). Furthermore, during the numerical computation of the conditional expected time for the first step, we find multiple integrals with singular integrands. These complexities make it difficult to use conventional integration techniques or standard numerical software for this purpose. Consequently, in Section 6, we present a numerical method that employs the Taylor series to address these challenges.

On the contrary, in the field of stochastic models, specific criteria are not commonly presented to determine the most convenient value of the stochastic parameter (see for example, [5–7]). In our approach, we rely on both the conditional expected value of the first step time and the corresponding value derived from the deterministic model. We undertake this scheme under the assumption that the deterministic model encapsulates the average behavior of the phenomenon under study. We show three models based on real world events, specifically, gonorrhea infection, pneumococcus infection, and the weight of guppy fish. One noteworthy observation is that the resulting model has a higher generality than the logistic model (see equation (2.4)), which makes our findings applicable in scenarios where the aforementioned model is employed. This claim is supported by our application of the model to the weight of guppy fish.

The article is organized in the following manner. In Section 2, we present the stochastic SIS model together with the fundamental concepts. Section 3 is dedicated to the calculation of various functionals, whose values will be used later. In Section 5, we deepen the examination of the asymptotic behavior of the stochastic model. The issue of the conditional expected value of the first passing time is addressed in Section 4. The specific integral expressions essential to calculate the first conditional expected step time are given in Section 6. Finally, Section 7 involves a discussion of three examples of real life phenomena that are modeled using the stochastic SIS model.

2. The stochastic SIS model and some preliminaries facts

If $\beta \geq 0$, the quantity $(\beta/N)S_t \cdot I_t\,dt$ in equations (1.1)-(1.2) can be interpreted as the total number of new infections occurring in the interval $[t, t+dt)$. Conversely, when $\beta < 0$, $(\beta/N)S_t \cdot I_t\,dt$ represents the total number of new recoveries within the same interval. In practical scenarios, the parameter β is not necessarily constant and can exhibit random fluctuations. In such cases, we assume that β has a mean value r and is subject to noise with a magnitude of $\sigma > 0$, yielding the expression

$$\beta = r + \sigma W_t, \tag{2.1}$$

where $W = \left\{W_t : t \geq 0\right\}$ represents white noise (see [8]).

Upon applying this parameter transformation in equations (1.1)-(1.2) and utilizing the relation

$$N = S_t + I_t, \quad t \geq 0,$$

we arrive at the stochastic differential equation:

$$dI_t = \left(a - bI_t\right) I_t\,dt + c\left(N - I_t\right) I_t\,dB_t, \tag{2.2}$$

where

$$a = r - (\tilde{b} + \gamma), \quad b = \frac{r}{N} \quad \text{and} \quad c = \frac{\sigma}{N}. \tag{2.3}$$

Here, the term $dB_t = W_t\,dt$ is understood as a stochastic differential with respect to the Brownian motion $B = \left\{B_t : t \geq 0\right\}$, defined within a complete probability space $(\Omega, \mathcal{F}, P)$, as described in [9] or [10].

In the deterministic case, when $\sigma = 0$, the corresponding ordinary differential equation is given by

$$dI_t = \left(a - bI_t\right) I_t\,dt. \tag{2.4}$$

The parameters a and b in equation (2.4) can be interpreted as measures of the environmental influence, where the former promotes the phenomenon under study while the latter inhibits it.

In the deterministic model, when studying an infection, the parameter

$$R_{de} = \frac{Nb}{Nb - a}$$

is referred to as the basic reproduction number of the infection. If $I_0 = i_0 \in (0, N)$, then the solution of the ordinary differential equation (2.4) is given by

$$I_t = \begin{cases} \frac{a}{b+\left(\frac{a}{i_0}-b\right)\exp(-at)}, & \text{if } a \neq 0, \\ \left(bt - \frac{1}{i_0}\right)^{-1}, & \text{if } a = 0. \end{cases} \tag{2.5}$$

The asymptotic behavior of the deterministic solution I is as follows

$$\lim_{t\to\infty} I_t = \begin{cases} \frac{a}{b}, & \text{if } a > 0, \\ 0, & \text{if } a \leq 0. \end{cases} \tag{2.6}$$

When $N > a/b$, we observe that $R_{de} > 1$ if and only if $a > 0$, thereby making the basic reproduction number R_{de} instrumental in determining the stable states of the disease. An important observation is: if we take a/b as the carrying capacity of the system (or environment) and a as its quality, the logistic model emerges as a special case. This remark empowers us to model phenomena where I does not necessarily signify an infection. In fact, as detailed in Section 7, we will illustrate this using the example of guppy fish weight, highlighting the potential of our approach.

In Theorem 3.1 of [4], it is demonstrated that if $I_0 = i_0 \in (0, N)$, the stochastic differential equation (2.2) possesses a unique global solution denoted as $I = \{I_t : t \geq 0\}$. Furthermore,

$$P[I_t \in (0, N) \text{ for all } t \geq 0] = 1.$$

In other words, if the process I commences at a point within the interval $(0, N)$, then it almost surely remains within this interval. This fact will be a particular case from Lemma 5.1, which is established through the utilization of Laplace transform techniques. Moreover, we will see in (5.8) that the infection remains below a level that is strictly less than N.

In the stochastic scenario, the reproduction number is given by

$$R_{st} := R_{de} - \frac{\sigma^2}{2(\tilde{b}+\gamma)}. \tag{2.7}$$

If $c \neq 0$, we define

$$\beta_1 = \frac{2a}{c^2 N^2},$$

noting that $R_{st} < 1$ if and only if $\beta_1 < 1$.

In accordance with our notation, we have the following result.

Theorem 2.1. *The disease dies out with probability one if*

$$\beta_1 < 1 \text{ and } \sigma^2 \leq r \quad \text{or} \quad \sigma^2 > \max\left\{r, \frac{r^2}{2(\tilde{b}+\gamma)}\right\}.$$

Moreover, if $\beta_1 > 1$, then the disease will almost surely persist indefinitely (there is recurrence).

Proof. The proof of this result is contained in Theorems 4.1, 4.3 and 5.1 of [4]. □

The authors [4] conjectured that if

$$\beta_1 < 1 \quad \text{and} \quad r \leq \sigma^2 \leq \frac{r^2}{2(\tilde{b}+\gamma)},$$

then the disease will die out with probability one. We will prove in Theorem 5.3 that this is certainly the case. The critical scenario $\beta_1 = 1$ is not investigated in [4]. However, this particular case is also encompassed by Theorem 5.3.

If $i \in [0, N]$, by T_i we will denote the first passage time of the level i. In symbols we express this as

$$T_i = \begin{cases} \infty, & \text{if } I_t \neq i, \text{ for all } t \geq 0, \\ \inf\{t \geq 0 : I_t = i\}, & \text{otherwise.} \end{cases} \tag{2.8}$$

Such time is the first time the stochastic process I hits the level i when it starts at i_0. With this notation we define the following conditional expected value

$$E[T_i | T_i < \infty] := \frac{E[T_i; T_i < \infty]}{P[T_i < \infty]} = \frac{1}{P[T_i < \infty]} \int_{[T_i<\infty]} T_i \, dP, \tag{2.9}$$

where $[T_i < \infty] = \{\omega \in \Omega : T_i(\omega) < \infty\}$.

3. Some basic functionals

Next, we will introduce several well-known functions from the theory of diffusion processes. Evaluating these functions at the boundaries, specifically at 0 and N, of the variable I, will provide us with valuable insights into the trajectory behavior of the process.

Take an arbitrary fixed point $i_0 \in (0, N)$ and let a, b and c as in (2.3). We set

$$s(x) = \exp\left\{ -\int_{i_0}^{x} \frac{2f(z)}{g(z)} dz \right\}, \quad x \in (0, N),$$

where

$$f(x) = (a - bx)x \quad \text{and} \quad g(x) = c^2 (N - x)^2 x^2. \tag{3.1}$$

We also introduce the function

$$m(x) = \frac{2}{s(x) g(x)}, \quad x \in (0, N). \tag{3.2}$$

From (2.3) we deduce the following fact that will be used several times

$$d := \frac{a}{N} - b = -\frac{\tilde{b} + \gamma}{N} < 0. \tag{3.3}$$

Lemma 3.1. *If $x \in (0, N)$, then*

$$s(x) = \beta_2 \cdot \left(\frac{N - x}{x} \right)^{\beta_1} \cdot \exp\left\{ \frac{\beta_3}{N - x} \right\}, \tag{3.4}$$

where

$$\beta_2 = \left(\frac{i_0}{N - i_0} \right)^{\beta_1} \exp\left\{ -\frac{\beta_3}{N - i_0} \right\} \quad \textit{and} \quad \beta_3 = -\frac{2d}{c^2}. \tag{3.5}$$

Proof. First we will assume that $i_0 \le x < N$. From (3.1) we have

$$\frac{2f(z)}{g(z)} = \frac{2}{c^2} \left\{ \frac{a}{N^2} \cdot \frac{1}{z} + \frac{a}{N^2} \cdot \frac{1}{N - z} + \left(\frac{a}{N} - b \right) \cdot \frac{1}{(N - z)^2} \right\},$$

this yields

$$\int_{i_0}^{x} \frac{f(z)}{g(z)} dz = \frac{2}{c^2} \left\{ \frac{a}{N^2} \ln \frac{x}{N - x} + \left(\frac{a}{N} - b \right) \frac{1}{N - x} - \frac{a}{N^2} \ln \frac{i_0}{N - i_0} - \left(\frac{a}{N} - b \right) \frac{1}{N - i_0} \right\},$$

from this we obtain (3.4). The case $0 < x < i_0$ can be worked in a similar way. □

Let us set the function

$$S(x) = \int_{i_0}^{x} s(y) dy, \quad x \in (0, N). \tag{3.6}$$

For the boundary point 0 we define

$$S(0) = \int_{i_0}^{0} s(x) dx,$$

$$\Sigma(0) = \int_{0}^{i_0} \left(\int_{0}^{y} s(z) dz \right) m(y) dy.$$

Similar definitions of $S(N)$ and $\Sigma(N)$, for the boundary point N, can be conceived (see [11]). Note that the continuity of the functions s and m implies that the convergence of the previous integrals does not depend on the point i_0.

Lemma 3.2. *Using the above notation we have, for the boundary point* 0*,*

$$S(0) > -\infty \quad \textit{if and only if} \quad \beta_1 < 1,$$

$$\Sigma(0) = \infty,$$

and, for the boundary point N*,*

$$S(N) = \infty \quad \textit{and} \quad \Sigma(N) = \infty.$$

Proof. In what follows we consider each of the cases.

$S(0)$: From (3.4) we have

$$-S(0) = \beta_2 \int_0^{i_0} \left(\frac{N}{x} - 1\right)^{\beta_1} \exp\left\{\frac{\beta_3}{N-x}\right\} dx.$$

Using Landau's notation (asymptotic order of convergence), we have

$$\left(\frac{N}{x} - 1\right)^{\beta_1} \exp\left\{\frac{\beta_3}{N-x}\right\} = O\left(x^{-\beta_1}\right), \quad \text{as} \quad x \downarrow 0.$$

Thus, $S(0) > -\infty$ if and only if $1 - \beta_1 > 0$.

$\Sigma(0)$: From (3.2) and (3.4) we assert that

$$\Sigma(0) = \frac{2}{c^2} \int_0^{i_0} \frac{y^{\beta_1 - 2}}{(N-y)^{\beta_1+2}} \exp\left\{-\frac{\beta_3}{N-y}\right\} \int_0^{y} \left(\frac{N}{z} - 1\right)^{\beta_1} \exp\left\{\frac{\beta_3}{N-z}\right\} dz dy.$$

The second integral, in the previous formula, is finite if and only if $\beta_1 < 1$. Thus, if $\beta_1 \geq 1$, then $\Sigma(0) = \infty$. Now let us suppose that $\beta_1 < 1$, then

$$\int_0^{y} \left(\frac{N}{z} - 1\right)^{\beta_1} \exp\left\{\frac{\beta_3}{N-z}\right\} dz = O\left(y^{1-\beta_2}\right), \quad \text{as} \quad y \downarrow 0,$$

consequently the convergence of $\Sigma(0)$ depends on the behavior of

$$\int_0^{i_0} \frac{y^{\beta_1 - 2}}{(N-y)^{\beta_1+2}} \exp\left\{-\frac{\beta_3}{N-y}\right\} y^{1-\beta_1} dy.$$

Since the order of the integrand is $O(y^{-1})$, as $y \downarrow 0$, then $\Sigma(0) = \infty$.

Now let us deal with the boundary point N.

$S(N)$: Considering (3.4) we get

$$S(N) \geq \beta_2 \min_{i_0 \leq x \leq N} \{x^{-\beta_1}\} \int_{i_0}^{N} (N-x)^{\beta_1} \exp\left\{\frac{\beta_3}{N-x}\right\} dx.$$

By (3.3) we know that $\beta_3 > 0$, thus $S(N) = \infty$.

$\Sigma(N)$: Inasmuch as $S(N) = \infty$, then

$$\Sigma(N) = \int_{i_0}^{N} \left(\int_{y}^{N} s(z) dz \right) m(y) dy = \infty.$$

In this way, all the required inequalities are verified. □

4. Conditional moments of the first passage time of the SIS epidemic model

As before, let us take some fixed arbitrary point $i_0 \in (0, N)$ and let us consider a parameter $\lambda > 0$. We begin by finding some monotone solutions of the ordinary differential equation

$$\frac{g(x)}{2} \cdot \frac{d^2}{dx^2}u(x) + f(x) \cdot \frac{d}{dx}u(x) - \lambda \cdot u(x) = 0, \quad x \in (0, N). \tag{4.1}$$

Let $\{u_n\}_{n=0}^{\infty}$ be a sequence of real-valued functions defined on $(0, N)$ as $u_0(x) = 1$ and

$$u_n(x) = \int_{i_0}^{x} s(y) \int_{i_0}^{y} u_{n-1}(z)m(z)dz\,dy, \tag{4.2}$$

for all n in $\mathbb{N}$.

Lemma 4.1. *For each $\lambda > 0$, the series*

$$u(\lambda, s) = \sum_{n=0}^{\infty} \lambda^n u_n(x), \quad x \in (0, N), \tag{4.3}$$

converges uniformly on compact subsets of $(0, N)$ and $u(\lambda, \cdot)$ satisfies the differential equation (4.1). For each $\lambda > 0$ and $x \in (0, N)$,

$$1 + \lambda u_1(x) \le u(\lambda, x) \le e^{\lambda u_1(x)}, \tag{4.4}$$

$$u_1(x) \le \frac{\partial}{\partial \lambda}u(\lambda, x) \le u_1(x)e^{\lambda u_1(x)}. \tag{4.5}$$

Proof. A proof can be consulted in [9], except for inequality (4.5) which can be verified by direct calculations. □

In what follows we will need only the first derivative of $u(\lambda, x)$ with respect to λ, but we can see that $u(\cdot, x)$ is in $C^\infty(0, \infty)$. This observation is useful if we want to calculate more conditional moments of the stopping time T_i, see Theorem 4.4.

Lemma 4.2. *Let $\lambda > 0$ be fixed. For $x \in (0, N)$ we set*

$$u^-(\lambda, x) := u(\lambda, x) \cdot \int_0^x \frac{s(y)}{u(\lambda, y)^2}dy, \tag{4.6}$$

$$u^+(\lambda, x) := u(\lambda, x) \cdot \int_x^N \frac{s(y)}{u(\lambda, y)^2}dy. \tag{4.7}$$

The functions u^- and u^+ are monotone increasing and decreasing, respectively. Moreover, they are solutions of (4.1). For each $\lambda > 0$ we also have

$$\frac{\partial}{\partial \lambda}u^-(\lambda, x) = \int_0^x \frac{s(y)}{u(\lambda, y)^2}dy \cdot \frac{\partial}{\partial \lambda}u(\lambda, x) - 2u(\lambda, x) \cdot \int_0^x \frac{s(y)}{u(\lambda, y)^3}\frac{\partial}{\partial \lambda}u(\lambda, y)dy. \tag{4.8}$$

Proof. The verification that u^- and u^+ are solutions of (4.1) is immediate, the same is for formula (4.8). The monotony of u^- and u^+ follows from the study of the sign of the derivatives, $\frac{\partial}{\partial x}u^+$, $\frac{\partial}{\partial x}u^-$, see for example [12] or [13]. □

Let I be the unique solution of (2.2) that starts at $i_0 \in (0, N)$. We will find the Laplace transform of the first time the process I hit the level $i \in (0, N)$, T_i.

Proposition 4.3. *Let $i_0 \in (0, N)$ and $I_0 = i_0$. For each $\lambda > 0$,*

$$E\left[e^{-\lambda T_i}; T_i < \infty\right] = \begin{cases} \frac{u^-(\lambda, i_0)}{u^-(\lambda, i)}, & \text{if} \quad i \in (i_0, N), \\ \frac{u^+(\lambda, i_0)}{u^+(\lambda, i)}, & \text{if} \quad i \in (0, i_0). \end{cases} \tag{4.9}$$

Proof. Let us suppose that $i_0 < i < N$. Using Itô's formula in $u^-(\lambda, I_{t\wedge T_i})e^{-\lambda(t\wedge T_i)}$ and that u^- is a monotone solution of (4.1) we obtain (for details see, for example, [13])

$$u^-(\lambda, i_0) = E\left[u^-(\lambda, I_{t\wedge T_i})e^{-\lambda(t\wedge T_i)}\right].$$

Letting $t \to \infty$, the dominated convergence theorem implies

$$u^-(\lambda, i_0) = u^-(\lambda, i)E\left[e^{-\lambda T_i}; T_i < \infty\right].$$

The case $0 < i < i_0$ can be worked similarly. □

We use the previous result to obtain the first conditional moment of T_i.

Theorem 4.4. *Let $i_0, i \in (0, N)$ with $i_0 < i$. If $\beta_1 < 1$ then*

$$E\left[T_i | T_i < \infty\right] = u_1(i) - u_1(i_0) + 2\frac{\int_0^{i_0} s(x)u_1(x)dx}{\int_0^{i_0} s(x)dx} - 2\frac{\int_0^{i} s(x)u_1(x)dx}{\int_0^{i} s(x)dx}.$$

Proof. Proposition 4.3 yields

$$\begin{aligned} P\{T_i < \infty\} &= \lim_{\lambda\downarrow 0} E\left[e^{-\lambda T_i}; T_i < \infty\right] \\ &= \lim_{\lambda\downarrow 0} \frac{u^-(\lambda, i_0)}{u^-(\lambda, i)}, \\ E\left[T_i; T_i < \infty\right] &= -\lim_{\lambda\downarrow 0} \frac{\partial}{\partial\lambda} E\left[e^{-\lambda T_i}; T_i < \infty\right] \\ &= -\lim_{\lambda\downarrow 0}\left\{\frac{1}{u^-(\lambda, i)}\cdot\frac{\partial}{\partial\lambda}u^-(\lambda, i_0) - \frac{u^-(\lambda, i_0)}{(u^-(\lambda, i))^2}\cdot\frac{\partial}{\partial\lambda}u^-(\lambda, i)\right\}. \end{aligned} \tag{4.10}$$

From (4.3) and (4.5) we get

$$\lim_{\lambda\downarrow 0} u(\lambda, x) = 1 \quad \text{and} \quad \lim_{\lambda\downarrow 0}\frac{\partial}{\partial\lambda}u(\lambda, x) = u_1(x).$$

Using the above limits, (4.6), (4.8) and Lemma 3.2 we deduce

$$\lim_{\lambda\downarrow 0} u^-(\lambda, x) = \int_0^x s(y)dy,$$

$$\lim_{\lambda\downarrow 0}\frac{\partial}{\partial\lambda}u^-(\lambda, x) = \int_0^x s(y)dy\, u_1(x) - 2\int_0^x s(y)u_1(y)dy.$$

Combining such limits we get the conditional expected value of T_i, see (2.9). □

Note that in the case $i < i_0$ it is not possible to calculate the conditional expected value of T_i, since in this case $u^+(\lambda, 0) = \infty$, by Lemma 3.2 we know that $S(N) = \infty$.

5. Asymptotic behavior of the infection in the SIS epidemic model

In order to study the asymptotic behavior of the infection in the SIS epidemic model it is convenient to introduce the following notation. If $0 \le i < j \le N$ are given we define the stopping time $T_{i,j}$ as $\min\{T_i, T_j\}$, $T_{i,j} := \min\{T_i, T_j\}$.

Lemma 5.1. *We have $P[T_{0,N} = \infty] = 1$.*

Proof. We know, see Lemma 3.2, that $\Sigma(0) = \infty$, then $u_1(0) = \infty$. The inequality (4.4) implies $\lim_{i\downarrow 0} u(\lambda, i) = \infty$. Since $\int_0^N s(y)u(\lambda, y)^{-2}dy > 0$, from (4.7) we deduce $\lim_{i\downarrow 0} u^+(\lambda, i) = \infty$. From (4.9) we see that (here we assume $i < i_0$)

$$\begin{aligned} 0 &= \lim_{i\downarrow 0}\frac{u^+(\lambda, i_0)}{u^+(\lambda, i)} \\ &= \lim_{i\downarrow 0} E[e^{-\lambda T_i}] \\ &= \lim_{i\downarrow 0}\left\{E[e^{-\lambda T_i}; T_0 = \infty] + E[e^{-\lambda T_i}; T_0 < \infty]\right\} \\ &\ge \lim_{i\downarrow 0} E[e^{-\lambda T_i; T_0 < \infty}], \end{aligned}$$

where we have used $T_i < T_0$. If $P[T_0 < \infty] > 0$, then $E[e^{-\lambda T_0}; T_0 < \infty] > 0$. But this is impossible, therefore $P[T_0 = \infty] = 1$. On the other hand, using $\Sigma(N) = \infty$, we can prove analogously that $P[T_N = \infty] = 1$. From this the result follows. □

The previous result means that the process I almost surely never touches the boundaries, 0 or N. In other words, the disease is almost surely not extinct, but neither is the entire population infected.

Theorem 5.2. *If $i \in (0, N)$, then*

$$P\left[\inf_{0\le t<\infty} I_t = 0\right] = 1. \tag{5.1}$$

If $\beta_1 \ge 1$, then

$$P\left[\sup_{0\le t<\infty} I_t = N\right] = 1, \tag{5.2}$$

and consequently the process I is recurrent, that is,

$$P[I_t = i, \text{ for some } 0 \le t < \infty] = 1. \tag{5.3}$$

Proof. Since $P[T_0 = \infty] = 1$, then $P[I_t > 0] = 1$. From this we have

$$P\left[\inf_{0\le t<\infty} I_t \ge 0\right] = 1. \tag{5.4}$$

We also know (see, for example, [9]) that $E[T_{i,j}] < \infty$, $0 < i < j < N$, then $P[T_{i,j} < \infty] = 1$. Using Itô's formula in $S(I(t))$ we obtain

$$S(x) = E[S(I_{\min\{t,T_{i,j}\}})|I_0 = x].$$

From here it follows that

$$P\left[\inf_{0\le t<\infty} I_t \le i\right] \ge P[I_{T_{i,j}} = i] = \frac{1-(S(i_0)/S(j))}{1-(S(i)/S(j))}. \tag{5.5}$$

If we let $j \uparrow N$, and using that $S(N) = \infty$, we get

$$P\left[\inf_{0\le t<\infty} I_t \le i\right] \ge 1. \tag{5.6}$$

Using (5.4) and letting $i \downarrow 0$, in (5.6), we get (5.1). If $\beta_2 \ge 1$, then $S(0) = -\infty$, thus a similar argument can be applied to prove (5.2). The statement (5.3) easily follows form (5.1) and (5.2). □

When $\beta_1 \ge 1$ the process I is recurrent, therefore $\lim_{t\to\infty} I_t$ does not exist. We will study the missing case, $\beta_1 < 1$.

Theorem 5.3. *If $\beta_1 < 1$, then*

$$P\left[\lim_{t\to\infty} I_t = 0\right] = 1. \tag{5.7}$$

Proof. The proof is divided in two steps, in the first we prove the existence of the limit and in the second we calculate it.

Step one: For each $n \in \mathbb{N}$ let us set

$$X_t^n := S\left(I_{\min\{t,\tau_n\}}\right) - S(0), \quad 0 \le t < \infty,$$

where $\tau_n := \min\{T_{1/n}, T_{N-1/n}\}$. We can use Itô's formula to see that X^n is a local martingale. Since $\lim_{n\to\infty} \tau_n = T_{0,N}$ and $T_{0,N} = \infty$ almost surely (a.s.), we have

$$X_t := S\left(I_t\right) - S(0) = \lim_{n\to\infty} X_t^n, \quad 0 \le t < \infty.$$

From (3.6) we see that $X \ge 0$, then we can use Fatou's lemma to see that X is a supermartingale. Since every positive supermartingale converges almost surely (see Corollary 2.11 in [14]), then $\lim_{t\to\infty} X_t$ exits almost surely. The function S is monotonically increasing and continuous, therefore it has a continuous inverse, from which it follows that $\lim_{t\to\infty} I_t$ exists a.s.

Step two: Now let us see that $\lim_{t\to\infty} I_t = 0$ a.s. Suppose otherwise. Let $\tilde{\Omega} \subset \Omega$ be such that $P[\tilde{\Omega}] > 0$ and $\lim_{t\to\infty} I_t(\omega) = l(\omega) > 0$, for each $\omega \in \tilde{\Omega}$. This implies that there exists a $t(\omega) > 0$ for which

$$I_t(\omega) > \frac{1}{2} l(\omega), \quad t \ge t(\omega).$$

Since $T_0 = \infty$ and I is continuous, we have $\inf_{0\le t<\infty} I_t > 0$ in $\tilde{\Omega}$. This is a contradiction to (5.1). From which the desired statement follows. □

It is interesting to observe that (5.7) and $P[T_N = \infty] = 1$ imply that

$$P\left[\sup_{0\le t<\infty} I_t < N\right] = 1. \tag{5.8}$$

In other words, the strict inequality implies that the infection remains below a level that is strictly less than N.

6. Numerical considerations

To obtain the conditional moment of the first passage time, T_i, it is essential to determine the value of the following integrals

$$\int_0^i s(x)dx, \quad u_1(i) = \int_{i_0}^i \int_{i_0}^y s(y)m(z)dzdy \quad \text{and}$$

$$\int_0^i s(x)u_1(x)dx = \int_0^i \int_{i_0}^x \int_{i_0}^y s(x)s(y)m(z)dzdydx.$$

The numerical computation of these integrals is not trivial. Indeed, the problem lies in the fact that the integrand has a singularity at 0. The main issue arises in the calculation of the third multiple integral. To address this problem, we embarked on a thorough exploration of robust numerical methods capable of computing multiple integrals with singular integrands.

Our meticulous review revealed that for “well-behaved integrals”, most methods or software are sufficient. However, when dealing with integrals featuring singularities or numerous oscillations, each integral or family thereof requires its customized approach. In our research, we found that the Taylor series expansion is particularly effective for evaluating these integrals. Below, we briefly describe this technique, which we are confident will prove beneficial for those applying the results of our work.

Given $n \in \mathbb{N}\cup\{\infty\}$, $k, j \in \mathbb{N}\cup\{0\}$, $m \in \{0,1\}$ and $\alpha, \beta \in \mathbb{R}$, we define

$$S_k^n(\alpha) = \sum_{k=0}^n \frac{\alpha^k N^{-k}}{k!}, \quad N > 0,$$

and

$$S_{k,j=m}^n(\alpha,\beta) = \sum_{k=0}^n \frac{\alpha^k}{k!} N^{-k} \sum_{j=m}^n \binom{\beta-k}{j} (-N)^{-j}, \quad N > 0.$$

Let us take $\beta_1 < 1$ and $i > i_0$. The functions s and m can be expressed in terms of power series, and their convergence is justified using Mertens' theorem for the product of series. Furthermore, due to uniform convergence, the interchange of the integral and the sum is justified. Using this, we have the following equalities

$$\int_0^i s(x)dx = \beta_2 N^{\beta_1} S_{k,j=0}^\infty(\beta_3,\beta_1)\, \frac{i^{j-\beta_1+1}}{j-\beta_1+1},$$

$$\begin{aligned}
u_1(i) = \frac{2}{(cN)^2} \Bigg[& S_k^\infty(\beta_3) S_{\tilde{k}}^\infty(-\beta_3) \frac{1}{\beta_1 - 1} \\
& \cdot \left\{ \ln(i) - \ln(i_0) - \frac{(i_0)^{\beta_1-1}}{1-\beta_1} \left(i^{1-\beta_1} - (i_0)^{1-\beta_1} \right) \right\} \\
& + S_k^\infty(\beta_3) S_{\tilde{k},\tilde{j}=1}^\infty(-\beta_3, -\beta_1 - 2) \frac{1}{\beta_1 - 1 + \tilde{j}} \\
& \cdot \left\{ \frac{i^{\tilde{j}} - (i_0)^{\tilde{j}}}{\tilde{j}} - \frac{(i_0)^{\beta_1 - 1 + \tilde{j}}}{1-\beta_1} (i^{1-\beta_1} - (i_0)^{1-\beta_1}) \right\} \\
& + S_{k,j=1}^\infty(\beta_3,\beta_1) S_{\tilde{k}}^\infty(-\beta_3) \frac{1}{\beta_1 - 1} \\
& \cdot \left\{ \frac{i^j - (i_0)^j}{j} - \frac{(i_0)^{\beta_1-1}}{j-\beta_1+1} (i^{j-\beta_1+1} - (i_0)^{j-\beta_1+1}) \right\} \\
& + S_{k,j=1}^\infty(\beta_3,\beta_1) S_{\tilde{k},\tilde{j}=1}^\infty(-\beta_3, -\beta_1 - 2) \frac{1}{\beta_1 - 1 + \tilde{j}}
\end{aligned}$$

$$\cdot \left\{ \frac{i^{j+\tilde{j}} - (i_0)^{j+\tilde{j}}}{j+\tilde{j}} - \frac{(i_0)^{\beta_1-1+\tilde{j}}}{j-\beta_1+1}\left(i^{j-\beta_1+1} - (i_0)^{j-\beta_1+1}\right) \right\} \Bigg]$$

and

$$\begin{aligned}
\int_0^i s(x)u_1(x)dx = \frac{2\beta_2}{c^2 N^{2-\beta_1}} \Bigg[& S^{\infty}_{\hat{k},\hat{j}=0}(\beta_3,\beta_1) S^{\infty}_{k}(\beta_3) S^{\infty}_{\tilde{k}}(-\beta_3) \frac{1}{\beta_1-1} \\
& \cdot \left\{ \frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1}\left(\ln(i) - \frac{1}{\hat{j}-\beta_1+1}\right) - \ln(i_0)\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right. \\
& \left. - \frac{(i_0)^{\beta_1-1}}{1-\beta_1}\left(\frac{i^{\hat{j}-2\beta_1+2}}{\hat{j}-2\beta_1+2} - (i_0)^{1-\beta_1}\frac{i^{\tilde{j}-\beta_1+1}}{\tilde{j}-\beta_1+1} \right) \right\} \\
& + S^{\infty}_{\hat{k},\hat{j}=0}(\beta_3,\beta_1) S^{\infty}_{k}(\beta_3) S^{\infty}_{\tilde{k},\tilde{j}=1}(-\beta_3,-\beta_1-2) \frac{1}{\beta_1-1+\tilde{j}} \\
& \cdot \left\{ \frac{1}{\tilde{j}}\left(\frac{i^{\hat{j}-\beta_1+\tilde{j}+1}}{\hat{j}-\beta_1+\tilde{j}+1} - (i_0)^{\tilde{j}}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right. \\
& \left. - \frac{(i_0)^{\beta_1-1+\tilde{j}}}{1-\beta_1}\left(\frac{i^{\hat{j}-2\beta_1+2}}{\hat{j}-2\beta_1+2} - (i_0)^{1-\beta_1}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right\} \\
& + S^{\infty}_{\hat{k},\hat{j}=0}(\beta_3,\beta_1) S^{\infty}_{k,j=1}(\beta_3,\beta_1) S^{\infty}_{\tilde{k}}(-\beta_3) \frac{1}{\beta_1-1} \\
& \cdot \left\{ \frac{1}{j}\left(\frac{i^{j+\hat{j}-\beta_1+1}}{j+\hat{j}-\beta_1+1} - (i_0)^{j}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right. \\
& \left. - \frac{(i_0)^{\beta_1-1}}{j-\beta_1+1}\left(\frac{i^{\hat{j}+j-2\beta_1+2}}{\hat{j}+j-2\beta_1+2} - (i_0)^{j-\beta_1+1}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right\} \\
& + S^{\infty}_{\hat{k},\hat{j}=0}(\beta_3,\beta_1) S^{\infty}_{k,j=1}(\beta_3,\beta_1) S^{\infty}_{\tilde{k},\tilde{j}=1}(-\beta_3,-\beta_1-2) \frac{1}{\beta_1-1+\tilde{j}} \\
& \cdot \left\{ \frac{1}{j+\tilde{j}}\left(\frac{i^{\tilde{j}-\beta_1+\hat{j}+j+1}}{\tilde{j}-\beta_1+\hat{j}+j+1} - (i_0)^{j+\tilde{j}}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right. \\
& \left. - \frac{(i_0)^{\beta_1-1+\tilde{j}}}{j-\beta_1+1}\left(\frac{i^{\hat{j}-2\beta_1+2+j}}{\hat{j}-2\beta_1+2+j} - (i_0)^{j-\beta_1+1}\frac{i^{\hat{j}-\beta_1+1}}{\hat{j}-\beta_1+1} \right) \right\} \Bigg].
\end{aligned}$$

During practical illustrations, it was observed that the conditional moment of the first passage time T_i became stable when n was less than 15. Therefore, in the upcoming section, we will establish the value of n as 15.

7. Some real world problems

In this section, we will showcase three real-life examples of phenomena that are investigated through the utilization of previously obtained theoretical outcomes.

Bear in mind that I_t represents the quantity of interest at time t and its dynamics is determined by the stochastic differential equation

$$dI_t = \left(a - bI_t\right) I_t\, dt + c\left(N - I_t\right) I_t\, dB_t. \tag{7.1}$$

Let us also notice that

$$R_{st} < 1 \Longleftrightarrow \beta_1 < 1 \Longleftrightarrow \frac{\sqrt{2a}}{N} < c, \tag{7.2}$$

where R_{st} represents the stochastic reproduction number. Then, in this case, Theorem 5.3 implies that the process I eventually dies out as time approaches infinity.

7.1. Gonorrhea

Gonorrhea is a sexually transmitted disease that has the potential to affect both men and women. It can lead to infections in the genital, rectal, and throat areas. This infection is quite common, especially among individuals aged 15 to 24. Notably, acquiring gonorrhea does not confer immunity, making it possible to become reinfected even after a prior infection has been treated.

Table 1
Several values of I^{-1}, within the context of gonorrhea, $i_0 = 1,000$.

i	1636	2198	2550	2725	2802	2835
$I^{-1}(i)$	124.86	249.77	374.73	499.35	622.89	749.48

Table 2
Probabilities and expected values of the first-passage time upper i_0 in the gonorrhea case.

c	i	1636	2198	2550	2725	2802	2835
1.5×10^{-5}	$P[T_i < \infty]$	0.82	0.72	0.67	0.65	0.65	0.64
	$E\left[T_i \vert T_i < \infty\right]$	**142.04**	**234.08**	283.60	306.69	316.59	320.79
1.45×10^{-5}	$P[T_i < \infty]$	0.84	0.75	0.71	0.69	0.68	0.67
	$E\left[T_i \vert T_i < \infty\right]$	175.51	289.65	**351.27**	380.06	392.42	397.67
1.4×10^{-5}	$P[T_i < \infty]$	0.86	0.78	0.74	0.72	0.72	0.71
	$E\left[T_i \vert T_i < \infty\right]$	227.21	375.72	456.28	**494.05**	**510.29**	**517.19**

The transmission of gonorrhea among homosexual individuals is investigated in [15]. In the aforementioned study, with our notation, the following parameters are taken into consideration:

$$N = 10,000, \quad \gamma = 0.018182, \quad \tilde{b} = 6.84463 \times 10^{-5} \quad \text{and} \quad r = 0.025550624.$$

Given these values, we have the parameters required by the stochastic differential equation (7.1)

$$a = 7.30017852 \times 10^{-3} \quad \text{and} \quad b = 2.555062482 \times 10^{-6}.$$

In the study of gonorrhea transmission, the initial population comprises $1,000$ infected individuals, $i_0 = 1,000$. Formula (7.2) provides us with a lower limit for the range of the parameter c in (7.1), indeed $c > 1.208319372 \times 10^{-5}$. In order to determine a more optimal value for such stochastic parameter, we employ the solution of the deterministic differential equation (2.4). The rationale behind this approach is our anticipation that, on average, the solution of the stochastic differential equation will closely align with its deterministic counterpart. Guided by this notion, we consider various values of the state i and ascertain the corresponding time required for the process to reach that state, denoted as $I^{-1}(i)$. Table 1 presents this computation for the case of gonorrhea.

Using Table 1, a range of possible values for the parameter c was explored, and the results of the conditional expectation of the first moment of arrival, along with the probability that this time is finite, are outlined in Table 1. The values in Table 2 that are closest to the deterministic value are highlighted in bold.

In Fig. 1, we display a simulation illustrating the growth of gonorrhea for different values of c, as listed in Table 2. As anticipated, these simulations vividly portray the erratic nature of the disease and its eventual extinction.

Based on the analysis performed, we deduce that for values of i that slightly exceed $i_0 = 1,000$, a suitable option for the parameter c could be 1.5×10^{-5}. Furthermore, this implies a probability of over 80 percent that more than $1,600$ people will be infected. This information can be used to launch an initial extensive infection awareness campaign, and gradually scale it down. We also observe that for prolonged periods of time, a favorable value for the parameter c could be 1.4×10^{-5}; in this case, the probability of reaching high levels of infection is around 70 percent, which indicates the importance of carrying out an initial awareness campaign.

7.2. Pneumococcus

Another contagious disease for which immunity is not developed is caused by the pneumococcus bacteria. This disease can be mild or severe, and the most common infections usually include: ear infections, sinusitis, pneumonia, sepsis and meningitis.

In the reference [16], pneumococcus infection is investigated within a child population. The parameters considered in this study are:

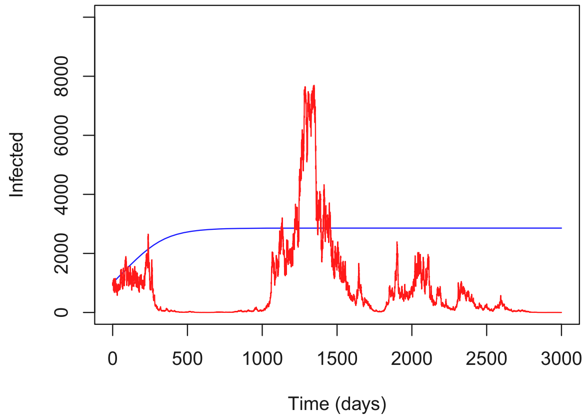
$$N = 150,000, \quad \gamma = 0.02011, \quad \tilde{b} = 1.3736 \times 10^{-3} \quad \text{and} \quad r = 0.042975.$$

The corresponding values for a and b, in equation (7.1), are

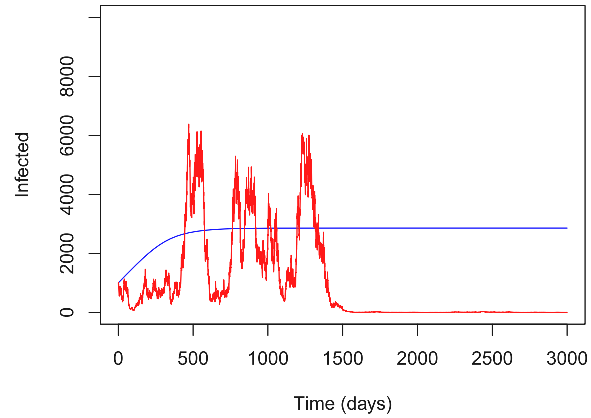
$$a = 0.0214914 \quad \text{and} \quad b = 2.865 \times 10^{-7}.$$

In this scenario, $i_0 = 50,000$, and as derived from equation (7.2), it can be observed that $c > 1.382 \times 10^{-6}$. Similar to the previous example, Table 3 provides several values of I^{-1} to aid in determining the most suitable stochastic parameter, c.

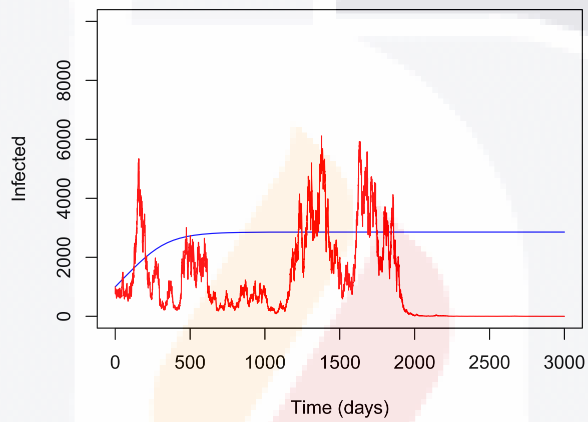
From Table 4, we observe that the optimal value of c is found in the middle row, specifically, in this case, $c = 1.5 \times 10^{-6}$. The probabilities are higher than 90, indicating that it is highly likely that starting with $i_0 = 50,000$ infected individuals, there will be either $72,000$ or $74,000$ infected individuals within a 4-month period. Additionally, it can be noted that there is a high probability of reaching a larger number of infected individuals, bigger than 90 percent.



(a) $c = 1.5 \times 10^{-5}$.



(b) $c = 1.45 \times 10^{-5}$.



(c) $c = 1.4 \times 10^{-5}$.

Fig. 1. Highlighted in red are simulated trajectories depicting the growth of gonorrhea, while the deterministic solution is depicted in blue.

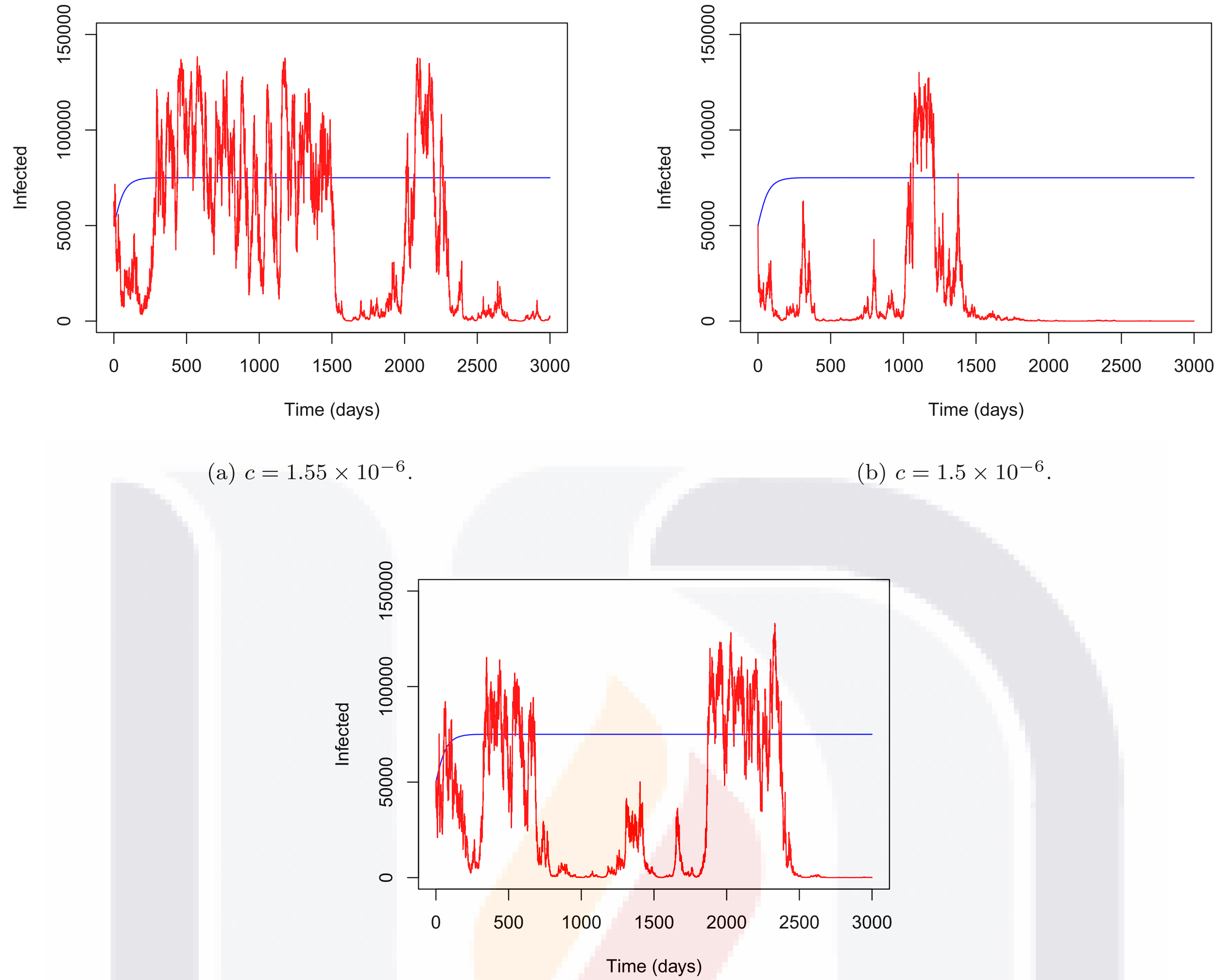
Table 3
Certain values of I^{-1} in the context of pneumococcus, $i_0 = 50,000$.

i	$60,000$	$70,000$	$72,000$	$74,000$	$75,000$	75010
$I^{-1}(i)$	32.24	90.44	115.44	167.41	368.60	430.33

Table 4
Probabilities and expected values for the infection caused by pneumococcus.

c	i	60,000	70,000	72,000	74,000	75,000	75010
1.55×10^{-6}	$P[T_i < \infty]$	0.95	0.92	0.91	0.91	0.90	0.90
	$E\left[T_i \mid T_i < \infty\right]$	**41.81**	**80.45**	87.99	95.51	99.26	99.30
1.5×10^{-6}	$P[T_i < \infty]$	0.97	0.94	0.93	0.93	0.93	0.93
	$E\left[T_i \mid T_i < \infty\right]$	59.04	113.45	**124.06**	**134.63**	139.91	139.96
1.45×10^{-6}	$P[T_i < \infty]$	0.98	0.96	0.96	0.95	0.95	0.95
	$E\left[T_i \mid T_i < \infty\right]$	101.07	193.86	211.93	229.93	**238.92**	**239.01**

Fig. 2 displays three simulations depicting potential trajectories of I for three distinct values of c. Similar to previous observations, it is evident that the infection ultimately diminishes over time. Nevertheless, it is worth highlighting that the likelihood of reaching a higher number of infected individuals is substantial. Therefore, a sensible health strategy in this scenario would involve implementing an informational campaign to reduce the contagion rate, because it is quite likely that a greater number of infected will be reached.



(a) $c = 1.55 \times 10^{-6}$.

(b) $c = 1.5 \times 10^{-6}$.

(c) $c = 1.45 \times 10^{-6}$.

Fig. 2. Simulated trajectories illustrating the growth of pneumococcus infection are highlighted in red, whereas the deterministic solution is represented in blue.

7.3. Guppy fish

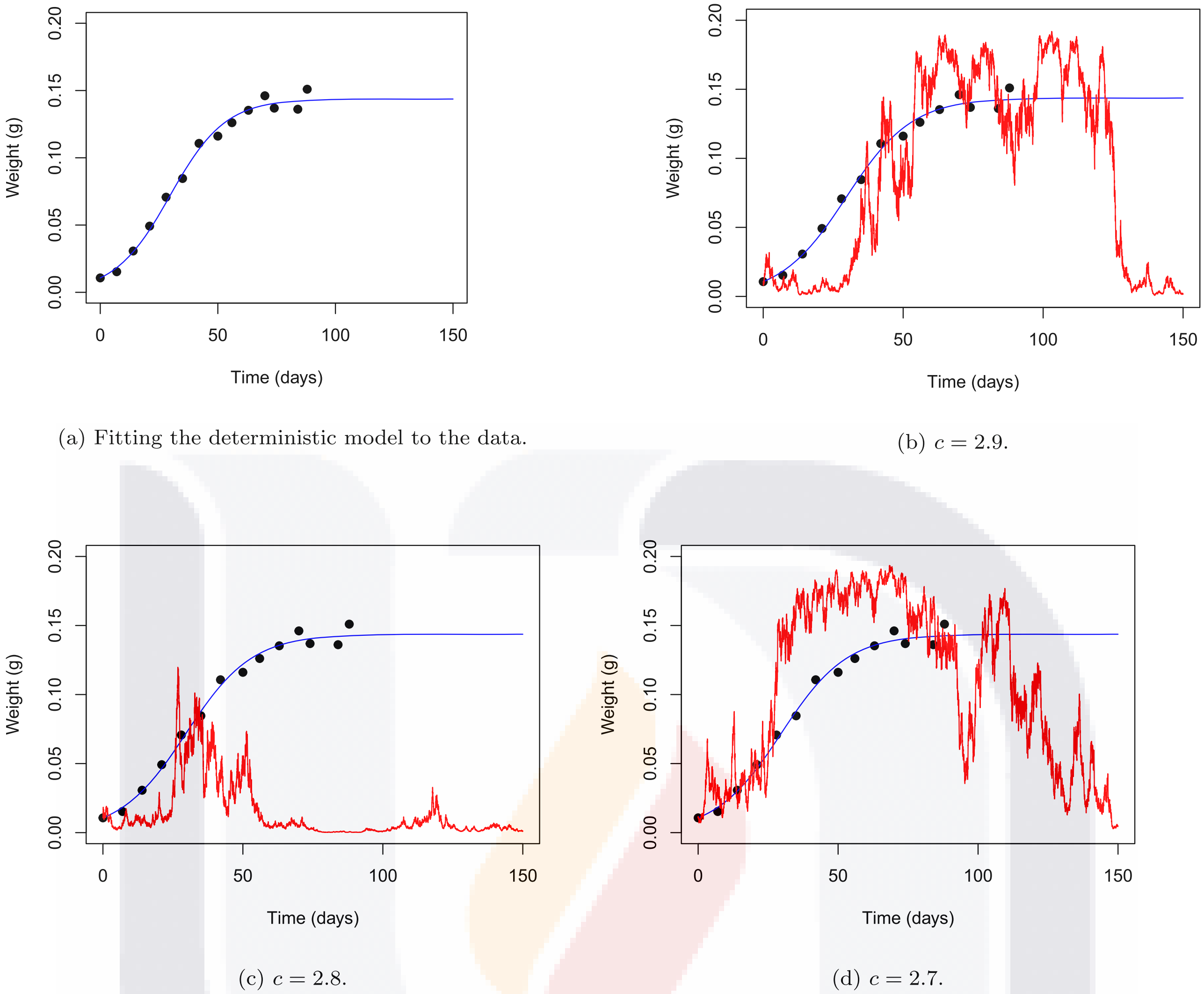
From expression (2.4), we observe that the model obtained is a generalization of a logistic model. Therefore, it is feasible to employ this stochastic model to simulate phenomena that display this type of behavior, namely sigmoidal growth. Thus, we can apply the stochastic SIS model to simulate various scenarios, such as machine learning, population growth, or phenomena with limited carrying capacity. In our specific case, we will utilize this model to examine the weight growth of guppy fish. For instance, in [17], the growth of dry weight in various plants and the development of other animals is studied, where the same technique we are about to develop can be aptly applied.

Typical weight-based growth data for guppy fish are provided in [17]. To fit the data to a growth pattern characterized by the function (2.5), we employ the nls function within the software R. Taking $i_0 = 0.0107$, this process yields the following values, see equation (7.1),

$$a = 8.491667 \times 10^{-2} \quad \text{and} \quad b = 0.59102452.$$

Fig. 3 (a) shows in blue the deterministic function I fitted to the data. Now, the next step is to add the stochastic component to it. From the deterministic case we see that (see (2.5)) a/b would be the load capacity of the system. And in our stochastic context we consider a parameter N, which represents the maximum value that the random variable I_t can take, therefore $N > a/b$. Let us take, $N = 0.2$. Using these values, we determine, as shown in (7.2), the lower bound for c to be $c > 2.0605$, ensuring the applicability of Theorem 5.3.

Table 5 presents values of $I^{-1}(i)$ for several values of i. We will use these values to represent the average behavior of the timeout of the first conditional passage. Table 6 provides the probability $P[T_i < \infty]$ and conditional expected time $E\left[T_i | T_i < \infty\right]$ for various values of the stochastic parameter c. The values closest to the deterministic value are highlighted in bold. From this, we can observe



(a) Fitting the deterministic model to the data.

(b) $c = 2.9$.

(c) $c = 2.8$.

(d) $c = 2.7$.

Fig. 3. Several simulations of guppy fish weight growth are shown in red, while deterministic growth is represented in blue.

Table 5
Values of I^{-1} in the context of guppy fish weight.

i	0.0227	0.0438	0.0728	0.1014	0.1219	0.1335
$I^{-1}(i)$	9.97	19.96	29.99	39.97	49.95	59.98

Table 6
Probabilities and expected values of first-passage times above $i_0 = 0.0107$ for guppy weight.

c	i	0.0227	0.0438	0.0728	0.1014	0.1219	0.1335
2.9	$P[T_i < \infty]$	0.69	0.50	0.39	0.34	0.31	0.30
	$E\left[T_i \mid T_i < \infty\right]$	**9.49**	18.27	25.80	31.54	35.39	37.62
2.8	$P[T_i < \infty]$	0.71	0.53	0.42	0.37	0.34	0.33
	$E\left[T_i \mid T_i < \infty\right]$	10.95	**21.04**	**29.63**	36.13	40.45	42.95
2.7	$P[T_i < \infty]$	0.73	0.56	0.46	0.40	0.38	0.37
	$E\left[T_i \mid T_i < \infty\right]$	12.87	24.66	34.63	**42.08**	**46.98**	**49.81**

that in the second row, two of the expected time values are closer to the deterministic values, and the probabilities are around 50 percent. In the third row, there are three values marked in bold, but the corresponding probabilities are smaller.

This kind of information can be utilized to monitor the health of a guppy fish population. For instance, it will become evident that things are not progressing well if, starting with a weight of $i_0 = 0.0107$ grams, the guppies have not doubled their weight ($i/i_0 = 0.0227/0.0107 \approx 2.12$) in approximately 10 (9.49) days. In such a scenario, the likelihood of making an assessment error will be around 30 percent. In Fig. 3 (b), (c), and (d), various simulations depicting the growth of guppy fish weight for different values of c are presented.

8. Conclusions

In this article, the first arrival time of the stochastic process I is analyzed. Under the premise that the stochastic reproduction number is less than one (see (2.7)), the probability that the first arrival time is finite is calculated. Conditioned on this event, its expected value is determined (see Theorem 4.4). In the context of this assumption, about the stochastic reproduction number, it is shown (see Theorem 5.3) that the stochastic process I approaches zero as time approaches infinity. In other words, the disease tends to disappear eventually over an infinite horizon. Thus, it is possible to solve the conjecture formulated in [4].

In addition, the stochastic SIS model is used in three real situations and a numerical scheme is proposed to find the conditional moment. In the first two examples, the deterministic solution is adopted as an average behavior of the phenomena, and with this approach possible values of the stochastic parameter are established. The merit of this method lies in the fact that it allows evaluating with what probability a value greater than the initial one is reached, which in turn facilitates decision making based on the probability of said value. In the third example, a non-linear fitting method is first used to identify the most suitable deterministic model. Then, from this adjusted model, we proceed in a similar way to the two previous examples.

In summary, the incorporation of random noise generates significantly different results when contrasted with the deterministic counterpart. In fact, simulations reveal dynamic trajectories that more accurately mirror the complexities of the real world. Additionally, we have successfully computed the conditional expectation of the first arrival time and explored the asymptotic behavior of the stochastic process I. We view these methodologies as versatile tools applicable across diverse scenarios, thereby expanding the practicality and adaptability of stochastic models.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors do not have permission to share data.

Acknowledgements

The research was partially supported by the grant PIM22-1 of Benemérita Universidad Autónoma de Aguascalientes and CONAHCYT. Thanks to José Miguel Villa Ocampo for his support in adjusting the data. Many thanks to the reviewer of the manuscript for appropriate and constructive comments.

References

[1] D. Mollison, Epidemic Models: Their Structure and Relation to Data, Cambridge University Press, 1995.
[2] G. Thompson, J. Berner, M. Frediani, J.A. Otkin, S.M. Griffin, A stochastic parameter perturbation method to represent uncertainty in a microphysics scheme, Mon. Weather Rev. 149 (5) (2021) 1481–1497.
[3] Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, Appl. Math. Comput. 243 (2014) 718–727.
[4] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math. 71 (2011) 876–902.
[5] R.K. McCormack, L.J. Allen, Stochastic sis and sir multihost epidemic models, in: Proceedings of the Conference on Differential and Difference Equations and Applications, Hindawi New York, 2006, pp. 775–786.
[6] Y. Chen, B. Wen, Z. Teng, The global dynamics for a stochastic sis epidemic model with isolation, Phys. A, Stat. Mech. Appl. 492 (2018) 1604–1624.
[7] W. Liu, Q. Zheng, A stochastic SIS epidemic model incorporating media coverage in a two patch setting, Appl. Math. Comput. 262 (2015) 160–168.
[8] E. Lungu, B. Øksendal, Optimal harvesting from a population in a stochastic crowded environment, Math. Biosci. 145 (1) (1997) 47–75.
[9] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, London, 1991.
[10] H. Durret, Stochastic Calculus, a Practical Introduction, CRC Press, London, 1996.
[11] S. Karlin, H.M. Taylor, A Second Course in Stochastic Process, Academic Press, London, 1981.
[12] P. Mandl, Analytical Treatment of One-Dimensional Markov Processes, Springer, London, 1968.
[13] G.J. Cabral-García, J. Villa-Morales, Conditional moments of the first-passage time of a crowed population, to appear in Appl. Math. Comput. (2024).
[14] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer, London, 1999.
[15] H. Hethcote, J. Yorke, Gonorrhea Transmission Dynamics and Control, Lecture Notes in Biomathematics, vol. 56, 1994.
[16] K. Lamb, D. Greenhalgh, C. Robertson, A simple mathematical model for genetic effects in pneumococcal carriage and transmission, J. Comput. Appl. Math. 235 (2011) 1812–1818.
[17] L. Cao, P.-J. Shi, L. Li, G. Chen, A new flexible sigmoidal growth model, Symmetry 11 (204) (2019) 1–20.

3 Crecimiento poblacional logístico y los momentos condicionales del tiempo de primera pasada

El estudio de los modelos de crecimiento demográfico se remonta a 1798, cuando Thomas Robert Malthus escribió un ensayo sobre el crecimiento exponencial de la población y el crecimiento lineal de los recursos [19]. La modelización del crecimiento de un sistema es un tema clásico de las matemáticas aplicadas; además, debido a su importancia, sus aplicaciones en el mundo real y la complejidad del problema, es un tema de actualidad (véanse, por ejemplo, [46], [26], [49], [27], [24], [17], [18], [35] y las referencias que contienen).

En este caso, se considera el crecimiento de una población en la que el medio (o ecosistema) tiene la propiedad de mantener como máximo una población de tamaño $K > 0$ indefinidamente, el cual denota la capacidad de carga del medio. Además se supone que la calidad del entorno viene determinada por un parámetro $\lambda \in \mathbb{R}$. En este caso, si X_t es el tamaño de la población en el tiempo $t > 0$, entonces es bien sabido que la dinámica o crecimiento de la población viene determinada por la ecuación diferencial logística

$$\frac{dX_t}{dt} = \lambda X_t \left(K - X_t\right), \quad t > 0,$$

donde $X_0 = x_0 > 0$ es la población inicial. En este caso, la solución está dada por la función sigmoidal

$$X_t = \frac{x_0 K}{x_0 + \left(K - x_0\right) e^{-\lambda K t}}, \quad t \geq 0.$$

Frecuentemente no hay un valor exacto del parámetro λ, hay más bien un valor medio del mismo el cual se denota con r. Además se denota con α al tamaño medio de las fluctuaciones alrededor de la media r. De este modo, el parámetro λ se sustituye por una expresión de la forma $r + \alpha dB_t$, donde dB_t es un ruido aleatorio (que representa el cambio en los medios o sus fluctuaciones). Sustituyendo esto en la ecuación anterior se obtiene la ecuación diferencial logístico estocástica

$$dX_t = rX_t \left(K - X_t\right) dt + \alpha X_t \left(K - X_t\right) dB_t.$$

Esta forma de obtener una ecuación diferencial estocástica es habitual (véase [1] o [25]) y se denomina variación estocástica de un parámetro. En este caso, para resolverla se utiliza la integral estocástica de Itô y $\{B_t, t \geq 0\}$ denota un movimiento Browniano definido en un espacio de probabilidad completo (Ω, A, P), véase [7].

Si $X_0 = x_0 \in (0, K)$ y $r \in \mathbb{R}$, el Lema 2.1 de [25] implica que la ecuación diferencial anterior tiene solución única $X = \{X_t, t \geq 0\}$. Para $x \in (0, K)$ el primer momento de pasada está definido por

$$T_x = \text{ínf}\{t \geq 0 : X_t = x\}.$$

Dicho tiempo es la primera vez que el proceso estocástico X_t alcanza el nivel x dado que comienza en x_0. Con esta notación se definen los siguientes valores esperados condicionales

$$\begin{aligned} E\left[T_x | T_x < \infty\right] =& \frac{E\left[T_x; T_x < \infty\right]}{P\left[T_x < \infty\right]}, \\ E\left[(T_x)^2 | T_x < \infty\right] =& \frac{E\left[(T_x)^2; T_x < \infty\right]}{P\left[T_x < \infty\right]}, \end{aligned}$$

donde

$$\begin{aligned} E\left[T_x; T_x < \infty\right] &= \int\limits_{\{\omega: T_x(\omega) < \infty\}} T_x(\omega) P(d\omega), \\ E\left[(T_x)^2; T_x < \infty\right] &= \int\limits_{\{\omega: (T_x(\omega))^2 < \infty\}} T_x(\omega) P(d\omega). \end{aligned}$$

El objetivo principal de este trabajo es encontrar estos valores esperados condicionales y presentar algunas aplicaciones. Además, el método puede utilizarse para calcular los momentos condicionales de T_x de cualquier orden.

La solución de la ecuación diferencial estocástica logística es un proceso de difusión y depende del comportamiento de los límites, en este caso 0 y K. De hecho, si los límites de una difusión son de tipo elástico o reflectante entonces puede utilizarse la ecuación de Kolmogórov hacia atrás para determinar formas recursivas de los momentos de T_x, véase por ejemplo [47], [7] y [50]. Otra técnica útil es encontrar la transformada de Laplace de T_x y diferenciarla. En estos casos, la transformada de Laplace se expresa como la función de transición de densidad del proceso estocástico X, es decir, se expresa en términos de la función de Green, véase [11], [20], [33], [28] o [36]. Es casi seguro que el primer tiempo de pasada al estado x, comenzando en x_0, es finito, es decir $P(T_x < \infty) = 1$. Sin embargo, se tiene que $P(T_x = \infty) > 0$, por lo que el valor esperado es infinito ($E\left[T_x\right] = \infty$) y de ahí que no se apliquen los resultados conocidos. Sin embargo, cabe mencionar que la teoría de Sturm-Liouville se ha introducido recientemente en el estudio de los tiempos de llegada que puede ser una opción interesante a explorar, especialmente cuando el valor esperado es infinito, como en este trabajo, véase [23].

Existen diferentes versiones de ecuaciones diferenciales logísticas estocásticas y para algunas de ellas es posible encontrar soluciones explícitas, véase por ejemplo [48] y [4]. El proceso considerado en [48] tiene $P(T_x = \infty) > 0$, por lo que encuentran el valor esperado $E\left[T_x; T_x < \infty\right]$. Los resultados de [48] no pueden aplicarse en este trabajo porque están especficamente diseñados para tratar con su versión de la ecuación diferencial logístico estocástica, que es diferente a la estudiada en este trabajo.

3.2. Artículo

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Conditional moments of the first-passage time of a crowed population

Gabriela de Jesús Cabral-García, José Villa-Morales *

Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Av. Universidad 940, C.P. 20131 Aguascalientes, Ags., Mexico

A R T I C L E I N F O

MSC:
92D25
60G53
60G17
92B15

Keywords:
Stochastic logistic differential equation
Conditional moments
First-passage time
Tumor
World population

A B S T R A C T

Using the method of stochastic variation of parameters in the logistic differential equation a stochastic logistic differential equation is obtained. For this stochastic differential equation, the first two conditional moments of the first-passage time are found. These results are applied in the study of the growth of cancerous tumors and in the study of the growth of the world population when a random perturbation is incorporated into the deterministic models.

1. Introduction

The study of population growth models dates back to 1798 when Thomas Robert Malthus wrote an essay on the exponential growth of population and the linear growth of resources. Modeling the growth of a system is a classic topic in applied mathematics; additionally, due to its importance, its applications in the real world, and the complexity of the problem, this is a current issue (see, for example, [25], [4], [27], [21], [20], [14], [11], [10] and the references contained therein).

In our case, we will consider the growth of a population in which the environment (or ecosystem) has the property of maintaining at most a population of size $K > 0$ indefinitely, we will say that the environment has a carrying capacity K. Furthermore, we will assume that the quality of the environment is determined by a parameter $\lambda \in \mathbb{R}$. In this case, if X_t is the size of the population at time $t > 0$, then it is well known that the dynamics or growth of the population is determined by the logistic differential equation

$$\frac{d}{dt}X_t = \lambda X_t(K - X_t), \quad t > 0, \tag{1.1}$$

where $X_0 = x_0 > 0$ is the initial population. In this case, the solution is given by the sigmoidal function

$$X_t = \frac{x_0 K}{x_0 + (K - x_0)e^{-\lambda K t}}, \quad t \geq 0. \tag{1.2}$$

* Corresponding author.
E-mail addresses: gabriela.cabral@edu.uaa.mx (G. de Jesús Cabral-García), jvilla@correo.uaa.mx (J. Villa-Morales).

https://doi.org/10.1016/j.amc.2024.128569
Received 11 April 2023; Received in revised form 14 December 2023; Accepted 19 January 2024

Frequently there is no exact value of the parameter λ, there is rather an average value of it, say r, and it is known that the fluctuations, around the mean r, have an average of size magnitude α. In this way, the parameter λ is replaced by an expression of the form $r + \alpha\, dB_t$, where dB_t is a white noise (representing the noise or fluctuations). Substituting this into (1.1) we get the stochastic logistic differential equation

$$dX_t = rX_t\left(K - X_t\right)dt + \alpha X_t\left(K - X_t\right)dB_t. \tag{1.3}$$

This way to obtain a stochastic differential equation is usual (see [1] or [15]) and is called stochastic variation of a parameter. In our case, to solve it we will use Itô's stochastic integral and $\{B_t, t \geq 0\}$ will denote a Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}, P)$, see [12].

If $X_0 = x_0 \in (0, K)$ and $r \in \mathbb{R}$, Lemma 2.1 in [15] implies the stochastic logistic differential equation (1.3) has a unique solution $X = \{X(t), t \geq 0\}$. For $x \in (0, K)$ the first-passage time of X to x is defined by

$$T_x = \inf\{t \geq 0 : X(t) = x\}.$$

Such time is the first time the stochastic process X hits the level x when it starts at x_0. With this notation we define the following conditional expected values

$$E[T_x | T_x < \infty] := \frac{E[T_x; T_x < \infty]}{P(T_x < \infty)},$$

$$E[(T_x)^2 | T_x < \infty] := \frac{E[(T_x)^2; T_x < \infty]}{P(T_x < \infty)},$$

where

$$E[T_x; T_x < \infty] := \int_{\{\omega : T_x(\omega) < \infty\}} T_x(\omega) P(d\omega),$$

$$E[(T_x)^2; T_x < \infty] := \int_{\{\omega : T_x(\omega) < \infty\}} (T_x(\omega))^2 P(d\omega).$$

The main objective of this work is to find these conditional expected values and present some applications. Furthermore, the method can be used to calculate the conditional moments of T_x of any order.

The solution of the stochastic logistic differential equation is a diffusion process and depending on the behavior of the boundaries, in this case 0 and K, the moments of T_x are known. Indeed, if the boundaries of a diffusion are of elastic or reflecting type then backward Kolmogorov equations can be used to determine recursive forms of the moments of T_x, see, for example, [26], [5], [28]. Another useful technique is to find the Laplace transform of T_x and differentiate it. In these cases, the Laplace transform is expressed as the Laplace transform of the density transition function of the stochastic process X, that is, it is expressed in terms of the Green's function, see [9], [16], [19], [8] or [22]. In these cases it is almost sure that the time of the first-passage to state x, starting at x_0, is finite, that is, $P(T_x < \infty) = 1$. However, in our case we have $P(T_x = \infty) > 0$, therefore the expected value is infinite ($E[T_x] = \infty$) and hence the known results do not apply. However, it is worth mentioning that Sturm-Liouville theory has recently been introduced in the study of arrival times and can be an interesting option to explore, especially when the expected value is infinite, as in our case, see [18].

There are different versions of stochastic logistic differential equations and for some of them it is possible to find explicit solutions, see, for example, [7], [2]. The process considered in [7] has $P(T_x = \infty) > 0$, thus they find the expected value $E[T_x; T_x < \infty]$. The results of [7] cannot be applied here because they are specifically designed to deal with their version of stochastic logistic differential equation, which is different from the one studied in this work (their equation has an explicit solution, contrary to our case (1.3)).

As we have said, since $P(T_x = \infty) > 0$, then it is necessary to consider conditional moments of T_x. To do this, we will first determine the conditional Laplace transform. In this part lies the problem, because in the general case the Laplace transform appears in terms of the Green function or of certain monotonic functions that are not given explicitly, but are solutions of a certain ordinary differential equation that is associated with the infinitesimal generator of the stochastic process. In our case, we solve this stumbling block by finding an explicit monotonic solution (see Lemma 3.2) and from there, using Itô's formula, we find the Laplace transform of T_x (see Proposition 4.1). It seems to us that it is the first time that the two conditional moments of the first-passage have been calculated explicitly for the solution of the stochastic differential equation (1.3).

As an application of the obtained results, we study two real world problems. The first models the growth of a cancerous tumor and the second models the growth of the population worldwide. In both cases, the difference between deterministic and stochastic models is highlighted. Moreover, in each of them we use the logistic behavior and the conditional expected value of the time of first arrival to determine the most appropriate stochastic model. In particular, it is observed that by adding a small noise, caused by climate change, to the classical logistic model of population growth, a short expected time (about 28 years) is obtained for the population to halve, with a relatively high probability (around 18%). This means that governments and people must become aware of how serious it can be to modify, even a little, our ecosystem.

The paper is organized in the following manner. First, we present in Section 2 some preliminary results that allow us to understand the asymptotic behavior of the solution of the stochastic logistic differential equation (1.3), these results will be applied in Section 5.

In Section 3 we find a helpful monotonic solution for a certain ordinary differential equation, see (5.6)-(3.3). In Section 4 we find the Laplace transform of the first-passage time and obtain its first two conditional moments in Theorems 4.2 and 4.3. Finally, in Section 5 we study the tumor growth model and the population growth model.

2. Preliminaries

In order to study the asymptotic behavior of the process X, solution of (1.3), we are going to introduce some auxiliary functions. For $y \in (0, K)$ we set

$$s(y) := \exp\left(-\int_{K/2}^{y} \frac{2f(z)}{g(z)} dz\right) \quad \text{and} \quad m(y) := \frac{2}{s(y)g(y)}, \tag{2.1}$$

where

$$f(y) := ry(K-y) \quad \text{and} \quad g(y) := \alpha^2 y^2 (K-y)^2. \tag{2.2}$$

The boundary points of the stochastic process X are 0 and K. For the boundary 0 we define the following functionals

$$S(0) := \int_0^{K/2} s(x)dx,$$

$$\Sigma(0) := \int_0^{K/2} \left(\int_0^y s(z)dz\right) m(y)dy,$$

$$N(0) := \int_0^{K/2} \left(\int_0^y m(z)dz\right) s(y)dy.$$

Analogous definitions can be introduced for the boundary K, see Section 6 of Chapter 15 in [13].

Let us start with the following elementary observation.

Lemma 2.1. *For $x \in (0, K)$ we have*

$$s(x) = \left(\frac{K-x}{x}\right)^{\beta},$$

where

$$\beta := \frac{2r}{K\alpha^2}. \tag{2.3}$$

Proof. First notice that

$$\frac{2f(z)}{g(z)} = \beta\left(\frac{1}{z} + \frac{1}{K-z}\right), \quad z \in (0, K).$$

Such equality implies

$$\int_{K/2}^{x} \frac{2f(z)}{g(z)} dz = \beta \ln\left(\frac{x}{K-x}\right).$$

From this the desired result follows immediately. □

The following analytical result will give us information about the trajectory behavior of the process X.

Lemma 2.2. *With the above notation we get the following facts for the boundary 0,*

$$S(0) < \infty \quad \textit{if and only if} \quad \beta < 1,$$

$$\Sigma(0) = \infty \quad \textit{and} \quad N(0) = \infty.$$

For the boundary K we have

$S(K) < \infty$ *if and only if* $\beta > -1$,

$\Sigma(K) = \infty$ *and* $N(K) = \infty$.

Proof. In what follows we deal with each case.

$S(0)$**:** From Lemma 2.1 we get

$$S(0) = \int_0^{K/2} \left(\frac{K-x}{x}\right)^{\beta} dx.$$

Since

$$\left(\frac{K-x}{x}\right)^{\beta} = O\left(x^{-\beta}\right), \quad \text{as } x \downarrow 0,$$

then $S(0) < \infty$ if and only if $1-\beta > 0$.

$\Sigma(0)$**:** The definition of $\Sigma(0)$ implies

$$\Sigma(0) = \frac{2}{\alpha^2} \int_0^{K/2} \frac{y^{\beta-2}}{(K-y)^{\beta+2}} \left(\int_0^{y} \left(\frac{K}{z} - 1\right)^{\beta} dz \right) dy.$$

Now observe that the integral

$$\int_0^{y} \left(\frac{K}{z} - 1\right)^{\beta} dz, \tag{2.4}$$

is finite if and only if $1-\beta > 0$. If $1-\beta \le 0$, then $\Sigma(0) = \infty$. Let us suppose that $1-\beta > 0$. Since the integral (2.4) is of order $y^{1-\beta}$, as $y \downarrow 0$, then the behavior of $\Sigma(0)$ depends on the integral

$$\int_0^{K/2} y^{\beta-2} \cdot y^{1-\beta} dy = \int_0^{K/2} \frac{dy}{y}.$$

From this, it is obvious that $\Sigma(0) = \infty$.

$N(0)$**:** The definition of s and m in (2.1) yield

$$N(0) = \frac{2}{\alpha^2} \int_0^{K/2} \left(\frac{K}{y} - 1\right)^{\beta} \left(\int_0^{y} \frac{z^{\beta-2}}{(K-z)^{\beta+2}} dz \right) dy.$$

The inner integral

$$\int_0^{y} \frac{z^{\beta-2}}{(K-z)^{\beta+2}} dz \tag{2.5}$$

is finite if and only if $\beta > 1$. When $\beta \le 1$ we have $N(0) = \infty$. However, if $\beta > 1$ then the integral (2.5) behaves like $y^{\beta-1}$, as $y \downarrow 0$. Therefore the behavior of $N(0)$ is the same as the integral

$$\int_0^{K/2} y^{-\beta} \cdot y^{\beta-1} dy = \int_0^{K/2} \frac{dy}{y}.$$

In this way, in any case we conclude that $N(0) = \infty$.

Now let us deal with the boundary K.

$S(K)$**:** By Lemma 2.1 we get

$$S(K) = \int_{K/2}^{K} \left(\frac{K-x}{x}\right)^{\beta} dx.$$

Using the fact that

$$\left(\frac{K-x}{x}\right)^{\beta} = O\left((K-x)^{\beta}\right), \quad \text{as } x \uparrow K,$$

we can conclude that $S(K) < \infty$ if and only if $1+\beta > 0$.

$\Sigma(K)$**:** In this case, the functional Σ at the point K is written in the following manner

$$\begin{aligned}
\Sigma(K) &= \int_{K/2}^{K} \left(\int_{y}^{K} s(z)dz \right) m(y)dy \\
&= \frac{2}{\alpha^2} \int_{K/2}^{K} \frac{y^{\beta-2}}{(K-y)^{\beta+2}} \left(\int_{y}^{K} \frac{(K-z)^{\beta}}{z^{\beta}} dz \right) dy \\
&\geq \frac{2}{\alpha^2} \left(\min_{\frac{K}{2} \leq y \leq K} y^{\beta-2} \right) \left(\min_{\frac{K}{2} \leq z \leq K} z^{-\beta} \right) \cdot \int_{K/2}^{K} (K-y)^{-\beta-2} \int_{y}^{K} (K-z)^{\beta} dz dy.
\end{aligned}$$

If $1+\beta \leq 0$, then $\Sigma(K) = \infty$. Now, if $1+\beta > 0$ then the behavior of $\Sigma(K)$ depends on the value of the integral

$$\int_{K/2}^{K} (K-y)^{-\beta-2} \cdot (K-y)^{\beta+1} dy = \int_{K/2}^{K} \frac{dy}{K-y},$$

which is infinite. Thus, in any case $\Sigma(K) = \infty$.

$N(K)$**:** In this case we have

$$\begin{aligned}
N(K) &= \int_{K/2}^{K} \left(\int_{y}^{K} m(z)dz \right) s(y)dy \\
&= \frac{2}{\alpha^2} \int_{K/2}^{K} \left(\int_{y}^{K} \frac{z^{\beta-2}}{(K-z)^{\beta+2}} dz \right) \left(\frac{K-y}{y} \right)^{\beta} dy \\
&\geq \frac{2}{\alpha^2} \left(\min_{\frac{K}{2} \leq z \leq K} z^{\beta-2} \right) \left(\min_{\frac{K}{2} \leq y \leq K} y^{-\beta} \right) \cdot \int_{K/2}^{K} (K-y)^{\beta} \int_{y}^{K} (K-z)^{-\beta-2} dz dy.
\end{aligned}$$

If $\beta+1 \geq 0$, then $N(K) = \infty$. In the case $\beta+1 < 0$, the behavior of $N(K)$ depends on

$$\int_{K/2}^{K} (K-y)^{\beta} \cdot (K-y)^{-\beta-1} dy = \int_{K/2}^{K} \frac{dy}{K-y}.$$

Therefore, we achieve, $N(K) = \infty$.

Thus, all cases have been considered, and the result is proven. □

With the above result we can elucidate the following probabilities.

Proposition 2.3. *Let X be the unique solution of (1.3), with $X_0 = x_0 \in (0, K)$ and $r \in \mathbb{R}$. For the boundary 0 we have*

$$P\left\{ \lim_{t\to\infty} X_t > 0 \right\} = 1, \quad \text{if} \quad \beta \geq 1,$$
$$P\left\{ \lim_{t\to\infty} X_t = 0 \right\} > 0, \quad \text{if} \quad \beta < 1,$$

and for the boundary K we obtain

$$P\left\{ \lim_{t\to\infty} X_t = K \right\} > 0, \quad \text{if} \quad \beta > -1,$$
$$P\left\{ \lim_{t\to\infty} X_t < K \right\} = 1, \quad \text{if} \quad \beta \leq -1.$$

Proof. By Lemma 2.2 we know that $\Sigma(0) = \infty$, $N(0) = \infty$ and $\Sigma(K) = \infty$, $N(K) = \infty$, then 0 and K are natural boundaries, see Section 6 of Chapter 15 in [13]. Consequently, 0 and K are not regular boundaries, then we can apply Theorem 7.3, of the previous reference, to obtain

$$S(0) < \infty \quad \text{if and only if} \quad P\left\{\lim_{t\to\infty} X_t = 0\right\} > 0,$$

and

$$S(K) < \infty \quad \text{if and only if} \quad P\left\{\lim_{t\to\infty} X_t = K\right\} > 0.$$

Using Lemma 2.2 and the above equivalences, we conclude the proof. □

In this part it is important to point out the following well known result.

Proposition 2.4. *Let X be the unique solution of (1.3), with $X_0 = x_0 \in (0, K)$ and $r \in \mathbb{R}$. Then*

$$P\left\{\lim_{t\to\infty} X_t = K\right\} = 1, \quad \text{if} \quad \beta > 1,$$

$$P\left\{\lim_{t\to\infty} X_t = 0\right\} = 1, \quad \text{if} \quad \beta < -1,$$

and for $\beta \in (-1, 1)$

$$P\left\{\lim_{t\to\infty} X_t = K\right\} = p^+(x_0), \tag{2.6}$$

$$P\left\{\lim_{t\to\infty} X_t = 0\right\} = 1 - p^+(x_0), \tag{2.7}$$

where

$$p^+(x_0) := \frac{\int_0^{x_0} s(y)dy}{\int_0^K s(y)dy}. \tag{2.8}$$

Proof. This result is contained in Theorem 2.5 of [15]. □

The results included in Proposition 2.4 are more accurate than those obtained in Proposition 2.3. Here our contribution is in the limiting case $\beta = 1$, which is not considered previously. On the other hand, if $-1 < \beta < 1$, then from Proposition 2.4 we see that the stochastic process X approaches 0 in infinite time, with probability $1 - p^+(x_0) > 0$. That is, in this case, with positive probability the stochastic process X approaches 0 at infinity. Therefore, if $x > x_0$ it can happen with positive probability that the stochastic process X does not approach x, this means that with positive probability the time of the first-passage time to x is infinite. Indeed, this occurs as proved in Theorem 4.2.

3. A special monotone solution

Let us consider the sequence $\{u_n\}_{n=0}^{\infty}$ of real-valued functions defined in $(0, K)$ as $u_0 \equiv 1$ and

$$u_n(x) := \int_{K/2}^{x} s(y) \int_{K/2}^{y} u_{n-1}(z)m(z)dzdy. \tag{3.1}$$

In what follows the function

$$u_1(x) := \int_{K/2}^{x} s(y) \int_{K/2}^{y} \frac{2}{s(z)g(z)}dzdy$$

will be of special interest.

Lemma 3.1. *For each $\lambda > 0$, the series*

$$u(\lambda, x) := \sum_{n=0}^{\infty} \lambda^n u_n(x), \quad x \in (0, K),$$

converges uniformly on compact subsets of $(0, K)$. Furthermore, $u(\lambda, \cdot)$ is decreasing (increasing) in $(0, K/2)$ (respectively, in $(K/2, K)$) and satisfies the differential equation

$$\frac{g(x)}{2}\frac{\partial^2 u}{\partial x^2}(\lambda, x) + f(x)\frac{\partial u}{\partial x}(\lambda, x) - \lambda u(\lambda, x) = 0, \quad x \in (0, K), \tag{3.2}$$

$$u(\lambda, K/2) = 1, \quad \frac{\partial u}{\partial x}(\lambda, K/2) = 0. \tag{3.3}$$

Even more, for each $\lambda > 0$ and $x \in (0, K)$,

$$1 + \lambda u_1(x) \le u(\lambda, x) \le e^{\lambda u_1(x)}, \tag{3.4}$$

$$\frac{\partial u}{\partial x}(\lambda, x) = \lambda s(x) \int_{K/2}^{x} u(\lambda, y) m(y) dy, \tag{3.5}$$

$$u_1(x) \le \frac{\partial u}{\partial \lambda}(\lambda, x) = \sum_{n=1}^{\infty} n\lambda^{n-1} u_n(x) \le u_1(x) e^{\lambda u_1(x)}, \tag{3.6}$$

$$2u_2(x) \le \frac{\partial^2 u}{\partial \lambda^2}(\lambda, x) = \sum_{n=2}^{\infty} n(n-1)\lambda^{n-2} u_n(x) \le \left(u_1(x)\right)^2 e^{\lambda u_1(x)}. \tag{3.7}$$

Proof. The monotony property of $u(\lambda, \cdot)$ follows easily from the definition of u_n. Note also that $u_n \ge 1$, $n \in \mathbb{N}$. For each $x \in (0, K)$ we can verify, by mathematical induction, that

$$u_n(x) \le \frac{\left(u_1(x)\right)^n}{n!}, \quad n = 0, 1, 2, \dots. \tag{3.8}$$

From (3.8) we have the first part of inequality (3.4),

$$u(\lambda, x) = \sum_{n=0}^{\infty} \lambda^n u_n(x) \le \sum_{n=0}^{\infty} \frac{\left(\lambda u_1(x)\right)^n}{n!} = e^{\lambda u_1(x)}, \quad x \in (0, K).$$

Using the M-test of Weierstrass we can conclude that $\sum_{n=0}^{\infty} \lambda^n u_n(x)$ converges uniformly on compact subsets of $(0, \infty)$, with respect to the parameter λ and uniformly on compact subsets of $(0, K)$, with respect to the variable x.

For each $k \in \mathbb{N}$ we deduce from (3.8) that

$$n! \lambda^{n-k} u_n(x) \le \left(u_1(x)\right)^k \left(\lambda u_1(x)\right)^{n-k}, \quad n \ge k. \tag{3.9}$$

We can use the inequality (3.9) to see that $\sum_{n=1}^{\infty} n\lambda^{n-1} u_n(x)$ converge uniformly on compact subsets of $(0, \infty)$, and by a classical result on analysis (see [24]) we get

$$\frac{\partial u}{\partial \lambda}(\lambda, x) = \sum_{n=1}^{\infty} n\lambda^{n-1} u_n(x) \le u_1(x) e^{\lambda u_1(x)}, \quad x \in (0, K).$$

This yields the inequality (3.6). Using (3.9) we can proceed in the same way to verify the inequality (3.7).

Now let us deal with the differentiability of u with respect to x. From (3.1) we get

$$u_n'(x) = s(x) \int_{K/2}^{x} u_{n-1}(z) m(z) dz, \quad x \in (0, K). \tag{3.10}$$

From (3.10), (3.8) and the monotone property of $u_1(x)$ we deduce

$$\left|u_n'(x)\right| \le \left|u_1'(x)\right| \frac{\left(u_1(x)\right)^{n-1}}{(n-1)!}, \quad n = 1, 2, \dots. \tag{3.11}$$

The inequality (3.11) implies that $\sum_{n=1}^{\infty} \lambda^n u_n'(x)$ converges uniformly on compact subsets of $(0, K)$, then

$$\frac{\partial u}{\partial x}(\lambda, x) = \sum_{n=1}^{\infty} \lambda^n u_n'(x), \quad x \in (0, K).$$

On the other hand, since the functions f and g are continuous in $(0, K)$ and $g > 0$ (see (2.2)) we can use

$$\frac{g(x)}{2}\left(\lambda^n u_n''(x)\right) = \lambda\left(\lambda^{n-1} u_{n-1}(x)\right) - f(x)\left(\lambda^n u_n'(x)\right), \quad n = 1, 2, \dots, \tag{3.12}$$

to verify that $\sum_{n=1}^{\infty} \lambda^n u_n''(x)$ also converges uniformly on compact subsets of $(0, K)$. Thus

$$\frac{\partial^2 u}{\partial x^2}(\lambda, x) = \sum_{n=1}^{\infty} \lambda^n u_n''(x), \quad x \in (0, K).$$

Using equality (3.12), we prove that $u(\lambda, \cdot)$ is a solution of the differential equation (3.2). From the definition of u_n we see that $u(\lambda, K/2) = 1$ (the first initial condition in (3.3)) and $u_n(x) \ge 0$ implies

$$u(\lambda, x) = \sum_{n=0}^{\infty} \lambda^n u_n(x) \geq 1 + \lambda u_1(x), \quad x \in (0, K),$$

which is the missing part in the inequality (3.4).

Since $u_n(x) \geq 0$ we have, by the monotone convergence theorem, and (3.10)

$$\begin{aligned}\frac{\partial u}{\partial x}(\lambda, x) &= \lambda s(x) \sum_{n=1}^{\infty} \int_{K/2}^{x} \lambda^{n-1} u_{n-1}(z) m(z) dz \\ &= \lambda s(x) \int_{K/2}^{x} \left(\sum_{n=1}^{\infty} \lambda^{n-1} u_{n-1}(z) \right) m(z) dz \\ &= \lambda s(x) \int_{K/2}^{x} u(\lambda, z) m(z) dz.\end{aligned}$$

This is precisely equality (3.5). Moreover, from here we obtain the initial condition that was missing in (3.3), $(\partial u/\partial x)(\lambda, K/2) = 0$. □

In what follows we will need only the second derivative of $u(\lambda, x)$ with respect to λ, but from (3.9) we can see that $u(\cdot, x)$ is in $C^\infty(0, \infty)$. As we have seen, the function $u(\lambda, \cdot)$ is a solution of (3.2) and is monotonous in parts, it is decreasing in $(0, K/2)$ and increasing in $(K/2, K)$. In order to apply the monotone convergence theorem, in the arguments that we will use in the next section, we look for a solution of (3.2) that is monotone in its entire domain, that is, $(0, K)$. This special solution is given in the following result.

Lemma 3.2. *Let $\lambda > 0$ be fixed. The function*

$$v(\lambda, x) := u(\lambda, x) \int_0^x \frac{s(y)}{u^2(\lambda, y)} dy, \quad x \in (0, K), \tag{3.13}$$

is monotone increasing and it is a solution of (3.2). For each $\lambda > 0$ and $x \in (0, K)$ we also have

$$\frac{\partial v}{\partial \lambda}(\lambda, x) = \int_0^x \frac{s(y)}{u^2(\lambda, y)} dy \, \frac{\partial u}{\partial \lambda}(\lambda, x) - 2 u(\lambda, x) \int_0^x \frac{s(y)}{u^3(\lambda, y)} \frac{\partial u}{\partial \lambda}(\lambda, y) dy \tag{3.14}$$

and

$$\begin{aligned}\frac{\partial^2 v}{\partial \lambda^2}(\lambda, x) &= \int_0^x \frac{s(y)}{u^2(\lambda, y)} dy \, \frac{\partial^2 u}{\partial \lambda^2}(\lambda, x) - 4 \int_0^x \frac{s(y)}{u^3(\lambda, y)} \frac{\partial u}{\partial \lambda}(\lambda, y) dy \, \frac{\partial u}{\partial \lambda}(\lambda, x) \\ &\quad + 6 u(\lambda, x) \int_0^x \frac{s(y)}{u^4(\lambda, y)} \left(\frac{\partial u}{\partial \lambda}(\lambda, y) \right)^2 dy - 2 u(\lambda, x) \int_0^x \frac{s(y)}{u^3(\lambda, y)} \frac{\partial^2 u}{\partial \lambda^2}(\lambda, y) dy.\end{aligned} \tag{3.15}$$

Proof. First we will see that the right hand side of (3.13) is monotonous. From (3.5), it follows that

$$\begin{aligned}\frac{\partial}{\partial y}(u(\lambda, y))^{-1} &= -(u(\lambda, y))^{-2} \frac{\partial u}{\partial y}(\lambda, y) \\ &= \frac{s(y)}{u^2(\lambda, y)} \lambda \int_y^{K/2} u(\lambda, z) m(z) dz.\end{aligned} \tag{3.16}$$

The derivative of the right hand side of (3.13) is

$$\begin{aligned}&\frac{s(x)}{u(\lambda, x)} + \frac{\partial u}{\partial x}(\lambda, x) \cdot \int_0^x \frac{s(y)}{u^2(\lambda, y)} dy \\ &= \frac{s(x)}{u(\lambda, x)} - \lambda s(x) \int_x^{K/2} u(\lambda, z) m(z) dz \cdot \int_0^x \frac{\frac{\partial}{\partial y}(u(\lambda, y))^{-1}}{\lambda \int_y^{K/2} u(\lambda, z) m(z) dz} dy\end{aligned}$$

$$= \frac{s(x)}{u(\lambda, x)} - s(x) \int_0^x \frac{\int_x^{K/2} u(\lambda, z) m(z) dz}{\int_y^{K/2} u(\lambda, z) m(z) dz} \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy$$

$$= s(x) \left\{ \int_0^x \left[1 - \frac{\int_x^{K/2} u(\lambda, z) m(z) dz}{\int_y^{K/2} u(\lambda, z) m(z) dz} \right] \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy + \frac{1}{u(\lambda, 0)} \right\}.$$

Observe that in the previous equality we have used that $u(\lambda, \cdot)$ is monotonously decreasing in $(0, K/2)$, therefore the limit $\lim_{x \downarrow 0} u(\lambda, x) = u(\lambda, 0) \geq 1$ exists, see (3.4).

If $0 < x < K/2$, from (3.16) we see that $(\partial/\partial y)(u(\lambda, y))^{-1} \geq 0$, $0 < y < x$, and

$$\int_x^{K/2} u(\lambda, z) m(z) dz \leq \int_y^{K/2} u(\lambda, z) m(z) dz,$$

implies that the right hand side of (3.13) is monotonously increasing in $(0, K/2)$. On the other hand, let us suppose that $K/2 \leq x < K$, and examine

$$\int_0^x \left[1 - \frac{\int_x^{K/2} u(\lambda, z) m(z) dz}{\int_y^{K/2} u(\lambda, z) m(z) dz} \right] \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy$$

$$= \left(\int_0^{K/2} + \int_{K/2}^x \right) \left[1 - \frac{\int_x^{K/2} u(\lambda, z) m(z) dz}{\int_y^{K/2} u(\lambda, z) m(z) dz} \right] \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy$$

$$= \int_0^{K/2} \left[1 + \frac{\int_{K/2}^x u(\lambda, z) m(z) dz}{\int_y^{K/2} u(\lambda, z) m(z) dz} \right] \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy$$

$$+ \int_{K/2}^x \left[1 - \frac{\int_{K/2}^x u(\lambda, z) m(z) dz}{\int_{K/2}^y u(\lambda, z) m(z) dz} \right] \frac{\partial}{\partial y} (u(\lambda, y))^{-1} \, dy. \tag{3.17}$$

The first integral in (3.17) is evidently positive. Additionally, if $K/2 \leq x < K$, then $(\partial/\partial y)(u(\lambda, y))^{-1} \leq 0$, $K/2 < y < x$, and

$$\int_{K/2}^x u(\lambda, z) m(z) dz \geq \int_{K/2}^y u(\lambda, z) m(z) dz,$$

therefore the integrand in the second integral in (3.17) is positive. In this way, we can see that $v(\lambda, \cdot)$ is well defined and it is monotonously increasing in $(0, K)$.

Performing a direct calculation we can verify that $v(\lambda, \cdot)$ is a solution of equation (3.2).

By Lemma 3.1 we see that the differentiability of $v(\cdot, x)$ follows from the differentiability of the function

$$\lambda \mapsto \int_0^x \frac{s(y)}{u^2(\lambda, y)} dy.$$

To verify that such function is differentiable we observe that

$$\left| \frac{\partial}{\partial \lambda} \frac{1}{u^2(\lambda, y)} \right| = \left| \frac{-2}{u^3(\lambda, y)} \frac{\partial}{\partial \lambda} u(\lambda, y) \right|$$
$$\leq 2 \left| \frac{\partial}{\partial \lambda} u(\lambda, y) \right| \leq 2 u_1(y) e^{\lambda u_1(y)}, \quad \lambda \in (0, \infty),$$

where we have used (3.4) and (3.6). Since $u_1 e^{\lambda u_1}$ is integrable in $(0, x)$, then we can apply the differentiation Lebesgue theorem to deduce the desired differentiability. From this (3.14) follows readily. Similarly, using (3.4) and (3.7) we can deduce the formula (3.15). □

4. The first and second moment of the first-passage time

Let X be the unique solution of stochastic differential equation (1.3) that starts at x_0, $X_0 = x_0$. Recall that by T_x we denote the time of the first visit of stochastic process X to state $x \in (0, K)$, when beginning at x_0. We begin this section by determining the Laplace transform of T_x, which is given in terms of the monotonic increasing function $v(\lambda, x)$, found in the previous section.

Proposition 4.1. *Let $x_0 \in (0, K)$ be the initial state of process X, $X_0 = x_0$, and let $v(\lambda, x)$ be the function defined in (3.13), then*

$$E\left[e^{-\lambda T_x}; T_x < \infty\right] = \frac{v(\lambda, x_0)}{v(\lambda, x)}, \quad x \in (x_0, K),$$

for each $\lambda \in (0, \infty)$.

Proof. By Theorem 3 in [15] we know that $X_t \in (0, K)$, $t \geq 0$, almost surely. Using Itô's formula we get

$$\begin{aligned} & v(\lambda, X_{t\wedge T_x})e^{-\lambda(t\wedge T_x)} \\ &= v(\lambda, x_0) + \int_0^{t\wedge T_x} \left[\frac{g(X_s)}{2}\frac{\partial^2 v}{\partial x^2}(\lambda, X_s) + f(X_s)\frac{\partial v}{\partial x}(\lambda, X_s) - \lambda v(\lambda, X_s)\right] e^{-\lambda s} ds \\ &\quad + \int_0^{t\wedge T_x} \frac{\partial v}{\partial x}(\lambda, X_s)\, e^{-\lambda s}\, \alpha(K - X_s)\, X_s\, dB_s. \end{aligned}$$

By Lemma 3.2 we know that $v(\lambda, \cdot)$ is a solution of (3.2), then

$$v(\lambda, X_{t\wedge T_x})e^{-\lambda(t\wedge T_x)} - v(\lambda, x_0) = \int_0^{t\wedge T_x} \frac{\partial v}{\partial s}(\lambda, X_s)\, e^{-\lambda s}\, \alpha(K - X_s)\, X_s\, dB_s. \tag{4.1}$$

Since $X_{t\wedge T_x} \leq x$, $t > 0$, Lemma 3.2 and the former equality implies

$$\left|\int_0^{t\wedge T_x} \frac{\partial v}{\partial s}(\lambda, X_s)\, e^{-\lambda s}\, \alpha(K - X_s)\, X_s\, dB_s\right| \leq v(\lambda, x),$$

thus the corresponding stochastic integral is a bounded local martingale. Because a bounded local martingale is a martingale (see Corollary 2.6 in Chapter 2 of [6]) we have that the stochastic integral is a martingale. Taken expectation in (4.1), we obtain

$$\begin{aligned} v(\lambda, x_0) &= E\left[v(\lambda, X_{t\wedge T_x})e^{-\lambda(t\wedge T_x)}\right] \\ &= E\left[v(\lambda, X_{t\wedge T_x})e^{-\lambda(t\wedge T_x)} 1_{\{T_x < \infty\}}\right] + E\left[v(\lambda, X_t)e^{-\lambda t} 1_{\{T_x = \infty\}}\right]. \end{aligned} \tag{4.2}$$

Using again Lemma 3.2 we get

$$\left|v(\lambda, X_{t\wedge T_x(\omega)}(\omega))\right| \leq \sup_{0<y\leq x} v(\lambda, y) = v(\lambda, x), \quad \omega \in \{T_x = \infty\}.$$

If we take the limit in (4.2), as $t \to \infty$, we have by the dominated convergence theorem

$$\begin{aligned} v(\lambda, x_0) &= E\left[v(\lambda, X_{T_x})e^{-\lambda T_x} 1_{\{T_x < \infty\}}\right] \\ &= E\left[v(\lambda, x)e^{-\lambda T_x}; T_x < \infty\right]. \end{aligned}$$

From here, the conclusion of the theorem is straightforward. □

Let us denote the Laplace transform of T_x by

$$\phi_x(\lambda) := E\left[e^{-\lambda T_x}; T_x < \infty\right], \quad \lambda > 0.$$

With this notation we have our main result.

Theorem 4.2. *Let X be the solution of (1.3), $X_0 = x_0 \in (0, K)$ and $x \in (0, K)$, with $x_0 < x$. If $\beta < 1$ then*

$$P\{T_x < \infty\} = \frac{\int_0^{x_0} s(y)dy}{\int_0^x s(y)dy},$$

$$\begin{aligned} E\left[T_x; T_x < \infty\right] &= \frac{\int_0^{x_0} s(y)dy}{\left(\int_0^x s(y)dy\right)^2} \int_0^x s(y)[u_1(x) - 2u_1(y)]dy \\ &\quad - \frac{1}{\int_0^x s(y)dy} \int_0^{x_0} s(y)[u_1(x_0) - 2u_1(y)]dy \end{aligned}$$

and

$$E\left[(T_x)^2; T_x<\infty\right] = \frac{1}{\int_0^x s(y)dy}\int_0^{x_0} s(y)[2u_2(x_0) - 4u_1(y)u_1(x_0) + 6(u_1(y))^2 - 4u_2(y)]dy$$
$$- \frac{\int_0^{x_0} s(y)dy}{\left(\int_0^x s(y)dy\right)^2}\int_0^x s(y)[2u_2(x) - 4u_1(y)u_1(x) + 6(u_1(y))^2 - 4u_2(y)]dy$$
$$- \frac{2}{\left(\int_0^x s(y)dy\right)^2}\int_0^x s(y)[u_1(x) - 2u_1(y)]dy\int_0^{x_0} s(y)[u_1(x_0) - 2u_1(y)]dy$$
$$+ \frac{2\int_0^{x_0} s(y)dy}{\left(\int_0^x s(y)dy\right)^3}\left(\int_0^x s(y)[u_1(x) - 2u_1(y)]dy\right)^2.$$

Proof. By estimating the integrals according to their order of convergence (as done in the proof of Proposition 2.3), we can verify that the expression on the right hand side of each moment is finite.

In what follows we will use the dominated convergence theorem repeatedly. We observe that

$$P\{T_x<\infty\} = \lim_{\lambda\downarrow 0}\phi_x(\lambda), \quad E\left[T_x; T_x<\infty\right] = -\lim_{\lambda\downarrow 0}\phi_x'(\lambda)$$

and

$$E\left[(T_x)^2; T_x<\infty\right] = \lim_{\lambda\downarrow 0}\phi_x''(\lambda).$$

Using Proposition 4.1 we obtain

$$E\left[T_x; T_x<\infty\right] = -\lim_{\lambda\downarrow 0}\left\{\frac{1}{v(\lambda,x)}\cdot\frac{\partial v}{\partial\lambda}(\lambda,x_0) - \frac{v(\lambda,x_0)}{(v(\lambda,x))^2}\cdot\frac{\partial v}{\partial\lambda}(\lambda,x)\right\},$$

and

$$E\left[(T_x)^2; T_x<\infty\right] = \lim_{\lambda\downarrow 0}\left\{\frac{1}{v(\lambda,x)}\cdot\frac{\partial^2 v}{\partial\lambda^2}(\lambda,x_0) - \frac{v(\lambda,x_0)}{(v(\lambda,x))^2}\cdot\frac{\partial^2 v}{\partial\lambda^2}(\lambda,x)\right.$$
$$\left. - \frac{2}{(v(\lambda,x))^2}\cdot\frac{\partial v}{\partial\lambda}(\lambda,x)\cdot\frac{\partial v}{\partial\lambda}(\lambda,x_0) + \frac{2v(\lambda,x_0)}{(v(\lambda,x))^3}\left(\frac{\partial v}{\partial\lambda}(\lambda,x)\right)^2\right\}.$$

From (3.4), (3.6) and (3.7) we calculate the following limits

$$\lim_{\lambda\downarrow 0} u(\lambda,x) = 1,$$
$$\lim_{\lambda\downarrow 0}\frac{\partial u}{\partial\lambda}(\lambda,x) = u_1(x),$$
$$\lim_{\lambda\downarrow 0}\frac{\partial^2 u}{\partial\lambda^2}(\lambda,x) = 2u_2(x).$$

Using the above limits and (3.13), (3.14), (3.15) we have

$$\lim_{\lambda\downarrow 0} v(\lambda,x) = \int_0^x s(y)dy,$$
$$\lim_{\lambda\downarrow 0}\frac{\partial v}{\partial\lambda}(\lambda,x) = \int_0^x s(y)dy\, u_1(x) - 2\int_0^x s(y)u_1(y)dy,$$
$$\lim_{\lambda\downarrow 0}\frac{\partial^2 v}{\partial\lambda^2}(\lambda,x) = 2\int_0^x s(y)dy\, u_2(x) - 4\int_0^x s(y)u_1(y)dy\, u_1(x)$$
$$+ 6\int_0^x s(y)(u_1(y))^2dy - 4\int_0^x s(y)u_2(y)dy.$$

Using such limits, the desired result is accomplished. □

Now let us deal with the moments of the first-passage time T_x of the processes X when $0 < x < x_0$. In this case, for $\lambda > 0$, we introduce the function

$$w(\lambda, x) := u(\lambda, x) \int_x^K \frac{s(y)}{u^2(\lambda, y)} dy, \quad x \in (0, K).$$

As before, we can see that $w(\lambda, \cdot)$ is monotone decreasing and it is a solution of (3.2), for each $\lambda > 0$. If $X_0 = x_0 \in (0, K)$ we have

$$E\left[e^{-\lambda T_x}; T_x < \infty\right] = \frac{w(\lambda, x_0)}{w(\lambda, x)}, \quad x \in (0, x_0).$$

We also achieve an analog of Theorem 4.2.

Theorem 4.3. *Let X be the solution of (1.3), $X_0 = x_0 \in (0, K)$ and $x \in (0, K)$, with $x_0 > x$. If $\beta > -1$ then*

$$P\{T_x < \infty\} = \frac{\int_{x_0}^K s(y)dy}{\int_x^K s(y)dy},$$

$$E\left[T_x; T_x < \infty\right] = \frac{\int_{x_0}^K s(y)dy}{\left(\int_x^K s(y)dy\right)^2} \int_x^K s(y)[u_1(x) - 2u_1(y)]dy - \frac{1}{\int_x^K s(y)dy} \int_{x_0}^K s(y)[u_1(x_0) - 2u_1(y)]dy$$

and

$$\begin{aligned} E\left[(T_x)^2; T_x < \infty\right] = & \frac{1}{\int_x^K s(y)dy} \int_{x_0}^K s(y)[2u_2(x_0) - 4u_1(y)u_1(x_0) + 6(u_1(y))^2 - 4u_2(y)]dy \\ & - \frac{\int_{x_0}^K s(y)dy}{\left(\int_x^K s(y)dy\right)^2} \int_x^K s(y)[2u_2(x) - 4u_1(y)u_1(x) + 6(u_1(y))^2 - 4u_2(y)]dy \\ & - \frac{2}{\left(\int_x^K s(y)dy\right)^2} \int_x^K s(y)[u_1(x) - 2u_1(y)]dy \int_{x_0}^K s(y)[u_1(x_0) - 2u_1(y)]dy \\ & + \frac{2\int_{x_0}^K s(y)dy}{\left(\int_x^K s(y)dy\right)^3} \left(\int_x^K s(y)[u_1(x) - 2u_1(y)]dy\right)^2. \end{aligned}$$

5. Some real world problems

In this section, we will address the estimation of the first arrival time of two stochastic processes related to real problems. The first deals with the study of the growth of cancerous tumors, and the second deals with the problem of world population growth. In both cases, there is the option that the process dies out at infinity or reaches its level of charge capacity at infinity. It should be noted that this dichotomy is not observed in the deterministic model. Here we can say with what probability one level 0 or K can be reached at infinity.

Calculating the time of first arrival, T_x, is crucial for our applications. In the subsequent subsection, we introduce a scheme that effectively tackles this task, a method we believe will prove highly valuable.

5.1. Numerical considerations

We need to evaluate integrals in the following form:

$$\int_0^{x_0} s(x)dx, \quad u_1(x) = \int_{x_0}^x \int_{x_0}^y s(y)m(z)dz dy \quad \text{and}$$

$$\int_0^x s(y)u_1(y)dy = \int_0^x \int_{x_0}^y \int_{x_0}^w s(y)s(w)m(z)dzdwdy.$$

Due to the presence of a singularity at 0 in the function $s(x) = (\frac{K-x}{x})^\beta$, its evaluation is not straightforward. Classical methods or software are typically tailored for computing integrals with smoother integrands. Here, we introduce an evaluation method that proves highly advantageous, particularly when computing integrals dependent on three variables. This method becomes crucial, given that even widely used software (such as Octave, Mathematica, or Matlab) has failed to evaluate it.

Given $m \in \mathbb{N} \cup \{\infty\}$, $n \in \mathbb{N} \cup \{0\}$, $j \in \{0,1\}$ and $\gamma \in \mathbb{R}$, we define the symbol

$$S_{n=j}^m(\gamma) = \sum_{n=j}^m (-K)^{-n} \binom{\gamma}{n}, \quad K > 0.$$

By employing Mertens' theorem for the product of series, we can justify the interchange of the integral and the sum in the subsequent evaluations:

$$\int_0^{x_0} s(x)dx = K^\beta S_{n=0}^\infty(\beta) \frac{(x_0)^{n-\beta+1}}{n-\beta+1},$$

$$\begin{aligned} u_1(x) = \frac{2}{(\alpha K)^2} \Bigg[& \frac{1}{\beta-1} \left\{ \ln x - \ln(K/2) - \frac{(K/2)^{\beta-1}}{1-\beta}(x^{1-\beta} - (K/2)^{1-\beta}) \right\} \\ & + S_{n=1}^\infty(-2-\beta) \frac{1}{n+\beta-1} \left\{ \frac{x^n - (K/2)^n}{n} - \frac{(K/2)^{n+\beta-1}}{1-\beta}(x^{1-\beta} - (K/2)^{1-\beta}) \right\} \\ & + S_{j=1}^\infty(\beta) S_{n=0}^\infty(-2-\beta) \frac{1}{n+\beta-1} \\ & \times \left\{ \frac{x^{n+j} - (K/2)^{n+j}}{n+j} - \frac{(K/2)^{n+\beta-1}}{j-\beta+1}(x^{j-\beta+1} - (K/2)^{j-\beta+1}) \right\} \Bigg] \end{aligned}$$

and

$$\begin{aligned} \int_0^x s(y)u_1(y)dy = \frac{2K^\beta}{(\alpha K)^2} S_{i=0}^\infty(\beta) \Bigg[& \frac{1}{\beta-1} \left\{ \frac{x^{i-\beta+1}}{i-\beta+1} \left(\ln x - \frac{1}{i-\beta+1} \right) \right. \\ & \left. - \frac{x^{i-\beta+1}\ln(K/2)}{i-\beta+1} - \frac{(K/2)^{\beta-1}}{1-\beta} \left(\frac{x^{i-2\beta+2}}{i-2\beta+2} - \frac{x^{i-\beta+1}(K/2)^{1-\beta}}{i-\beta+1} \right) \right\} \\ & + S_{n=1}^\infty(-2-\beta) \frac{1}{n+\beta-1} \left\{ \frac{1}{n} \left(\frac{x^{n+i-\beta+1}}{n+i-\beta+1} - \frac{x^{i-\beta+1}(K/2)^n}{i-\beta+1} \right) \right. \\ & \left. - \frac{(K/2)^{n+\beta-1}}{1-\beta} \left(\frac{x^{i+2-2\beta}}{i+2-2\beta} - \frac{x^{i-\beta+1}(K/2)^{1-\beta}}{i-\beta+1} \right) \right\} \\ & + S_{j=1}^\infty(\beta) S_{n=0}^\infty(-2-\beta) \frac{1}{n+\beta-1} \\ & \times \left\{ \frac{1}{n+j} \left(\frac{x^{n+j+i-\beta+1}}{n+j+i-\beta+1} - \frac{x^{i-\beta+1}(K/2)^{n+j}}{i-\beta+1} \right) \right. \\ & \left. - \frac{(K/2)^{n+\beta-1}}{j-\beta+1} \left(\frac{x^{j-2\beta+2+i}}{j-2\beta+2+i} - \frac{x^{i-\beta+1}(K/2)^{j-\beta+1}}{i-\beta+1} \right) \right\} \Bigg]. \end{aligned}$$

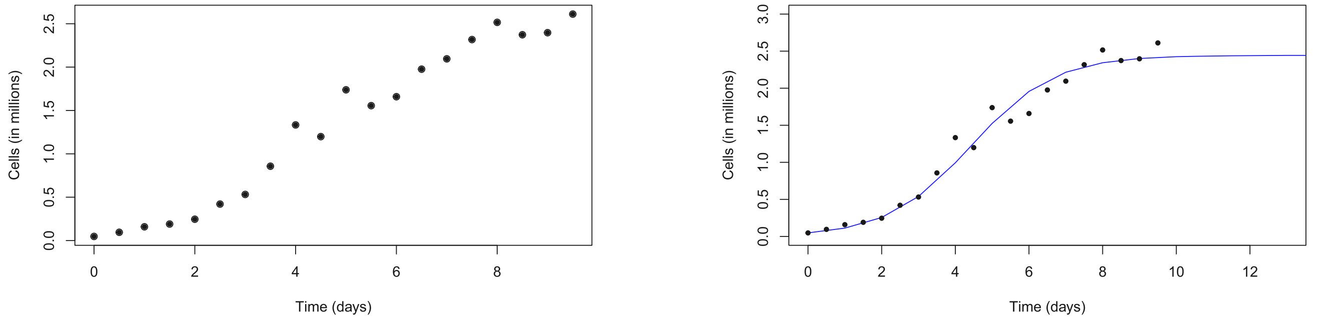
5.2. Cancer tumor growth

It is well known that the growth of tumors in humans is highly dependent on the type of tumor. The most frequent types of growth are exponential and of sigmoidal type growth (see [17]). In our case, we are interested in tumors that grow sigmoidally, since this type of growth is modeled very well by a logistic differential equation. Indeed, this is the case of the growth of a Jurkat T human leukemia, the data from a sample is plotted in Fig. 1(a) (see [23]).

To fit the data to a logistic function of the form (1.2) we consider a new variable $\tilde{r} = rK$ and use the function *nls* (nonlinear least squares) of software R. To be able to apply this function we assume initial conditions $(0, 47.619 \times 10^{-3})$, $K = 3$ and $\tilde{r} = 1$. Doing the above, we find the fit values $K = 2.442$ and $\tilde{r} = 0.885$. With these quantities we obtain the sigmoidal function

$$X_t = \frac{11.633 \times 10^{-2}}{47.619 \times^{-3} + 2.395\, e^{-0.885\, t}}, \quad t \geq 0. \tag{5.1}$$

Its graph appears in blue in Fig. 1(b).



(a) A sample of Jurkat T human leukemia growth.

(b) In blue the graph of the sigmoidal function (5.1).

Fig. 1. Modeling the growth of a cancerous tumor.

Table 1
Probabilities and expected values of first-passage times upper x_0 of a tumor growth.

α	x	0.1	0.5	1	1.5	2
0.65	$P[T_x < \infty]$	0.804	0.512	0.432	0.398	0.382
	$E\left[T_x \mid T_x < \infty\right]$	1.985	6.349	8.313	9.554	10.607
0.75	$P[T_x < \infty]$	0.706	0.340	0.255	0.220	0.203
	$E\left[T_x \mid T_x < \infty\right]$	0.949	3.095	4.144	4.886	5.617
0.85	$P[T_x < \infty]$	0.648	0.257	0.177	0.146	0.130
	$E\left[T_x \mid T_x < \infty\right]$	0.595	1.959	2.652	3.170	3.718

As we have mentioned in the introduction, in this case, the medium promotes a tumor growth factor of $r = 0.362$. If we add the uncertainty to this factor, we obtain the stochastic differential equation

$$dX_t = 0.362\, X_t(2.443 - X_t)dt + \alpha X_t(2.443 - X_t)dB_t, \quad t > 0, \tag{5.2}$$

with initial condition $X_0 = 47.619 \times 10^{-3}$ millions cells and

$$\alpha > 0.544.$$

This inequality is due to the fact that in the Theorem 4.2 it is required that $\beta < 1$, see (2.3).

Using Theorem 4.2 we calculate, for several values of α, the probability $P[T_x < \infty]$ and the conditional expected value $E\left[T_x|T_x < \infty\right]$. These values are summarized in Table 1.

From the definition (5.1) of the sigmoidal function we see that

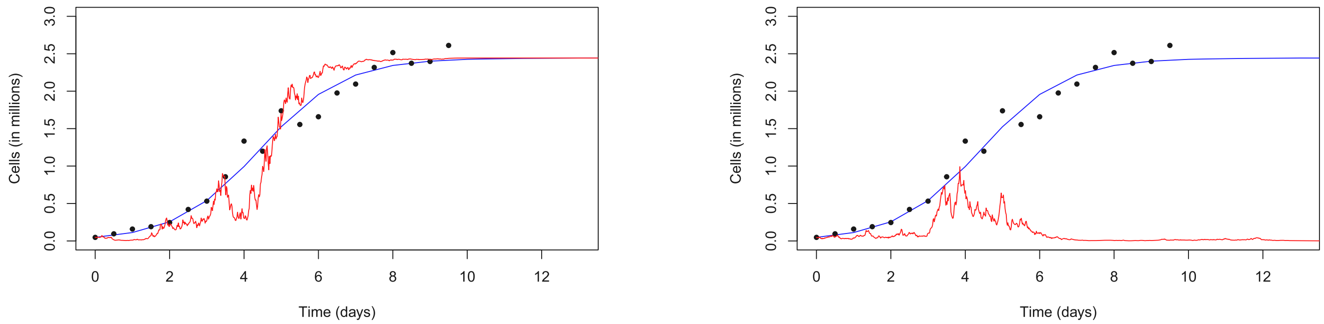
$$X_{0.863} \approx 0.1,\ X_{2.892} \approx 0.5,\ X_{4.012} \approx 1,\ X_{4.950} \approx 1.5,\ X_{6.129} \approx 2. \tag{5.3}$$

On the other hand, from Table 1 we see that for small values of α the probabilities are higher than for slightly larger values. An opposite effect is observed in the conditional expected values. In both cases ($\alpha = 0.65$ or $\alpha = 85$), we note that the conditioned expected values are outside the range of the sigmoidal function, see (5.3), which we can use in principle as a starting point to model the behavior of the average growth of cancerous tumors when they reach their maximum volume (the carrying capacity, $K = 2.442$) or minimum volume (the cure of the cancerous tumor). Thus, we see that if we consider the fluctuation of the conditional expected values and the corresponding probabilities, then a feasible value for α can be 0.75. With this value of α we observe that when $x = 0.1$ then the probability to reach this value in finite time is high (0.706) and the estimated conditional expected value is 0.949 which is close to the sigmoidal value 0.863. A similar behavior occurs with the other values of x, $0.5, 1, 1.5$ and 2, see (5.3) and Table 1.

For the data plotted in Fig. 1(a) we observe that at the beginning there was a tumor of 47.619×10^{-3} million cells. Since $\alpha = 0.75$, then $\beta = 0.527$ therefore from (2.6) it follows that with probability 0.377 the tumor reaches its maximum volume at infinity, that is, K million of cells, and from Table 1 we see that under the condition that it will take a finite time to reach level K, then it takes an average of a little more than 5.617 days to reach level K. Using the function *sde.sim* from the library *sde* of software R we give in Fig. 2 two paths in red of the stochastic process X defined in (5.2).

From (2.7) we see that with probability 0.622 the tumor is extinguished at infinity, that is, it is not necessarily true that it grows. Furthermore, according to the stochastic model, there is a probability of 0.293 that the tumor will never reach a volume of 0.1 million cells. Therefore, a recovery of the patient. Similarly, we can say that on the fifth day the tumor is expected to have an average volume of 2 million cells with a probability close to 0.203.

Remember that $X_0 = 47.629 \times 10^{-3}$ million cells and $\alpha = 0.75$, so we use the Theorem 4.3 to build the Table 2. We see that the probabilities of the first arrival of the stochastic process X, solution of (5.2), at $X_0/\cdot$, when it starts at 47.619×10^{-3} million cells,



(a) A path of X tending to $K = 2.4429482$.

(b) A path of X tending to 0.

Fig. 2. In red some simulated paths of a tumor growth X, given in (5.2).

Table 2
Probabilities and expected values of first-passage times below x_0 of a tumor growth.

x	$X_0/2$	$X_0/4$	$X_0/8$	$X_0/16$	$X_0/32$
$P[T_x < \infty]$	0.898	0.830	0.781	0.745	0.719
$E\left[T_x \mid T_x < \infty\right]$	0.758	1.598	2.521	3.527	4.613

are high, which indicates that there are more possibilities of recovery than of death. In fact, it tells us that the behavior of the growth or decrease of the tumor is immediately reflected. For example, with a probability of 0.8988 the first-passage time at half the initial size of the tumor is finite and under this assumption the expected time is 0.758 days.

Here we note that the deterministic model does not consider the possibility of a cure, unlike the stochastic model; furthermore, conditional estimates of the expected time to tumor growth are given and probabilities for said estimates are obtained.

5.3. *World population*

It is clearly evident that the earth's natural resources are limited, so it is important to know what is the maximum population the earth can sustain indefinitely. This amount is called carrying capacity. Drinking water, fertile land and fisheries are insufficient to sustain the growth of the world's population. Furthermore, waste is increasing considerably, thus forming a harmful cycle for the planet. Twenty studies consider that the carrying capacity of the earth is less than or equal to 8 billion inhabitants and another fourteen studies affirm that the carrying capacity is 16 billion inhabitants (see [3]). The carrying capacity of 8 billion has already been exceeded, the world population as of February 7, 2023 is 8,015,493,917 people.

As we have said, as the population increases, the damage to the different ecosystems of the earth increases, which makes their recovery increasingly difficult. In turn, in recent years climate change has caused rising temperatures, more powerful storms, increased droughts, rising ocean levels, disappearance of species, food shortages, more health risks, poverty and displacement, among other effects. We know that, the parameter r of the deterministic logistic model (1.1) is interpreted as a measure of the quality of the environment. Therefore, it makes sense to introduce some noise, caused for example by climate change, in said parameter. The purpose of this subsection is to study this new stochastic model.

Fig. 3(a) shows the world population from 1950 to 2023 (see [29]). To determine the carrying capacity we will use this information. We proceed as in the previous example, setting the initial conditions $(2023, 8.015$, $K = 11$ and $\tilde{r} = 0.02$. With this we find the fit values $K = 11.981$ and $r = 2.369 \times 10^{-3}$. Therefore, we obtain a sigmoidal function that models the deterministic growth of the world population

$$X_t = \frac{96.032}{8.015 + 3.965\, e^{-28.388\times 10^{-3}\,(t-2023)}}, \quad t \geq 0. \tag{5.4}$$

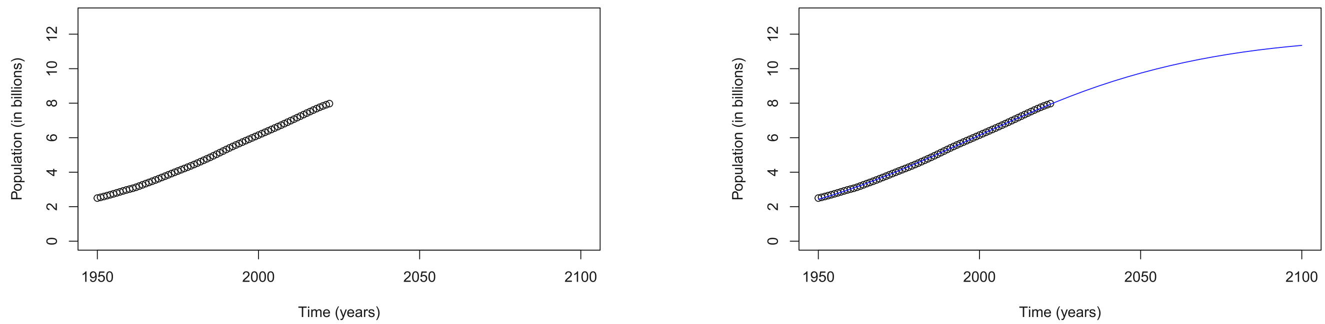
We observe in particular that $X_{2023} = 8.015$ billion people and

$$X_{2024.128} \approx 8.1,\;\; X_{2029.657} \approx 8.5,\;\; X_{2037.133} \approx 9,\;\; X_{2055.241} \approx 10,\;\; X_{2083.357} \approx 11. \tag{5.5}$$

The blue curve in Fig. 3(b) is the graph of (5.4).

Unlike the previous case, where we could take several samples of the growth of the cancerous tumor, that is, consider several patients, in this case it is reasonable to adjust the parameters of the logistic model and simulate the possible future scenarios when we incorporate a small amount of noise into the deterministic model. By incorporating an additive noise in the parameter that measures the quality of the environment, $r = 2.369 \times 10^{-3}$, we obtain the stochastic differential equation

$$dX_t = 2.369 \times 10^{-3}\, X_t(11.981 - X_t)dt + \alpha X_t(11.981 - X_t)dB_t, \quad t > 0, \tag{5.6}$$

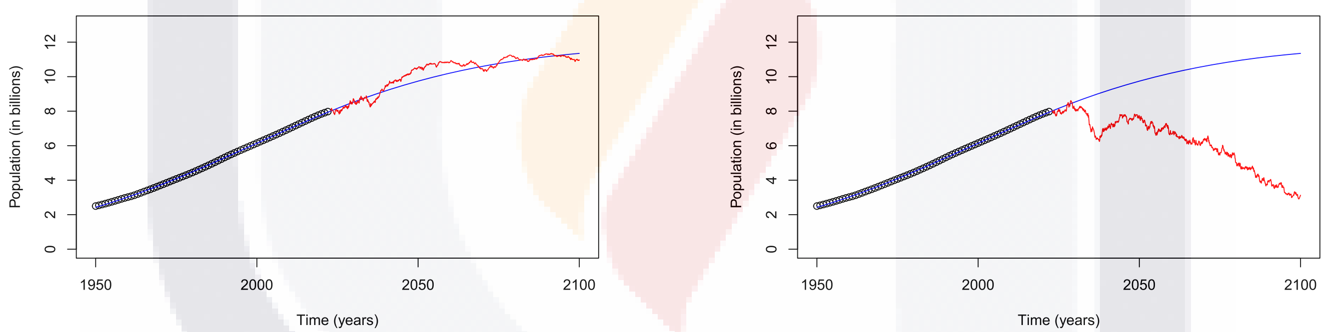


(a) World population until 2023. (b) In blue the graph of sigmoidal function (5.4).

Fig. 3. Modeling the growth of the world population.

Table 3
Probabilities and expected values of first-passage times upper x_0 of the world population.

α	x	8.1	8.5	9	10	11
0.022	$P[T_x < \infty]$	0.999	0.995	0.991	0.985	0.982
	$E\left[T_x \mid T_x < \infty\right]$	2024.520	2031.560	2040.077	2056.888	2076.151
0.023	$P[T_x < \infty]$	0.998	0.993	0.987	0.977	0.972
	$E\left[T_x \mid T_x < \infty\right]$	2024.253	2030.115	2037.346	2052.121	2069.859
0.024	$P[T_x < \infty]$	0.998	0.990	0.982	0.969	0.961
	$E\left[T_x \mid T_x < \infty\right]$	2024.083	2029.187	2035.567	2048.915	2065.446



(a) A path of X tending to $K = 11.981$. (b) A path of X tending to 0.

Fig. 4. In red some simulated paths of the world population X, given in (5.6).

with initial condition $X_0 = 8.015$ billion people. It is worth noting that (in time) the initial point of equation (5.6) is 0, which actually represents the year 2023. Taking this observation into account and that

$$\alpha > 19.888 \times 10^{-3},$$

we vary the parameter α and obtain the values in Table 3. If we consider that the logistic model (5.4) represents the average behavior of the world population, then of the values of the parameter α that appear in Table 3, the most appropriate is $\alpha = 0.023$, see (5.5). From the values in this table we see that the world population will almost surely grow. For example, in approximately 14 more years, that is, in the year 2037, there will be an average of 9 billion people with a probability of 0.987. Furthermore, since $\beta = 0.747 < 1$, then from Proposition 2.4 we see that the carrying capacity K is reached at infinity with a probability of 0.995. According to this stochastic logistic model, the probability of extinction is 4.76×10^{-3}. We recall that in this model, at infinity the stochastic process tends almost surely to 0 or K, see for example the trajectories in Fig. 4.

On the other hand, since $\beta = 0.747 > -1$ we can use Theorem 4.3, with $\alpha = 0.023$ and $X_0 = 8.015$ billion people, to get Table 4. Here the opposite behavior is observed, that is, the probabilities are small and the values of the expected times are a little higher. However, we do get insightful information; for example, if climate change produces adverse effects over a continuous cycle, then the world's population will on average halve over the next twenty eight years, and the probability of this happening is 0.177.

Table 4
Probabilities and expected values of first-passage times below x_0 of the world population.

x	$X_0/2$	$X_0/4$	$X_0/8$	$X_0/16$	$X_0/32$
$P[T_x < \infty]$	0.177	0.083	0.052	0.037	0.029
$E[T_x \mid T_x < \infty]$	2051.020	2072.795	2094.833	2118.966	2146.132

6. Conclusions

In the present work, the first and second moments of the first arrival times have been obtained under the assumption that these times are finite. In order to illustrate these results, two real-world problems have been considered. In the first one, the growth of a Jurkat T human leukemia tumor was modeled; as a relevant aspect, it was obtained that it is more likely to recover from this disease than to die from it. Indeed, in the course of the first day, the tumor is reduced by half, with a probability of almost 90%. In the second, the growth of the world population is modeled, considering climate change as a possible disturbing effect. In this case, we determined the carrying capacity of the planet and saw that, although unlikely (a probability of the order of 18%), if climate change causes a continuous series of natural disasters, then it is feasible that in a period of twenty eight years the population is cut in half. This seems quite worrying to us, and it is necessary to implement policies at the global level to reduce this probability.

Data availability

The authors do not have permission to share data.

Acknowledgement

The research was partially supported by the grant PIM22-1 of Benemérita Universidad Autónoma de Aguascalientes and CONAHCYT. We thank the reviewers of the article and the responsible Associate Editor for their constructive feedback. We also appreciate José Miguel Villa-Ocampo's help in implementing R software.

References

[1] L.J. Allen, A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis, Infect. Dis. Model. 2 (2017) 128–142.
[2] C.A. Braumann, Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance, Wiley, New York, 2019.
[3] J.E. Cohen, Population growth and earth's human carrying capacity, Science 269 (1995) 341–346.
[4] A. Di Crescenzo, P. Paraggio, P. Román-Román, F. Torres-Ruiz, Applications of the multi-sigmoidal deterministic and stochastic logistic models for plant dynamics, Appl. Math. Model. 92 (2021) 884–904.
[5] M. Domine, Moments of the first-passage time of a wiener processes with drift between two elastic barriers, J. Appl. Probab. 32 (1995) 1007–1013.
[6] R. Durret, Stochastic Calculus, A Practical Introduction, CRC Press, Boca Raton, 1996.
[7] J.S. Giet, P. Vallois, S. Wantz-Méziéres, The logistic s.d.e., Theory Stoch. Process. 20 (2015) 28–62.
[8] H. Hulley, E. Platen, Laplace transform identities for diffusions, with applications to rebates and barrier options, Adv. Math. Finance 83 (2008) 139–157.
[9] K. Itô, H.P. McKean, Diffusion Processes and Their Sample Paths, Springer, New York, 1964.
[10] D. Jiang, N. Shi, A note on nonautonomous logistic equation with random perturbation, Math. Anal. Appl. 303 (2005) 164–172.
[11] D. Jiang, N. Shi, X. Li, Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation, J. Math. Anal. Appl. 340 (2008) 588–597.
[12] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, New York, 1991.
[13] S. Karlin, H.M. Taylor, A Second Course in Stochastic Processes, Academic Press, New York, 1981.
[14] R. Kryscio, C. Lefevre, On the extinction of the s-i-s stochastic logistic epidemic, J. Appl. Probab. 27 (1989) 685–694.
[15] E.M. Lungu, B. Øksendal, Optimal harvesting from a population in a stochastic crowded environment, Math. Biosci. 145 (1997) 47–75.
[16] P. Mandl, Analytical Treatment of One-Dimensional Markov Processes, Springer, New York, 1968.
[17] H. Murphy, H. Jaafari, H. Dobrovolny, Differences in predictions of ode models of tumor growth: a cautionary example, BMC Cancer 16 (2016) 1–10.
[18] M.N.P. Nilsson, The moving-eigenvalue method: hitting time for itô processes and moving boundaries, J. Phys. A, Math. Theor. 53 (2020) 1–32.
[19] A.G. Nobile, L. Ricciardi, L. Sacerdote, A note on first-passage time and some related problems, J. Appl. Probab. 22 (1985) 346–359.
[20] R. Norden, On the distribution of the time of extinction in the stochastic logistic population model, Adv. Appl. Probab. 14 (1982) 687–708.
[21] S. Pasquali, The stochastic logistic equation: stationary solutions and their stability, Rend. Semin. Mat. Univ. Padova 106 (2001) 165–183.
[22] M. Pollak, D. Siegmund, A diffusion process and its applications to detecting a change in the drift of Brownian motion, Biometrika 72 (1985) 267–280.
[23] I.A. Rodriguez-Brenes, N.L. Komarova, D. Wodarz, Tumor growth dynamics: insights into evolutionary processes, Trends Ecol. Evol. 28 (2013) 597–604.
[24] W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, Auckland, 1976.
[25] A. Tsoularis, J. Wallace, Analysis of logistic growth models, Math. Biosci. 179 (2002) 21–55.
[26] H.C. Tuckwell, F.Y.M. Wan, First-passage time of Markov processes to moving barriers, J. Appl. Probab. 21 (1984) 695–709.
[27] G. Wake, S. Watt, The relaxation of May's conjecture for the logistic equation, Appl. Math. Lett. 9 (1996) 50–62.
[28] H. Wang, C. Yin, Moments of the first passage time of one-dimensional diffusion with two-side barriers, Stat. Probab. Lett. 78 (2008) 3373–3380.
[29] WP, https://population.un.org/wpp/Download/Standard/Population/, 2023.

4 Conclusión

- **Capítulo 1:** Los teoremas de punto fijo son especialmente útiles para demostrar la existencia de soluciones de ecuaciones diferenciales e integrales. En este trabajo, se vió que el Teorema 1.1 se aplica para demostrar la unicidad y existencia de ciertos tipos de ecuaciones de Fredholm y algunas ecuaciones diferenciales fraccionarias en el sentido de Caputo, que es un área de interés reciente en la teoría de punto fijo. En la última sección del artículo se mostraron algunas las aplicaciones del teorema del punto fijo.

- **Capítulo 2:** En este artículo se analizó el primer tiempo de llegada del proceso estocástico I. Bajo la premisa de que el número de reproducción estocástica es menor que uno, se calcula la probabilidad de que el primer tiempo de llegada sea finito. Condicionado a esto, se determinó su valor esperado de dicho momento. En el contexto de este supuesto se demuestró que el proceso estocástico I se aproxima a cero a conforme el tiempo tiende a infinito. En otras palabras, la enfermedad tiende a extinguirse. Así, es posible resolver la conjetura formulada en [9].

  Además, el modelo SIS estocástico se aplicó en tres situaciones con datos reales y se propone un esquema numérico para hallar el momento condicional. En los dos primeros ejemplos, la solución determinista se adoptó como un comportamiento medio de los fenómenos, y con este enfoque se establecen los posibles valores del parámetro estocástico. El mérito de este método reside en que permitió evaluar con qué probabilidad se alcanza un valor superior al inicial, lo que a su vez facilita la toma de decisiones en función de la probabilidad de dicho valor. En el tercer ejemplo, se utilizó en primer lugar un método de ajuste no lineal para identificar el modelo determinista más adecuado. Posteriormente, a partir de este modelo ajustado, se procedió de forma similar a los dos ejemplos anteriores.

  En resumen, la incorporación de ruido aleatorio genera resultados significativamente distintos a los de la contraparte determinista. De hecho, las simulaciones revelan distintas trayectorias aleatorias que reflejan con mayor precisión las complejidades del mundo real. Además, se calcularon con éxito la esperanza condicional del primer tiempo de llegada y se exploró el comportamiento asintótico del proceso estocástico I. Se considera que estas metodologías son herramientas versátiles aplicables a diversos escenarios, ampliando así la practicidad y adaptabilidad de los métodos estocásticos.

- **Capítulo 3:** En el trabajo presentado, el primer y segundo momento de primera pasada se obtevieron suponiendo que dichos tiempos son finitos. Para ilustrar estos resultados, se han considerado dos problemas con datos obtenidos bajo circunstancias reales. En el primero, el crecimiento de células cancerígenas de leucemia humana Jurkat T; como aspecto relevante, se obtuvo que es más probable recuperarse de esta enfermedad que morir por ella. De hecho, en el transcurso del primer día, el tumor se reduce a la mitad, con una probabilidad de casi el 90 %. En el segundo ejemplo, se modela el crecimiento de la población mundial, considerando el cambio climático como un posible efecto perturbador. En este caso se determinó la capacidad de carga del planeta y se vió que, aunque improbable (una probabilidad del orden del 18 %), si el cambio climático provoca una serie continua de catástrofes naturales, entonces es factible que en un periodo de veintiocho años la población se reduzca a la mitad. Esto dato es preocupante, y es necesario aplicar políticas a escala mundial para reducir esta probabilidad.

Bibliografía

[1] Linda J.S. Allen. A primer on stochastic epidemic models: Formulation, numerical simulation and analysis. *Infectious Disease Modelling*, 2(2):128–142, 2017.

[2] R. K. McCormack & L.J. Allen. *Stochastic SIS and SIR multihost epidemic models. In Proceedings of the Conference on Differential and Difference Equations and Applications.* Hindawi, New York, 2006.

[3] Gregory Thompson & Judith Berner. A stochastic parameter perturbation method to represent uncertainty in a microphysics scheme. *Monthly Weather Review*, 149(5):1481–1497, 2021.

[4] Carlos A. Braumann. *Introduction to stochastic differential equations with applications to modelling in biology and finance.* John Wiley & Sons, Inc, New York, 2019.

[5] Felix E. Browder. Fixed-point theorems for noncompact mappings in hilbert space. 53:1272–1276, 1965.

[6] Ronald E. Bruck. Properties of fixed-point sets of nonexpansive mappings in banach spaces. *Transactions of the American Mathematical Society*, 179:251–262, 1973.

[7] Marco Dominé. Moments of the first-passage time of a wiener processes with drift between two elastic barriers. *Journal of Applied Probability*, 32(4):1007–1013, 1995.

[8] Avner Friedman. *Stochastic Differential Equations.* Springer., 2010.

[9] A. Gray & D. Greenhalgh. A stochastic differential equation sis epidemic model. *SIAM Journal on Applied Mathematics*, 71:876–902, 2011.

[10] Siegfried Carl & Seppo Heikkila. *Fixed point theory in ordered sets and applications. From differential and integral equations to game theory.* Springer, 2011.

[11] Kiyosi Itô & P. Henry. *Diffusion processes and their sample paths.* Springer, New York, 1964.

[12] Yanan Zhao & Daqing Jiang. The threshold of a stochastic sis epidemic model with vaccination. *Applied Mathematics and Computation*, 243:718–727, 2014.

[13] Erdal Karapınar. Generalized c-conditions and related fixed point theorems. *Comp. Math. App.*, (61):3370–3380, 2011.

[14] Erdal Karapınar. Edelstein type fixed point theorems. *Fixed Point Theory and Applications*, 75(107), 2012.

[15] Mohamed A. Khamsi. Generalized metrics paces: A survey. *Journal of Fixed Point Theory and Applications*, (17):455–475, 2015.

[16] Abdul Latif. *Banach Contraction Principle and Its Generalizations.* Springer., 2013.

[17] Richard J. Kryscio & Claude Lefèvre. On the extinction of the s–i–s stochastic logistic epidemic. *Journal of Applied Probability*, 26(4), 2016.

[18] Daqing Jiang & Ningzhong Shi & Xiaoyue Li. Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation. *Journal of Mathematical Analysis and Applications*, 340(1):588–597, 2008.

[19] Thomas Robert Malthus. *An essay on the principle of population.* Pickering & Chatto Publishers., 1798.

[20] Petr Mandl. *Analytical Treatment of One-dimensional Markov Processes.* Springer, Berlin, Heidelberg, 1968.

[21] D. Mollison. *Epidemic models: their structure and relation to data.* Cambridge University Press., 1995.

[22] Ioannis Farmakis & Martin Moskowitz. *Fixed Point Theorems and Their Applications.* 2013.

[23] Martin N. P. Nilsson. The moving-eigenvalue method: hitting time for itô processes and moving boundaries. *Journal of Physics A: Mathematical and Theoretical*, 53(40):1–32, 2020.

[24] R. H. Norden. On the distribution of the time to extinction in the stochastic logistic population model. *Advances in Applied Probability*, 14(4), 2016.

[25] E.M. Lungu & B. Oksendal. Optimal harvesting from a population in a stochastic crowded environment. *Mathematical Biosciences*, 145(1):47–75, 1997.

[26] Antonio Di Crescenzo & Paola Paraggio. Applications of the multi-sigmoidal deterministic and stochastic logistic models for plant dynamics. *Applied Mathematical Modelling*, 92:884–904, 2021.

[27] Sara Pasquali. The stochastic logistic equation: Stationary solutions and their stability. *Rendiconti del Seminario matematico della Universita di Padova*, (16):165–183, 2001.

[28] Hardy Hulley & Eckhard Platen. Laplace transform identities for diffusions, with applications to rebates and barrier options. *Banach Center Publications*, 83(1):139–157, 2008.

[29] Maurizio Pratelli. A minimax theorem without compactness hypothesis. *Mediterranean Journal of Mathematics*, (2):103–112, 2005.

[30] Mortaza Abtahil & Zoran Kadelburg & Stojan Radenović. Fixed points of Ćirić-matkowski-type contractions in v-generalized metric spaces. *RACSAM*, 10(1007), 2017.

[31] I. R. Sarma & J. M. Rao & S. S. Rao. Contractions over generalized metric spaces. *The Journal of Nonlinear sciences and applications*, 2:180–182, 2009.

[32] Karen E. Lamb & David Greenhalgh & Chris Robertson. A simple mathematical model for genetic effects in pneumococcal carriage and transmission. *Journal of Computational and Applied Mathematics*, 235:1812–1818, 2011.

[33] A. G. Nobile & L. M. Ricciardi & L. Sacerdote. A note on first-passage time and some related problems. *Journal of Applied Probability*, 22(2), 2016.

[34] Mohamed Jleli & Bessem Samet. A generalized metric space and related fixed point theorems. *Fixed Point Theory and Applications*, (61), 2015.

[35] Daqing Jiang & N. Shi. A note on non-autonomous logistic with random perturbation. *Journal of Mathematical Analysis and Applications*, 303(1):164–172, 2005.

[36] Moshe Pollak & David Siegmund. A diffusion process and its applications to detecting a change in the drift of brownian motion. *Biometrika*, 72(2):267–280, 1985.

[37] Yiliang Chen & Buyu Wen & Zhidong Teng. The global dynamics for a stochastic sis epidemic model with isolation. *Physica A: Statistical Mechanics and its Applications*, 492:1604–1624, 2018.

[38] Naseer Shahzad & Oscar Valero. Nemytskii-edelstein type fixed point theorem for partial metric spaces. *Fixed Point Theory and Applications*, (26), 2015.

[39] Bessem Samet & Calogero Vetro & Pasquale Vetro. Fixed point theorems for $\alpha - \psi$-contractive type mappings. *Nonlinear Analysis: Theory, Methods and Applications*, 75(4):2154–2165, 2012.

[40] Daniela Paesano & Pasquale Vetro. Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces. *Topology and its Applications*, 159(3):911–920, 2012.

[41] D. Hernández-Hernández & J. Villa-Morales. On the minimax theorem for the space of probability measures on metric spaces. *Banach Center Publications*, (122):159–168, 2020.

[42] J. Villa-Morales. A fixed point theorem and some properties of v-generalized metric spaces. *Journal of Fixed Point Theory and Applications*, 20(3), 2018.

[43] J. Villa-Morales. Subordinate semimetric spaces and fixed point theorems. *Journal of Mathematics*, 2018.

[44] J. Villa-Morales. An extreme value theorem on closed convex subsets of l^p spaces and two applications. *Journal of Mathematical Analysis and Applications*, 491(1), 2020.

[45] J. Villa-Morales. Dominated weak solutions for a second order ordinary differential equation with boundary conditions in $(0, \infty)$. *Mediterranean Journal of Mathematics*, 19(128), 2022.

[46] A. Tsoularis & J. Wallace. Analysis of logistic growth models. *Mathematical Biosciences*, 179(1):21–55, 2002.

[47] Henry C. Tuckwell & Frederic Y. M. Wan. First-passage time of markov process to moving barriers. *Journal of Applied Probability*, 21(4):695–709, 1984.

[48] Jean-Sébastien Giet & Pierre Vallois & Sophie Wantz-Mézières. *The logistic s.d.e. Theory of Stochastic*. 1964.

[49] G.C. Wake & S.D. Watt. The relaxation of may's conjecture for the logistic equation. *Applied Mathematics Letters*, 9(5):59–62, 1996.

[50] Huiqing Wang & Chuancun Yin. Moments of the first passage time of one-dimensional diffusion with two-sided barriers. *Statistics and Probability Letters*, 78(18):3373–3380, 2008.

[51] Wenbin Liu & Qiben Zheng. A stochastic sis epidemic model incorporating media coverage in a two patch setting. *Applied Mathematics and computation*, 262:160–168, 2015.