CENTRO DE CIENCIAS BÁSICAS
DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

HOMOTOPY CLASSIFICATION OF BILINEAR MAPS ON SPHERES

PRESENTA

Luis Arturo García Macías

PARA OBTENER EL GRADO DE MAESTRO EN CIENCIAS CON OPCIÓN A MATEMÁTICAS APLICADAS

TUTOR
Dr. Hugo Rodríguez Ordóñez

COMITÉ TUTORAL
Dr. Jorge Eduardo Macías Díaz
Dr. José Villa Morales

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L.M.A. LUIS ARTURO GARCÍA MACÍAS
ALUMNO (A) DE LA MAESTRIA EN CIENCIAS
CON OPCIÓN MATEMÁTICAS APLICADAS,
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Asesor de tesis

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Resumen

El problema de composición de formas cuadráticas, originado por la fórmula de Euler, nos dirige a hacernos algunas preguntas, la primera es: ¿Para qué dimensiones pueden existir dichas sumas de cuadrados? Las siguientes preguntas involucran la clasificación y el análisis de las formas cuadráticas relacionadas en una fórmula de composición. Históricamente, se ha encontrado una conexión entre fórmulas de composición y mapeos bilineales sobre esferas, donde dichos mapeos sobre esferas aportan una clase de homotopía de esferas. El objetivo principal es continuar con el trabajo desarrollado hasta ahora en la búsqueda de nuevos mapeos bilineales ya sean no singulares o normados asociados a fórmulas de composición, así como clasificar mapeos tanto nuevos como algunos ya existentes. Esto último ayudado por **Cobordismo enmarcado**, la cual es una herramienta reciente no muy utilizada para lidiar con este problema. La noción de cobodismo consiste en un método para clasificación de variedades mediante una parametrización llamada *marco*. Este estudio se relaciona con ciertos grupos de homotopía a traves del cálculo del grupo de cobordismo no orientado, dicho cálculo seguido por algunos pasos que, a la vez proveen un nuevo algoritmo para resolver el problema presentado para grupos finitos. En este trabajo exploto las propiedades de este método aplicadas en mapeos existentes y entonces clasificarlos (es decir, encontrar la clase de homotopía representada por el mapeo) mediante los resultados sustentados por el cobordismo enmarcado. Siendo más específico, trato con variedades enmarcadas asociadas a un mapeo bilineal, en concreto, el mapeo de contrucción de Hopf de un mapeo bilineal dado, ya sea no singular o normado, obtenido por la operación del producto en estructuras algebraicas superiores.

Abstract

The problem of composition of quadratic forms, originated from Euler's fomula, addresses some basic questions, the initial question is: For what dimensions can such sums of squares formulas exist? Subsequent questions involve classification and analysis of quadratic forms which can occur in a composition formula. Historically, a connection has been found between composition formulas and bilinear maps on spheres, and such bilinear maps gives a homotopy class of spheres. The main objective is to continue on with the work developed all along in search for new nonsingular or normed bilinear maps associated to composition formulas, also to classify both new and existing maps. This aided by **Framed cobordism**, which is a recent tool not quite used before to treat this problem. The notion of cobordism consists in a method to classify manifolds, this by using a parametrisation called *frame*. This study is related with certain homotopy groups through the computation of the unoriented cobordism groups, whose computation is followed by a couple of steps, these steps provide a new-fashioned algorithm to solve the problem for finitely-presented groups. In this work I exploited the properties of this method, applied to existing mappings and then made their the classification (i.e. to find the homotopy class represented by the mapping) through the results supported by framed cobordism. To be more specific, I treated with framed manifolds associated to a bilinear map, namely, the Hopf construction map of a given either nonsingular or normed bilinear map, obtained through the product operation of elements on further algebraic structures.

Introduction

In algebraic topology the classification of continuous maps between spheres has been a basic problem for the last 50 years. Despite of the fact that the sphere intuitively involve a simple space, the experience has shown that the computation of the homotopy groups of spheres $\pi_{n+k}(S^n)$ is hard as in any branch in mathematics. However there are some remarkable breakthroughs in the subject such as the EHP sequence and the spectral sequences of Serre and Adams. Henri Poincaré exploited an alternative idea, namely cobordism.

The main objective of this work is to use the tools developed by the cobordism theory used by K. Y. Lam and H. Rodríguez [14], [24] in order to classify bilinear maps between spheres which is based on the computation of a homology invariant called the Arf-Kervaire invariant which essentially determines whether the homotopy class representing the bilinear map is trivial or not.

In Chapter 2 we introduce the basics on cobordism and vector bundles to construct one of our main objects of study, the framed cobordism ring besides of its relation with the homotopy problem through the Pontrjagin–Thom theorem establishing an isomorphism between the framed cobordism ring and the stable homotopy group of spheres.

In Chapter 3 we define the Arf invariant and then proceed to the construction of the particular case of the Kervaire invariant developed by Pontrjagin [20] for surfaces of genus g.

Chapter 4 deals with an introduction to the Cayley-Dickson algebras. These algebras constitute a generalisation of the already known normed algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$ which are used to construct new bilinear maps such as the modified polynomial multiplications.

Finally, Chapters 5 and 6 present how the framed cobordism tools take form to classify two kinds of bilinear maps: nonsingular and normed. The two kind of maps are taken as survey from Lam [14].

0.1 The Composition Formula Problem: Historical Review

A brief review about the theory of composition of quadratic forms over fields can be associated with the old problem of searching for n-square identities of the type:

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) = (z_1^2 + z_2^2 + \cdots + z_n^2)$$

where $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ consist of systems of variables and each $z_k = z_k(X, Y)$ is a bilinear form in the variables X and Y.

For example if $n = 2$ there is the ancient identity:

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2.$$

In this case, $z_1 = x_1 y_1 + x_2 y_2$ and $z_2 = x_1 y_2 - x_2 y_1$ are the bilinear forms in X and Y with real coefficients. Such a formula can be interpreted as the *law of moduli* for complex numbers: $|\alpha| \cdot |\beta| = |\alpha\beta|$ where $\alpha = x_1 + x_2 i$ and $\beta = y_1 + y_2 i$.

A similar formula was found by Euler (1748) in his attempt to prove the Fermat's conjecture that every positive integer is a sum of four integer squares. This occurs for a 4-square identity:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$$

where

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\ z_2 &= x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 \\ z_3 &= x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2 \\ z_4 &= x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1, \end{aligned}$$

and some time later, it was Hamilton (1843) who discovered this 4-square formula interpreted as the law of moduli for *quaternions*. Some mathematicians spent years searching for more of these identities like Legendre, who proved the impossibility of a 3-square identity. John Graves discovered an algebra of *octaves*, the multiplication of these elements satisfies the law of moduli and gives rise to a 8-square identity, which eventually would be introduced as the elements in the algebra of *Cayley numbers*.

It was until 1898 when Adolf Hurwitz [11] proved that there exists an n-square identity with real coefficients if and only if $n = 1, 2, 4$ or 8. At the end of the paper Hurwitz posed the general problem: For which integers r, s, n does there exist a **composition formula** (or a formula of size $[r, s, n]$):

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_1, y_2, \ldots, y_s)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form with real coefficients X and Y?

In an attempt to determine such integers r, s and n. Johannes Radon (1922) developed a function that actually determines the exact conditions on particular integers r and n for the existence of a $[r, n, n]$ formula over the real field $\mathbb{R}$ while this very same condition was found independently by Hurwitz for formulas over the complex field $\mathbb{C}$. This condition is known as the *Hurwitz-Radon Theorem* and stated as in Bochnak [5] (as well as the proof):

Theorem 0.1. A formula of size $[r, n, n]$ exists if and only if $r \leqslant \rho(n)$.

Where $\rho(n)$ represents the **Hurwitz-Radon function** and is defined for intergers $n = 2^{4a+b}n_0$ where n_0 is odd and $0 \leqslant b \leqslant 3$, then $\rho(n) = 8a + 2^b$. There are several different ways this function can be described, for example:

$$\text{If } n = 2^m n_0 \text{ where } n_0 \text{ is odd then } \rho(n) = \begin{cases} 2m + 1 \text{ if } m \equiv 0, \\ 2m \text{ if } m \equiv 1, 2, \\ 2m + 2 \text{ if } m \equiv 3, \end{cases} \pmod 4.$$

In particular, $\rho(n) = n$ if and only if $n = 1, 2, 4$ or 8, as seen in Hurwitz' theorem. Also $\rho(16) = 9$, $\rho(32) = 10$, $\rho(64) = 12$ and general calculations display $\rho(16n) = 8 + \rho(n)$.

Some of the new proofs of the Hurwitz-Radon Theorem for composition formulas of size $[r, n, n]$ show applications of matrix methods such as representations of Clifford algebras, generalisations to quadratic forms over arbitrary fields and others are motivated by geometry problems to classify the type of solutions. Some recent studies on Hurwitz' theorem are compiled by Shapiro [27] for arbitrary quadratic forms over any field of characteristic $\neq 2$ which admit composition.

0.2 Preliminares on Bilinear Maps

Let us describe the two main types of maps and some of their pertinent features which we are about to study all along this work.

Definition 0.1. Consider a bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$.

- f is said to be *nonsingular*, if $f(x, y) = 0$ implies $x = 0$ or $y = 0$.
- We say the map is *normed* if it satifies $||f(x, y)|| = ||x|| \cdot ||y||$ for every $(x, y) \in \mathbb{R}^r \times \mathbb{R}^s$.
- The *Hopf construction map* associated to a nonsingular map f is defined as the map $H = H_f : \mathbb{R}^{r+s}\backslash\{0\} \to \mathbb{R}^{n+1}\backslash\{0\}$ given by:

$$H(x, y) = (||x||^2 - ||y||^2, 2f(x, y)).$$

From these definitions we can notice that the Hopf construction map is nonconstant which is an important feature for this kind of maps and it turns out more interesting how we can restrict a Hopf construction map to the unit spheres i.e. $H_f : S^{r+s-1} \to S^n$ since $\mathbb{R}^k\backslash\{0\}$ and S^{k-1} are homotopy equivalent. Moreover, by applying this idea to normed bilinear maps we can establish how, in particular, the multiplication of complex numbers, quaternions and Cayley numbers provide us examples of maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 4, 8$, respectively and their associated Hopf maps are the classical Hopf fibrations $S^{2n-1} \to S^n$ for $n = 2, 4, 8$. Thus, the existence of a normed bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ is equivalent to the existence of a formula for the product of sums of squares type as a formula of size $[r, s, n]$ as seen at the previous section.

In the search for a size $[r, s, n]$ of a composition formula consider $r * s$ to denote the smallest positive integer n such that there exists a normed map from $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ and in a similar fashion let $r\#s$ denote the smallest positive integer n such that there exists a nonsingular map from $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. Since the class of nonsingular maps is larger than the normed maps we obtain $r\#s \leqslant r * s$. The proof for this and some other properties and values already found between $r * s$ and $r\#s$ are detailed in Bochnak [5].

There are some sizes for a $[r, s, n]$ formula which can be eliminated by applying algebraic topology to the problem. Stiefel and Hopf used those topological connections to heighten the interest in the study of composition formulas such as the theory of characteristic classes of vector bundles developed by Stiefel where we can find the **Stiefel-Hopf Condition**. On the other hand, some observations made by Hopf showed how the *normed* bilinear maps induce maps between spheres. Moreover, this notion can be extended to a larger class of bilinear maps, namely *nonsingular*, as it was established by K. Y. Lam. So, the tools provided by Stiefel and Hopf exhibit a more sophisticated method to study nonsingular bilinear maps. Of course aided by algebraic topology and K-theory.

Chapter 1

Framed cobordism and its relation with homotopy groups of spheres

Cobordism is a notion which can be traced back to the end of the 19^{th} century with Henri Poincaré in his study of homology theory using smooth manifolds. One of the main ideas on cobordism is to extract algebraic structures from smooth manifolds. The modern theorems identify these algebraic structures explicitly. In cobordism theory: *cycles* are replaced by closed smooth manifolds mapped continuously into a topological space say C and *chains* by a compact smooth manifold X and a continuous map $X \to C$; the *boundary* of this chain is the restriction $\partial X \to C$ to the boundary. Finally, the *cobordism invariants* constitute homomorphisms from a cobordism group (category) into an abstract group (category) used to extract information in two ways about the domain or codomain. This is our motivation to study cobordism.

1.1 Basics on Cobordism

This chapter contains the basics notions to introduce the main tools provided by *Framed Cobordism* in order to classify manifolds and how much is related with our work.

Manifolds will be consider as smooth, compact and closed, unless otherwise specified. To have a deeper review on the basics for smooth manifolds see Guillemin-Pollack [8] or Lee [17].

Definition 1.1. Two n-dimensional manifolds M and N are said to be **cobordant**, or to belong to the same **cobordism class**, if there exists a $(n+1)$-dimensional manifold-with-boundary X such that ∂X is diffeomorphic to the disjoint union $M \sqcup N$).

This cobordism notion is sometimes called *un-oriented cobordism* due to the fact that there is no emphasis on the orientability of the manifold. Moreover, “to be cobordant” or “to belong to the same cobordism class”, as is pretended, establishes an equivalence relation. Just recall the gluing property for manifolds along the common boundary to prove it.

Now let us consider the set of all cobordism classes of n-dimensional manifolds. This set with the disjoint union of manifolds $\sqcup$ as operation constitutes an abelian group, usually denoted by Ω_n. The zero element of this group corresponds to the vacuous manifold.

Furthermore, if M and N are m and n-dimensional manifolds respectively, the cartesian product map between them $(M, N) \mapsto M \times N$ gives rise to an associative operation, which distributives the disjoint union and respects the cobordism relation and yield a bilinear product:

$$\Omega_m \times \Omega_n \to \Omega_{m+n}.$$

Therefore the sequence

$$\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \dots)$$

of groups of cobordism classes has the structure of a graded ring where the degree is indexed by the dimension. This ring possesses a 2-sided identity element $1 \in \Omega_0$, namely, the one-point manifold. The unorientability notion makes Ω_* a commutative graded ring.

Definition 1.2. The graded ring of all groups of cobordism classes Ω_* is called the **Cobordism Ring**.

1.2 Vector Bundles and their Application to Framed Cobordism

The first step to use topology in approach to the classification of the Hopf map associated to a nonsingular bilinear map is to relate it with certain vector bundle on the projective space. For instance, the called Stiefel-Whitney classes which are vector bundle invariants used in the Stiefel-Hopf condition. This section offers a brief introduction to basic content on vector bundles. Details for the proofs of the results presented on this section can be checked in Milnor-Stasheff [19].

Consider B to be a fixed topological space, in terms of vector bundles B will be called the *base space.*

Definition 1.3. A **real vector bundle** ξ over B consists of:

i) a topological space $E = E(\xi)$ called the *total space*,

ii) a continuous map $\pi : E \to B$ called the *projection map*, and

iii) for each $b \in B$ the structure of a real vector space in the set $\pi^{-1}(b)$ called the *fibre* over b.

Additionally it must satisfy the *local triviality condition.* This condiion states that for each point $b \in B$ there should exist a neighbourhood $U \subset B$, an integer $n > 0$, and a homeomorphism:

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that, for each $b \in U$, the correspondence $x \mapsto h(b, x)$ defines an isomorphism between the vector spaces $\mathbb{R}^n$ and $\pi^{-1}(b)$. The integer n determines ξ to be a n-plane bundle or $\mathbb{R}^n$-bundle for short.

From the definition above, in case we can choose $U = B$, the vector bundle ξ is called a *trivial bundle.* A fibre $\pi^{-1}(b)$ over a point b can also be denoted by F_b or $F_b(\xi)$.

Definition 1.4. Let ξ be a vector bundle with base space B.

a) A *cross-section* of a vector bundle ξ is a continuous function

$$s : B \to E(\xi)$$

which takes each $b \in B$ to an element in the corresponding fibre $F_b(\xi)$.

b) A cross-section is *nowhere zero* if $s(b)$ is a non-zero vector of $F_b(\xi)$ for each $b \in B$.

c) A collection $\{s_1, \ldots, s_n\}$ of cross-sections of a vector bundle ξ is *nowhere dependent* if, for each $b \in B$, the vectors $s_1(b), \ldots, s_n(b)$ are linearly independent.

In terms of smooth manifolds, we can define similarly a *smooth vector bundle* by taking B and E smooth manifolds, the projection map π would be a smooth map, and the map h in the local triviality condition would be a diffeomorphism for each $b \in U \subset B$. The same line of thought applies to the cross-sections to be smooth functions.

The cross-sections help us to characterise vector bundles in such a local or global way depending whether the vector bundle is trivial or not as it is shown in the following result.

Theorem 1.1. An $\mathbb{R}^n$-bundle ξ is trivial if and only if ξ admits n cross-sections $s_1, \ldots, s_n$ which are nowhere dependent.

Now, in order to fulfil our need to compare two vector bundles. It seems natural to say *isomorphism*, as expected, since there is a notion of vector space within them.

Definition 1.5. Let ξ and η be two vector bundles over the same base space B. We say that ξ is **isomorphic** to η, denoted $\xi \cong \eta$, if there exists a homeomorphism $f : E(\xi) \to E(\eta)$ between total spaces which maps each fibre $F_b(\xi)$ isomorphically onto the corresponding fibre $F_b(\eta)$.

Example 1.2.1. Consider a trivial bundle with total space $B \times \mathbb{R}^n$, the projection map $\pi(b, x) = b$ and with the vector space structure in the fibres defined by:

$$t_1(b, x_1) + t_2(b, x_2) = (b, t_1 x_1 + t_2 x_2),$$

denoted by ϵ_B^n. Then, given a $\mathbb{R}^n$-bundle over B is trivial if and only if it is isomorphic to ϵ_B^n.

Example 1.2.2. The *tangent bundle* τ_M of a n-dimensional smooth manifold M consists of:

- the total space $E(\tau_M)$ is the manifold DM, the set of all pairs (x, v) with $x \in M$ and v tangent M at x,
- the projection map $\pi : DM \to M$ defined by $\pi(x, v) = x$, and
- the vector space structure in $\pi^{-1}(x)$ defined by:

$$t_1(x, v_1) + t_2(x, v_2) = (x, t_1 v_1 + t_2 v_2),$$

If the tangent bundle τ_M is trivial, then the manifold M is called *parallelizable*. A cross-section of the tangent bundle of a smooth manifold M is usually called a *vector field* on M.

The unit 2-sphere $S^2 \subset \mathbb{R}^3$ provides an example of a manifold which is not parallelizable.

Example 1.2.3. The *normal bundle* ν of a smooth manifold $M \subset \mathbb{R}^n$ is obtained by taking as total space $E = E(\nu) \subset M \times \mathbb{R}^n$ the set of all pairs (x, v) such that v is orthogonal to the tangent space DM_x. The projection map $\pi : E \to M$ defined by $\pi(x, v) = x$ and the vector space structure in $\pi^{-1}(x)$ as the defined in example 1.2.2.

Example 1.2.4. Consider the circle $S^1 \subset \mathbb{R}^2$. We can notice that the tangent bundle of S^1 admits one nowhere zero cross-section (see figure), namely, for each point $x = (x_1, x_2) \in S^1$ given by:

$$s(x) = (x, v) = ((x_1, x_2), (x_2, -x_1)).$$

Therefore S^1 is parallelizable.

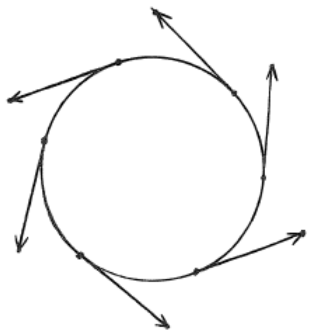

Figure 1.1: Nowhere Zero Cross Sections of S^1.

Example 1.2.5. In a similar way we can see how the 3 dimensional sphere $S^3 \subset \mathbb{R}^4$ admits three nowhere dependent vector fields:

$$s_1(x) = (x, (x_2, -x_1, x_4, -x_3))$$
$$s_2(x) = (x, (x_3, -x_4, -x_1, x_2))$$
$$s_3(x) = (x, (x_4, x_3, -x_2, -x_1))$$

where $x = (x_1, x_2, x_3, x_4) \in S^3$. Hence S^3 is parallelizable.

Remark 1.2. *It is quite interesting to have a look at the previous cross-sections which actually are pretty related with the complex number and quaternion multiplications, this by setting the components of these cross-sections as the entries of a matrix multiplying an arbitrary vector obtaining as result the imaginary part of the multiplication. By adding as first row the array* $\begin{pmatrix} x_1 & -x_2 \end{pmatrix}$ *and* $\begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \end{pmatrix}$ *for each case. We have obtained an encoded matrix which represents the multiplication of complex and quaternion numbers in the form* $A \cdot v$.

We need to pay special attention to the normal bundle of a smooth manifold that as we defined before consists of all pairs (x, v) such that the vector v is orthogonal to the tangent space of the manifold at x. So, it is needed to have a notion of orthogonality. This is why we are about to introduce *Euclidean vector bundles* which are vector bundles whose fibres have the structure of a Euclidean vector space. This way allows us to determine the normal bundle of a manifold (our goal of this section) which eventually will help us to define the so called *framed manifolds.*

An *Euclidean vector space* is a real vector space equipped with a positive definite quadratic form q (a quadratic form will be properly introduced in chapter 3). For brief, a quadratic form in a real vector space V determines a symmetric bilinear form B such that $q(v) = B(v, v)$ for all $v \in V$. The quadratic form q is said to be *positive definite* if $q(v) > 0$ for $v \neq 0$ and the real number $B(u, v)$ is called the *inner product* of the vectors u and v, which for real vector spaces is abbreviated $u \cdot v$. Now we are ready to define:

Definition 1.6. A **Euclidean vector bundle** is a real vector bundle ξ together with a continuous map $q : E(\xi) \to \mathbb{R}$ such that the restriction of q to each fibre of ξ is a positive definite and quadratic.

The function q itself will be called a *Euclidean metric* on the vector bundle ξ.

Example 1.2.6. The trivial bundle ϵ^n_B can be given the Euclidean metric:

$$q(b, x) = x_1^2 + \cdots + x_n^2.$$

Lemma 1.3. *Let ξ be a trivial vector bundle of dimension n over B, and let q be any Euclidean metric on ξ. Then there exist n cross-sections $s_1, \ldots, s_n$ of ξ which are normal and orthogonal in the sense that*

$$s_i(b) \cdot s_j(b) = \delta_{ij} \qquad (= \textit{Kronecker delta})$$

for each $b \in B$. See Milnor-Stasheff [19].

As we said before this review of vector bundles have the objective to introduce the so far called a *framed manifolds*.

Definition 1.7. Let M be an n–dimensional smooth manifold. A trivialisation of the normal bundle ν of M is called a *framing* of M. The pair (M, ν) is called **framed manifold**.

We can notice that framed manifolds are just a particular kind of manifolds, so we can proceed to construct cobordism classes between framed manifolds, as well as the cobordism groups and so on, just as we did at the beginning of this chapter.

In order to make it. we would need to consider the next pointers:

- Two framed manifolds (M, ν) and (N, σ) are framed cobordant (or belong to the same framed cobordism class) if M and N are cobordant and the frames ν and η are isomorphic vector bundles. The framed cobordism class of (M, ν) is denoted by $[M, \nu]$.
- The collection of all framed cobordism along with the correspondence:

  $$[M, \nu] + [N, \sigma] \mapsto [M \sqcup N, \nu \oplus \sigma],$$

  constitutes an abelian group, denoted Ω^{fr}_n, where $\sqcup$ represents the disjoint union whilst $\oplus$ is the *Whitney sum* of vector bundles.
- The operation defined as $[M, \nu] \times [N, \sigma] = [M \times N, \nu \otimes \sigma]$ with $\times$ as the cartesian product and $\otimes$ the tensor product, which distributives the Whitney sum, is associative and commutative provides us a bilinear map between framed manifolds of any dimensions Milnor-Stasheff [19].
- This product operation allows to the sequence $\Omega^{fr}_* = (\Omega^{fr}_0, \Omega^{fr}_1, \ldots)$ to acquire the structure of graded ring.

Definition 1.8. The graded ring Ω^{fr}_* is called the **Framed Cobordism Ring**.

1.3 The Stable Homotopy Group and its Relation with Framed Cobordism

Let us recall a very important result on homotopy theory best known as the **Freudenthal suspension theorem** stated in a particular version:

Theorem 1.4. The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.

In more a general way the theorem establishes the isomorphism for topological spaces, specifically *CW* complexes depending on its connectivity which is not a problem in the case of spheres. By iterating the suspension map on spheres we get a sequence:

$$\pi_i(S^n) \to \pi_{i+1}(\Sigma S^n) \to \pi_{i+2}(\Sigma^2 S^n) \to \cdots$$

Notice that all these maps eventually become isomorphisms and then yield the **stable homotopy group**, denoted $\pi_i^s(S^n)$. There is a special interest in the group $\pi_i^s(S^0)$ which is equal to $\pi_{n+i}(S^n)$ for $i+1 < 2n$. This particular group is often abbreviated π_i^s and called **stable i-stem**. Some studies about this group have shown that π_i^s is always finite for $i > 0$ but in general, the stable homotopy does not present any pattern as long as we run off the i's.

On the other hand, the main result which binds the so far framed manifolds and the homotopy theory, specifically, in terms of classification relies on the next theorem.

Theorem 1.5 (Pontrjagin-Thom)**.** There is an isomorphism

$$\varphi : \pi_n^s \to \Omega_n^{fr}$$

for each $n \geqslant 0$.

This remarkable theorem is the key step which turns the homotopy problem into a cobordism problem, and a particular result provided by Pontrjagin turns it into a homology problem. Although, the homotopy problem is still far from trivial, this generalisation provides the progress made in classification of manifolds up to cobordism.

For low dimension cases, some of the stable homotopy groups of spheres are (see Hatcher [9]):

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
π_n^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_6$	$\mathbb{Z}_{504}$	0	$\mathbb{Z}_3$

We can convince ourselves that these stable groups do not follow an obvious pattern. However, it is interesting how in actuality there is evidence of patterns in a deeper sense but not so uniform by projecting π_i^s onto its *p-components*, the quotient groups obtained by factoring out all elements of order relatively prime p. Eventually leading us to study *spectral sequences* but for instance we will just keep it mentioned. For a more intensive review is recommended to check Ravenel [23].

Chapter 2

Quadratic forms, framed manifolds and homotopy group of spheres

In 1940 Cahit Arf introduced an invariant of quadratic forms over a field of characteristic 2 (see [2]). In the literature, this invariant is known as the *Arf invariant* and turns out to be an interesting key for the solution of several classical problems in algebraic and differential topology about the topology of manifolds. The application of the Arf invariant in topology was introduced by Michael Kervaire as an invariant of $(4k+2)$-dimensional framed manifolds and defined as the Arf invariant of certain quadratic form on the middle-dimensional homology group $H_{2k+1}(M;\mathbb{Z}_2)$ of the manifold M and is called the *Arf–Kervaire invariant* or just *Kervaire invariant.* This is why the Arf invariant is the main tool to use in this work.

2.1 The Arf Invariant of Quadratic Forms

Our motivation to study quadratic forms lies on their construction since they determine maps between Euclidean spheres. Of course,our interest will be on those maps which are not constant. So far, the Hopf construction map consists of the most systematic construction of quadratic maps between Euclidean spheres and the Arf invariant is the tool to classify this maps. Now, in order to define Arf invariant we introduce some basic concepts on quadratic and bilinear forms.

Definition 2.1. Let V be an n-dimensional vector space over any field R of characteristic $\neq 2$.

- A mapping $q : V \to R$ is called *quadratic form* on V if for every basis $\{v_1, \dots, v_n\}$ of V there is a matrix $A = (a_{ij}) \in M(R)^{n\times n}$ such that

$$q(x_1v_1 + \cdots + x_nv_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j = x^\top Ax.$$

  for all $x \in R^n$.

- We say $B : V \times V \to R$ is a *bilinear form* if for all $u, v, w \in V$ and $\lambda \in R$:

  i) $B(u + v, w) = B(u, w) + B(v, w)$.
  ii) $B(u, v + w) = B(u, v) + B(u, w)$.
  iii) $B(\lambda u, v) = B(u, \lambda v) = \lambda B(u, v)$.

  A bilinear form B is said to be *symmetric* if $B(u, v) = B(v, u)$ for all $u, v \in V$. And B is called *nondegenerate* if $B(u, v) = 0$ for all $v \in V$ implies $u = 0$.

Remark 2.1. *There is a condition which states that for a quadratic form* q *there exists a bilinear form* B *such that* $q(v) = B(v, v)$ *for all* $v \in V$*. Moreover, it also satisfies*

$$q(u + v) - q(u) - q(v) = 2B(u, v) \text{ for all } u, v \in V.$$

The purpose of this work on the study of quadratic forms lies on the ones taking values in the field $\mathbb{Z}_2$, notice that in order to re-define a quadratic form and the associated bilinear form this fashion, it is needed to make some changes because of the division by 2 is not allowed in this field and the identity in remark 2.1 above turns the quadratic form into linear which contradicts the fact of being quadratic.

Definition 2.2. We define a map $q : V \to \mathbb{Z}_2$ as a *quadratic form* if there exists a bilinear form $B(x, y)$ for which:

$$q(x + y) + q(x) + q(y) = B(x, y). \tag{2.1}$$

And a quadratic form q is said to be *nondegenerate* if the bilinear form B is nondegenerate.

Remark 2.2. *For a quadratic form* q *with values in* $\mathbb{Z}_2$*, the signs displayed in 2.1 are not as important as the fact how* $2B(x, y)$ *is replaced by* $B(x, y)$*. We can also check that* $B(x, y) = B(y, x)$ *and* $B(x, x) = q(2x) + 2q(x) = 0$ *for all* $x, y \in V$ *which, for the latter identity, does not hold as in the real case.*

Finally, we are in conditions to define our main tool, the Arf invariant.

Definition 2.3.

- Let q be a nondegenerate quadratic form on V with values in $\mathbb{Z}_2$. A set of elements $e_1, f_1, \dots, e_n, f_n \in V$ such that:

  $$B(e_i, e_j) = B(f_i, f_j) = 0 \quad \text{and} \quad B(e_i, f_j) = \delta_{ij},$$

  is called a **symplectic basis** of V.

- The element in $\mathbb{Z}_2$ defined as:

  $$\text{Arf}(q) = \sum_{i=1}^{n} q(e_i)q(f_i)$$

  is called the *Arf invariant* of the nondegenerate quadratic form q.

Some of the features around the Arf invariant $\text{Arf}(q)$, such as the following, are stated in the original Arf paper [2] and they can also be found in a detailed survey of Prasolov [21].

Theorem 2.3. Any nondegenerate quadratic form q can be reduced to the form $x_1y_1 + \cdots + x_ny_n + \text{Arf}(q)(x_n^2 + y_n^2)$. Moreover, the Arf invariant $\text{Arf}(q)$ does not depend on the choice of a symplectic basis.

It is also seen that q gives rise to nondegenerate quadratic forms q_0 and q_1 which have $\text{Arf}(q_0) = 0$ and $\text{Arf}(q_1) = 1$. Besides, by setting φ_i to be the restriction of q to the subspace spanned by the vectors e_i and f_i i.e. the quadratic form $q|_{U_i} = \varphi : U_i \to \mathbb{Z}_2$ where $U_i = span(e_i, f_i)$ for $i = 1, \dots, n$. In this fashion, the symplectic basis admits to be decompose the quadratic form $q = \varphi_1 \oplus \cdots \oplus \varphi_n$ through the direct sum of the subspaces U_i and now each form φ_i turns out to be equivalent either to q_0 or to q_1. Then we can make sense to introduce the notation:

$$\underbrace{q \oplus \cdots \oplus q}_{\text{n-times}} = nq,$$

in terms of the direct sum of subspaces U_i and together with the next result.

Lemma 2.4. *The quadratic forms $\psi_0 = q_0 \oplus q_0$ and $\psi_1 = q_1 \oplus q_1$ are equivalent.*

This shows how the direct sum of a pair of subspaces whose restriction φ_i, φ_j with $\mathrm{Arf}(\varphi_i) = \mathrm{Arf}(\varphi_j) = 1$ produce a subspace with Arf invariant zero as well as it happens for the direct sum of subspaces with Arf invariant zero.

2.2 Classification of Maps of Spheres: S^{n+2} into S^n

The purpose for this section is to offer a brief extract of the work developed by Lev Pontrjagin [20] in his attempt to classify maps of spheres, more precisely, maps from S^{n+k} into S^n. The main contribution of his work consisted in expressing the *Hopf invariant*, a homotopy invariant of maps between spheres, as a homology invariant associated to a k-dimensional framed manifold. Moreover, he applied this technique to compute the stable homotopy groups π_k^s for $k = 0, 1$ and 2 and the particular cases of the Hopf fibrations $S^3 \to S^2$ and $S^7 \to S^4$.

The classification of maps of S^{n+2} into S^n is based on the construction of a homology invariant $\delta(M, U)$ of the 2-dimensional framed manifold M in the Euclidean space E^{n+2} taking either of the values 0 or 1. The construction of δ is described as follows:

Let $U(x) = \{u_1(x), \ldots, u_n(x)\}$ be an orthonormal frame for a 2-dimensional manifold M and let C be a smooth simple and closed curve on M. Denote $u_{n+1}(x)$ the unit normal vector to C touching the surface M at the point $x \in C$ and set $V(x) = \{u_1(x), \ldots, u_{n+1}(x)\}$.

The invariant δ is defined for 1-dimensional manifolds (C, V) and denoted by $\delta(C)$. Suppose that M is a connected surface whose genus we designate by g. There exists a system of smooth simple closed curves $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ such that A_i and B_i intersect at a single point, $i = 1, \ldots, g$, but no two other curves intersect at all. As result:

$$\delta(M, U) = \sum_{i=1}^{g} \delta(A_i)\delta(B_i).$$

is a homology invariant of the framed manifold (M, U). We can notice how this invariant δ looks similar to the Arf invariant saw at previous section. However, δ takes values on curves. It turns out that δ is actually the *Kervaire invariant* of a 2-dimensional framed manifold i.e. $\delta(M, U) = \mathrm{Arf}(q)$ with q being a quadratic form defined on $H_1(M; \mathbb{Z}_2)$.

The more important details on the construction of the invariant δ are explained next. Let M be an orientable surface i.e. a smooth closed and orientable 2-dimensional manifold, and let N be a curve, meaning a smooth closed 1-dimensional manifold. Let f be a regular map of N into M such that no three distinct points of N are mapped to the same point of M. Let $C = f(N)$ be a curve on M, a point of the form $x = f(a) = f(b)$ with $a \neq b$ is called a *double point* of C.

Consider a curve C on the surface M to be *nullhomologous* (or more precisely nullhomologous mod 2), denoted $C \sim 0$ or $C = \Delta G$, if there exists an open set G on the surface M such that $C = \overline{G} - G$ and that in any neighbourhood of a point $x \in C$ there are points of M not belonging to the closure $\overline{G}$.

Let $C_1 = f_1(N)$ and $C_2 = f_2(N)$ be two curves on M such that the double points do not belong to the other and at each point of intersection of the two curves the tangents to them are nonparallel. In this case $C_1 \cup C_2$ is again a curve and it is said that C_1 and C_2 *admit addition* and conveniently written $C_1 + C_2$ for $C_1 \cup C_2$. Of course, we can notice how the intention of these notions is to establish an equivalence relation of smooth closed curves on M. Indeed, if $C_1 = \Delta G_1$ and $C_2 = \Delta G_2$ and admit addition we can define the relation $C_1 \sim C_2$ by $C_1 + C_2 = \Delta G$, where $G = (G_1 \cup G_2) - (G_1 \cap G_2)$. Apparently this relation makes sense only if the curves C_1 and C_2 admit addition. However, we can find a curve C_1' such that $C_1' \sim C_2$ via a homotopy between the curves C_1 and C_1'. In other words, if there are two curves that do not admit addition we can *enlarge* one of them along M to get a pair which that do. As a geometrical interpretation of the equivalence classes of the relation $\sim$.

The totality of all these classes is called *connectivity group* of the surface M. As expected, if z_1, z_2 represent the curves C_1 and C_2 which admit addition then the class defined $z = z_1 + z_2$ represents the curve $C_1 + C_2$. The associativity law for elements z_1, z_2, z_3 can be checked by transitivity of admitting addition relation and the associative law on $(C_1 + C_2) + C_3 = \Delta G$ where:

$$G = [(G_1 \cup G_2) \cup G_3] - [(G_1 \cap G_2) \cap G_3].$$

The identity element of this group consists of the null class z_0 representing the nullhomologous curve $C \sim 0$ and the inverse element of any z representing a curve $C = f(N)$ will be the one which represents the reversed oriented curve $-C$. This makes the connectivity group, as its name suggests, a group which can for short be denoted by $\Delta^1 = \Delta^1(M)$.

Definition 2.4.

- A finite collection of curves $C_1, \ldots, C_q$ on M is called a **homology basis** if given any curve C on the surface M, we have a relation:

$$C \sim \sum_{i=1}^{q} \epsilon_i C_i,$$

  where $\epsilon_i \equiv 0$ or $1 \mod 2$ and the coefficients ϵ_i are zero whenever $C \sim 0$.

- Let C_1 and C_2 be two curves on M admitting addition. We define the *intersection index* between two curves, denoted $I(C_1, C_2)$, as the number of points of intersection of C_1 and C_2 reduced $\mod 2$.

Notice that:

$$I(C_1 + C_2, C_3) = I(C_1, C_3) + I(C_2, C_3),$$

that is, the intersection index behaves as a bilinear map. Moreover $C_1 \sim 0$ implies $I(C_1, C_2) = 0$ meaning that the nullhomologous class of curves do not intersect any other, as expected. It also satisfy that if $C_1 \sim D_1$ and $C_2 \sim D_2$, then $I(C_1, C_2) = I(D_1, D_2)$ i.e. I is well-defined. We can use this intersection index in terms of the elements in the connectivity group, if $z_1, z_2 \in \Delta^1$ and $C_1 \in z_1$, $C_2 \in z_2$ then, defining $I(z_1, z_2) = I(C_1, C_2)$, we obtain a well-defined intersection index of two homology classes.

This is how we get to the most known notion of a canonical homology basis.

Definition 2.5. For a genus g closed surface M, a set of curves $A_1, \ldots, A_g, B_1, \ldots, B_g$ such that

$$I(A_i, A_j) = I(B_i, B_j) = 0 \qquad \text{and} \qquad I(A_i, B_j) = \delta_{ij}; i, j = 1, \ldots, g$$

and is called **canonical homology basis**.

Example 2.2.1. Consider the 2-dimensional torus $T = S^1 \times S^1$ built up as the quotient space of the square $[0,1] \times [0,1]$. The opposite sides of the square form the curves A_1, A_2 and B_1, B_2 satisfy the intersection index $I(A_i, A_j) = I(B_i, B_j) = 0$ and $I(A_i, B_j) = \delta_{ij}$ via the quotient identification since A_1 and A_2 became the same curve (resp. B_1 and B_2) as seen in the figure. This is how the curves A_1, B_1 is a canonical homology basis for T which are indeed the axes of the torus.

Example 2.2.2. We can obtain canonical homology basis to any surface of genus g extending this notion by considering the construction of the surface as the quotient of a $4g$-sided polygon.

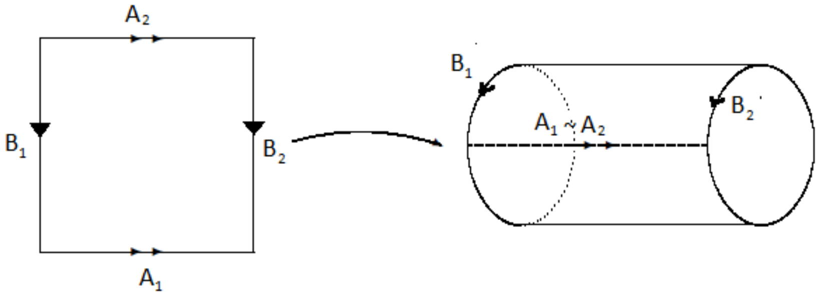

Figure 2.1: Canonical Homology Basis of T^2.

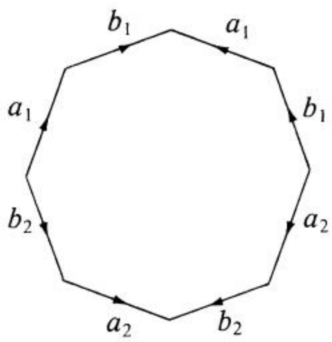

Figure 2.2: Canonical Homology Basis of the torus of genus 2.

At this point we can realise how the construction of the intersection index and the canonical homology basis is a particular representation of the standard dot product on the Euclidean space and the symplectic basis of a finite vector space seen at the previous section.

Moreover, it can be verified that for any homology class $z \in \Delta^1$:

$$I(z, z) = 0,$$

and if z_1 is a non-zero class, then there exist a class z_2 such that:

$$I(z_1, z_2) = 1.$$

We will now define the invariant δ for 2-dimensional framed manifolds (M, U) which is applied to a canonical homology basis and stated as:

$$\delta(M, U, C) = \delta(C) \equiv \beta(h) + r(C) + s(C),$$

where $r(C)$ is the number of connected components of the curve C, $s(C)$ is the number of its *double points*, and β is obtained as follows:

To each map h of a 1-dimensional manifold N into the Lie group $SO(n+1)$ of all rotations of a Euclidean space E^{n+1}, $n \geqslant 2$, $\beta(h)$ associates a residue class $\mod 2$. If $n \geqslant 3$ and N is connected the class $\beta(h)$ is zero if h is nullhomotopic and non-zero otherwise. If $n = 2$ we define $\beta(h)$ to be the degree of the map h reduced $\mod 2$.

From δ by setting $\delta(z) = \delta(C)$ we obtain a homology invariant of the class $z \in \Delta^1$ containing the curve C. Moreover for two arbitrary classes z_1 and z_2 of M we have:

$$\delta(z_1 + z_2) = \delta(z_1) + \delta(z_2) + I(z_1, z_2).$$

Finally we can rewrite the δ in terms of an arbitrary canonical basis $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ of the surface M by:

$$\delta = \delta(M, U) = \sum_{i=1}^{g} \delta(A_i)\delta(B_i).$$

The invariant δ constructed this way is known as the *Kervaire invariant* but it is formally defined as follows.

Definition 2.6. Let (M, ν) a $2n$-dimensional framed manifold and let $q : H_n(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ be a quadratic map. The **Kervaire invariant** of the framed manifold (M, ν) is defined to be the Arf invariant $\mathrm{Arf}(q)$ of the quadratic form on the middle-dimensional homology group $H_n(M; \mathbb{Z}_2)$.

Actually, the invariant δ constructed by Pontrjagin consists of a particular case of the Kervaire invariant which is applied to 2-dimensional framed manifolds (i.e. $n = 1$) and helped Pontrjagin to compute the homotopy group $\pi_{n+2}(S^n) = \mathbb{Z}_2$ of maps between $S^{n+2} \to S^n$ for $n \geqslant 2$, which is the cobordism group of surfaces embedded in S^{n+2} with trivialised normal bundle. This is how the Kervaire invariant is used to classify maps of spheres.

Chapter 3

Cayley–Dickson algebras and the modified polynomial multiplication

Our motivation to study more about algebras is concerned in an attempt to extend the range (on dimension) of a bilinear maps such as those constructed by José Adem [1] and K.Y. Lam [14]. The Cayley–Dickson algebras provide algebraic structures in a recursive way and at the same time shows how some *nice* algebraic properties are lost in the process. However these algebras constitute a vast collection of coefficients and hence a greater collection of bilinear maps to construct.

3.1 Basics on Algebras

Let us recall some basics on algebras over a field, our focus is to consider the field of real numbers $\mathbb{R}$.

Definition 3.1.

- An **algebra** $\mathcal{A}$ over the field $\mathbb{R}$ is a vector space over $\mathbb{R}$ equipped with a multiplication $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(\mathcal{A}, +, m)$ is a ring. As usual, we abbreviate $m(x, y)$ as $x \cdot y$ or simply xy.
- An algebra $\mathcal{A}$ is a **division algebra** if given $x, y \in \mathcal{A}$ such that $xy = 0$, then either $x = 0$ or $y = 0$.
- A **normed division algebra** is an algebra $\mathcal{A}$ that is also a normed vector space and satisfies $n(ab) = n(a)n(b)$.

Remark 3.1. *The above multiplication m is also assumed to satisfy $m(x, 1) = m(1, x) = x$ for a nonzero element $1 \in \mathbb{A}$, where 1 is called identity. And we can see that $\mathcal{A}$ is a division algebra if the left and right multiplication by a nonzero element is invertible.*

Remark 3.2. *Notice that every normed division algebra is a division algebra. To see this suppose $xy = 0$ for some elements x, y. Then $n(xy) = n(x)n(y) = 0$. Since n takes real values this implies $n(x) = 0$ or $n(y) = 0$ which implies $x = 0$ or $y = 0$. It is known that the only normed division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$ as stated by Hurwitz (Th. 3.4).*

3.2 The Cayley–Dickson Algebras and a Matrix Representation for their Multiplication

The Cayley–Dickson Algebras consist of an infinite sequence of algebras constructed inductively, each time doubling in dimension. Such a construction is called the *Cayley–Dickson process*.

Consider the algebra of the real numbers $\mathbb{R}$ with usual multiplication, the involution $\bar{x} = x$ (recall an involution as a linear operator such that $\bar{\bar{x}} = x$ for all $x, y \in \mathbb{R}$), where $\bar{x}$ is called *conjugate* of x, and the correspondences induced by this involution:

$$tr(x) = x + \bar{x} \text{ , and } n(x) = x\bar{x},$$

known as the *trace* and the *norm* of x respectively (see Adem [1]).

Definition 3.2. Let $\mathbb{A}_0 = \mathbb{R}$ together with the previous equipment. For $n \geqslant 1$ we define the n-th **Cayley–Dickson algebra** over $\mathbb{R}$, denoted $\mathbb{A}_n = \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ as the set of ordered pairs $(x, y) \in \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ with multiplication given by:

$$(x, y) \cdot (z, w) = (xz - \bar{w}y, wx + y\bar{z})$$

and conjugation:

$$\overline{(x, y)} = (\bar{x}, -y).$$

the trace tr and the norm n are the maps from $\mathbb{A}_n \to \mathbb{R}$ given by:

$$tr((x, y)) = (x, y) + \overline{(x, y)} = tr(x), \text{and } n((x, y)) = (x, y)\overline{(x, y)} = n(x) + n(y).$$

It is very useful to mention that the conjugation of a product of two elements x, y in the Cayley-Dickson algebras $\mathbb{A}_n$ satisfies $\overline{xy} = \bar{y}\bar{x}$.

This construction yields as the first Cayley-Dickson algebras: $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ and $\mathbb{A}_3 = \mathbb{K}$ of the complex numbers, quaternions and Cayley numbers respectively, each one with a quite particular feature such as the loss of the order in the elements in $\mathbb{C}$, the non–commutativity in the case of the multiplication of quaternions $\mathbb{H}$ and the loss of associativity in $\mathbb{K}$. However these Cayley–Dickson algebras belong to the special class of the normed division algebras.

For instance consider the complex numbers $\mathbb{C}$, it is a real vector space having dimension two with basis $\{1, i\}$ i.e. we can write each complex number x uniquely as a linear combination $x = a + bi$ or as well as a pair (a, b) with $a, b \in \mathbb{R}$. The known rule for the complex multiplication is given by:

$$xy = (a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{3.1}$$

which can be checked by its correspondence on the multiplication as Cayley–Dickson algebra above by setting x and y as pairs of real numbers and recall the conjugation for reals as the identity operator. This multiplication turns the real vector space $\mathbb{C}$ into an algebra over $\mathbb{R}$.

On the other hand, we can check the identity for conjugation by means of the Cayley–Dickson process:

$$\overline{(a + bi)} = \bar{x} = \overline{(a, b)} = (\bar{a}, -b) = (a, -b) = a - bi.$$

as well as $\overline{xy} = \bar{y}\bar{x}$. The trace and the norm in $\mathbb{C}$ obtained via $x \mapsto x + \bar{x}$ and $n(x) = x\bar{x}$. The identity (3.1) allows us to prove that actually $n(xy) = n(x)n(y)$ for $x, y \in \mathbb{C}$ so that $\mathbb{C}$ is a normed division algebra.

Moving on the construction, the second Cayley–Dickson algebra corresponds to the quaternions $\mathbb{H}$ seen in Chapter 1 as a four–dimensional real vector space having basis $\{1, i, j, k\}$ which means that any quaternion q is uniquely represented by four real numbers as $q = a+bi+cj+dk$ or as a pair of complex numbers $(a+bi, c+di)$. It can be checked how the quaternion multiplication is governed by the Hamilton's rules:

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik,$$

and

$$i^2 = j^2 = k^2 = -1.$$

As a consequence of these rules, we can see how the multiplication of quaternions is *non–commutative*, more precisely it is *anti–commutative*.

The quaternions can be conjugated same as happened in $\mathbb{C}$ under the Cayley–Dickson process. Let $q = (x, y) \in \mathbb{H}$ with $x = a + bi$ and $y = c + di$ elements of $\mathbb{C}$:

$$\overline{a + bi + cj + dk} = \bar{q} = \overline{(x, y)} = (\bar{x}, -y) = (a - bi, -c - di) = a - bi - cj - dk$$

which is the expected fashion just as in $\mathbb{C}$ for the conjugate, and again as in $\mathbb{C}$ we have the corresponding identity $\overline{pq} = \bar{q}\bar{p}$ on $\mathbb{H}$ by some calculations.

The norm and trace are defined in a similar way in $\mathbb{H}$, namely $n(p) = p\bar{p}$ and $tr(p) = p + \bar{p}$. And to verify that $n(pq) = n(p)n(q)$ we use the Euler's formula (mentioned in Chapter 1) which proves that $\mathbb{H}$ is a normed division algebra and then a division algebra.

To have an illustration or better said to verify how the quaternion multiplication actually works. Let us calculate some basic products such as ij, jk, ki and ji with $i, j, k \in \mathbb{H}$. So we write each element as it corresponds in $\mathbb{H}$, in other words as the pairs of complex numbers $i = (i, 0), j = (0, 1)$ and $k = (0, i)$, then by using the formula for multiplication in the Cayley–Dickson algebra defined above we obtain:

$$\begin{aligned}
ij =& (i, 0) \cdot (0, 1) = (0 - 0, 1 \cdot i + 0) = (0, i) = k. \\
jk =& (0, 1) \cdot (0, i) = (0 - \bar{i} \cdot 1, 0 + 0) = (i, 0) = i \\
ki =& (0, i) \cdot (i, 0) = (0 - 0, 0 + i \cdot \bar{i}) = (0, 1) = j \\
ji =& (0, 1) \cdot (i, 0) = (0 - 0, 0 + 1 \cdot \bar{i}) = (0, -i) = -(0, i) = -k
\end{aligned}$$

Our main interest is focused on the Cayley numbers $\mathbb{K}$ which corresponds to the third Cayley–Dickson algebra $\mathbb{A}_3$ over $\mathbb{R}$ seen as an eight– dimensional real vector space with basis say $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$. In the sense as in $\mathbb{H}$ the multiplication of these basis elements is obtained from the above rule:

Table 1. Cayley Numbers Multiplication Table

	1	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
1	1	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
ε_1	ε_1	-1	ε_3	$-\varepsilon_2$	ε_5	$-\varepsilon_4$	ε_7	$-\varepsilon_6$
ε_2	ε_2	$-\varepsilon_3$	-1	ε_1	$-\varepsilon_6$	ε_7	ε_4	$-\varepsilon_5$
ε_3	ε_3	ε_2	$-\varepsilon_1$	-1	ε_7	ε_6	$-\varepsilon_5$	$-\varepsilon_4$
ε_4	ε_4	$-\varepsilon_5$	ε_6	$-\varepsilon_7$	-1	ε_1	$-\varepsilon_2$	ε_3
ε_5	ε_5	ε_4	$-\varepsilon_7$	$-\varepsilon_6$	$-\varepsilon_1$	-1	ε_3	ε_2
ε_6	ε_6	$-\varepsilon_7$	$-\varepsilon_4$	ε_5	ε_2	$-\varepsilon_3$	-1	ε_1
ε_7	ε_7	ε_6	ε_5	ε_4	$-\varepsilon_3$	$-\varepsilon_2$	$-\varepsilon_1$	-1

Notice that the multiplication of Cayley numbers is non–associative. For example it can checked

$$(\varepsilon_1\varepsilon_2)\varepsilon_3 = \varepsilon_1(\varepsilon_2\varepsilon_3) = -1 \qquad \text{while} \qquad (\varepsilon_1\varepsilon_2)\varepsilon_4 = \varepsilon_7 \neq -\varepsilon_7 = \varepsilon_1(\varepsilon_2\varepsilon_4).$$

However, the Cayley number restricted to those with coefficient zero on the components $\varepsilon_4, \varepsilon_5, \varepsilon_6$ and ε_7 i.e. $x = (p, 0)$ with $p \in \mathbb{H}$ take us back to the multiplication on $\mathbb{H}$, the same occurs if we have $p = (z, 0)$ with $z \in \mathbb{C}$ where we recover the commutative property. In other words, we can obtain the complex and quaternion multiplications seen each of $\mathbb{C}$ and $\mathbb{H}$ as a copy inside $\mathbb{K}$ more precisely as vector subspaces. Some other examples of these subspaces are those with the form $\{z = a + b\varepsilon_k \backslash a, b \in \mathbb{R}\}$ for any $k = 1, 2, \ldots, 7$ which are subspaces of $\mathbb{K}$ isomorphic to $\mathbb{C}$.

Despite of the fact about the non–associativity on the Cayley numbers, they satisfy a particular type of associativity called *alternate associativity.* An algebra is **alternate associative** or *alternative* for brief if any two elements generate an associative algebra.

A more common condition used to illustrate this definition can be stated as:

$$x(yx) = (xy)x \tag{1}$$
$$x(xy) = (xx)y \tag{2}$$
$$x(yy) = (xy)y \tag{3}$$

for all elements $x, y \in \mathbb{K}$. These identities are usually called the *flexible law* (1) and the *alternative laws* for (2) and (3). All these properties turn $\mathbb{K}$ into an alternative algebra, moreover into a *normed alternative algebra* through the conjugation defined for:

$$x = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4 + x_5\varepsilon_5 + x_6\varepsilon_6 + x_7\varepsilon_7$$

by:

$$\bar{x} = x_0 - x_1\varepsilon_1 - x_2\varepsilon_2 - x_3\varepsilon_3 - x_4\varepsilon_4 - x_5\varepsilon_5 - x_6\varepsilon_6 - x_7\varepsilon_7$$

We can gather these alternate laws and the restrictions of the Cayley numbers to obtain some interesting subalgebras.

Example 3.2.1. We can prove how the lateral multiplication of complex numbers acts as an associative subalgebra over $\mathbb{K}$ i.e. $z_1(z_2x) = (z_1z_2)x$ for $z_1, z_2 \in \mathbb{C}$ and $x \in \mathbb{K}$. Consider the elements $z_1, z_2 \in \mathbb{C}$ and $x \in \mathbb{K}$ as pairs of quaternions $z_1 = (u_1, 0), z_2 = (v_1, 0)$ and $x = (p_1, p_2)$ which we should recall they are associative. Then:

$$\begin{aligned}(z_1z_2)x &= [(u_1, 0) \cdot (v_1, 0)](p_1, p_2)\\ &= (u_1v_1, 0) \cdot (p_1, p_2)\\ &= ((u_1v_1)p_1, p_2(u_1v_1)).\end{aligned}$$

$$\begin{aligned}z_1(z_2x) &= (u_1, 0)[(v_1, 0) \cdot (p_1, p_2)]\\ &= (u_1, 0) \cdot (v_1p_1, p_2v_1)\\ &= (u_1(v_1p_1), (p_2v_1)u_1)\\ &= ((u_1v_1)p_1, p_2(u_1v_1)) = (z_1z_2)x.\end{aligned}$$

We can also prove how this property holds for the right multiplication by complex numbers.

To conclude the features about the alternative algebras, these are used to prove the generalisation on the *Cayley–Dickson algebras* in the particular cases of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$.

Theorem 3.3. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$ are the only alternative division algebras.

Theorem 3.4. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{K}$ are the only normed division algebras.

Theorem 3.5. All division algebras have dimension $1, 2, 4$ or 8.

There are several versions of all these theorems and some these reviews can be seen in a 1930 paper by Zorn [31] for the first. About the second takes us back to the 1898 paper by Hurwitz [11] and it is also in a compilation made by Shapiro [27] and for a modern proofs of both these theorems (see Schafer [26]). Finally, the last theorem is considered as the top result about division algebras and the two independent proofs are given in Kervaire [12] and by Bott-Milnor [6].

The construction of the Cayley–Dickson algebras provides us certain peculiarities as we have seen it already, in terms of the multiplication of its elements, such as the loss commutativity and associativity. These features present a great source for the construction of both normed and nonsingular bilinear maps just as some of the given by Adem, Lam and Rodríguez (see [24]). Essentially we keep our focus on Cayley numbers $\mathbb{A}_3 = \mathbb{K}$.

However, it could be tedious to manipulate the elements in $\mathbb{K}$ in their explicit form so we introduce some standard notation based on matrices which seems pretty natural by construction of these algebras.

The multiplication in the n-th Cayley–Dickson algebra $\mathbb{A}_n$ can be encoded into the product between a matrix of size $2^n \times 2^n$ and a coordinate vector (i.e. $A \cdot y$) as mentioned in 1.2. The conjugation operator can be seen as a matrix $\kappa = diag(1, -1, \ldots, -1)$.

Now let $L_a : \mathbb{A}_n \to \mathbb{A}_n$ (resp. R_a) be the linear map given by left (resp. right) multiplication by the element $a \in \mathbb{A}_n$. Setting A (resp A') as the matrix of size $2^n \times 2^n$ with the real entries associated to L_a (resp. R_a), the maps L_a and R_a are actually given by the matrix products:

$$L_a(x) = ax = Ax \qquad \text{and} \qquad R_a(x) = xa = A'x.$$

The polynomial multiplication of finite degree comprise a vast collection of bilinear maps, more specifically they are nonsingular.

Let $a = (a_0, \ldots, a_{r-1})$, and $b = (b_0, \ldots, b_{s-1})$ be the vectors containing the coefficients of the polynomials $a_0 + \cdots + a_{r-1}t^{r-1}$ and $b_0 + \cdots + b_{s-1}t^{s-1}$ of degree $r-1$ and $s-1$ respectively, and denote $\phi(a, b)$ the vector which contains the coefficients of the polynomial $(a_0 + \cdots + a_{r-1}t^{r-1})(b_0 + \cdots + b_{s-1}t^{s-1})$ of degree $r+s-2$. If we consider the multiplication of these polynomials we can represent it in terms of the product of a matrix say α containing the coefficients a and the coordinate vector b or a matrix β with the coefficients of b multiplied by the vector a:

$$\alpha b = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \vdots & \\ \vdots & \vdots & \vdots & a_0 \\ a_{r-1} & a_{r-2} & \vdots & \vdots \\ & a_{r-1} & \vdots & \vdots \\ & & & a_{r-1} \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_{s-1} \end{bmatrix} \quad \text{or} \quad \beta a = \begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & b_1 & \vdots & \\ \vdots & \vdots & \vdots & b_0 \\ b_{s-1} & b_{s-2} & \vdots & \vdots \\ & b_{s-1} & \vdots & \vdots \\ & & & b_{s-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{r-1} \end{bmatrix}$$

We can carry out this very same representation if we consider the polynomials with coefficients in any Cayley–Dickson algebra by placing the corresponding coefficient coded matrices A_i and B_i instead of a_i and b_i as above. Of course being cautious about the importance of the left and right multiplication of these coefficients matrices.

Example 3.2.2. To illustrate this notion, consider the nonsingular bilinear map constructed by Lam [14] $\phi : \mathbb{R}^{16} \times \mathbb{R}^{16} \to \mathbb{R}^{23}$ from $\mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}^3$ and given by the formula:

$$\phi((x_1, x_2), (y_1, y_2)) = (x_1y_1 - \bar{y_2}x_2, y_2x_1 + x_2\bar{y_1}, x_2y_2 - y_2x_2)$$

and we can check it with the matrix:

$$\begin{bmatrix} X_1 & -X_2'\kappa \\ X_2\kappa & X_1' \\ 0 & X_2 - X_2' \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

At this point we are motivated to introduce some modifications of this matrix representation which can be done by elementary operations on the matrix. Just observe first that we replace the third entry by the commutator $x_1y_1 - y_1x_1$:

$$\begin{bmatrix} X_1 & -X_2'\kappa \\ X_2\kappa & X_1' \\ X_1 - X_1' & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and then subtracting the first row to the third, obtaining as a equivalent matrix:

$$\begin{bmatrix} X_1 & -X_2'\kappa \\ X_2\kappa & X_1' \\ -X_1' & X_2' \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Example 3.2.3. Proceeding similarly, we can check the nonsingular bilinear map constructed by Adem $\psi : \mathbb{K}^2 \times \mathbb{K}^4 \to \mathbb{K}^5$ defined by:
$\psi_1(x, y) = x_1y_1 + x_2\bar{y_4}$
$\psi_2(x, y) = x_2y_2 - x_1\bar{y_3}$
$\psi_3(x, y) = \bar{y_1}x_2 - x_1\bar{y_2}$
$\psi_4(x, y) = y_3x_2 - x_1y_4$
$\psi_5(x, y) = x_1y_1 - y_1x_1$

Can be represented by the matrices (the one on the right is obtained by interchange of rows, addition, etc.):

$$\begin{bmatrix} X_1 & 0 & 0 & X_2\kappa \\ 0 & X_2 & -X_1\kappa & 0 \\ X_2'\kappa & -X_1\kappa & 0 & 0 \\ 0 & 0 & X_2' & -X_1 \\ X_1 - X_1' & 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} X_1 & 0 & 0 & X_2\kappa \\ X_2'\kappa & -X_1\kappa & 0 & 0 \\ 0 & X_2 & -X_1\kappa & 0 \\ 0 & 0 & X_2' & -X_1 \\ -X_1' & 0 & 0 & -X_2\kappa \end{bmatrix}$$

This last matrix seems quite similar to the one representing a polynomial multiplication seen at the beginning of the section. In this particular case the multiplication of a linear polynomial $x_1 + x_2t$ and a cubic polynomial $y_1 + y_2t + y_3t^2 + y_4t^3$ with coefficients in $\mathbb{K}$ and the parameter t commuting with all coefficients.

So then the map ψ (as well as ϕ) representation looks like a polynomial product up to apparently not-so random details such as:

1. Some minus signs included.
2. Conjugations for some elements y_i, but none of the x_i but either way conjugation transposes the corresponding block.
3. The *priming* of some blocks, meaning the change of left multiplication by right multiplications.
4. The addition of some possibly nonzero blocks in the off–diagonal corners.

This is why we say that the bilinear maps ψ and ϕ can be seen as **modified polynomial multiplications**. Something interesting about this kind multiplication lies on the fact that most of the examples of nonsingular maps known to date are a modified polynomial multiplication (Rodríguez [24]) and these maps can actually be a source of new nonsingular bilinear maps. The actual purpose of these modifications lies on keeping the nonsingular property of the map and decrease the dimension of its image and hence improving the upper bound for $r\#s$.

Chapter 4

Discussion of results

From now on we will show some applications of the tools developed all to classify both nonsingular and normed bilinear maps provided by Kee Yuen Lam on the composition formula problem all along the 70s and 80s.

Recall a composition formula of size $[r, s, n]$:

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = \phi_1^2 + \cdots + \phi_n^2. \qquad (*)$$

As seen before, in order that a nonsingular bilinear map f can exist, certain numerical conditions must be satisfied by r, s and n such as the Stiefel–Hopf condition or the given by the Hurwitz–Radon function which leads us to translate it into estimations for $r\#s$ and $r * s$.

Lam illustrated how the normed property of a bilinear map is identifiable with a composition formula using as an example a sum of squares of size $[2, 2, 3]$ which after coordinate changes he obtained a composition formula of size $[2, 2, 2]$ i.e. the formula for the complex multiplication (see[16]). However this does not hold for polynomial multiplication which is a nonsingular bilinear map and contrary to the complex multiplication its image is not contained in itself due to the addition of the degree.

The most remarkable property of the Hopf map studied by Lam about normed maps is that any inverse image $H^{-1}(q)$, $q \in S^n$ is a linear subsphere of S^{r+s-1}, that is, a subsphere cut out from S^{r+s-1} by a linear subspace in $\mathbb{R}^r \times \mathbb{R}^s$. So then it turns out that the Hopf construction map has features quite similar to the classical Hopf fibration maps $S^{2n-1} \to S^n$ for $n = 2, 4, 8$ previously mentioned. This is how the definition of the Hopf construction map for a nonsingular bilinear map makes sense and motivates an homotopy classification.

4.1 The Nonsingular Map $\mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^9$

In [14], Lam constructed some examples of nonsingular bilinear maps from $f : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}^3$, where $\mathbb{K}$ represents the Cayley numbers or the 3^{rd} Cayley–Dickson algebra as mentioned in chapter 4, this by restricting f to certain subspaces of $\mathbb{K}^2 \times \mathbb{K}^2$ and used general theory of **framed cobordism** to classify the nonsingular bilinear map $f : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{R} \oplus \mathbb{C}^4$ [15] in which determined that the Hopf construction map h associated to f represents the homotopy class $\eta^2 \in \pi_{11}(S^9)$.

Note 4.1. The classical Hopf fibrations introduced all along are usually denoted by: $\eta : S^3 \to S^2$, $\nu : S^7 \to S^4$ and $\sigma : S^{15} \to S^8$ and the symbol η^2 represents the homotopy class of the composition $(\Sigma^8\eta) \circ (\Sigma^7\eta) : S^{11} \to S^9$ which lies on the stable 2^{nd}-stem.

The bilinear map $f : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{R} \oplus \mathbb{C}^4$ defined by sending $z \times w = (z_1, z_2, z_3) \times (w_1, w_2, w_3)$ to the vector:

$$\begin{aligned}
f_1(z,w) =& \overline{w_1} z_1 + \overline{z_1} w_1, \\
f_2(z,w) =& \overline{w_1} z_1 + \overline{w_2} z_2 + w_3 z_3, \\
f_3(z,w) =& w_2 \overline{z_1} + z_2 \overline{w_1}, \\
f_4(z,w) =& z_2 \overline{w_3} - w_2 \overline{z_3}, \\
f_5(z,w) =& z_1 \overline{w_3} - w_1 z_3.
\end{aligned}$$

Besides, this bilinear map f can be represented as a matrix in the sense of a modified polynomial multiplication as seen at Chapter 4, in particular:

$$\begin{bmatrix} Z_1'\kappa + \overline{Z_1} & 0 & 0 \\ Z_1'\kappa & Z_2'\kappa & Z_3' \\ Z_2\kappa & \overline{Z_1'} & 0 \\ 0 & -\overline{Z_3'} & Z_2\kappa \\ -Z_3' & 0 & Z_1\kappa \end{bmatrix} \simeq \begin{bmatrix} \overline{Z_1} & -Z_2'\kappa & -Z_3' \\ Z_1'\kappa & Z_2'\kappa & Z_3' \\ Z_2\kappa & \overline{Z_1'} & 0 \\ 0 & -\overline{Z_3'} & Z_2\kappa \\ -Z_3' & 0 & Z_1\kappa \end{bmatrix} \simeq \begin{bmatrix} Z_2\kappa & 0 & \overline{Z_1'} \\ -Z_3' & Z_1\kappa & 0 \\ Z_1'\kappa & Z_3'\kappa & Z_2'\kappa \\ \overline{Z_1}\kappa & -Z_3'\kappa & -Z_2'\kappa \\ 0 & Z_2\kappa & -\overline{Z_3'} \end{bmatrix}, \tag{4.1}$$

Now, let h be the Hopf construction map associated to f. In order to prove the class $[h]$ is η^2 we must analyse first of all, the type of the fibers for h. In this case, we must check the point $m = (0,0,0,1,0,0)$ in $\mathbb{R}^2 \oplus \mathbb{C}^4$ is a regular value of h and such fiber $M = h^{-1}(m)$ is a smooth manifold diffeomorphic to a 2-dimensional torus. Then compute its Kervaire invariant seen as a framed manifold (M, σ) for some frame σ of M. This is the main interest on this work.

First, the diffeomorphism recovered from Lam [15] from $S^1 \times S^1 \to M$:

$$(u_1, u_2) \to \zeta = \frac{\sqrt{2}}{4}(u_1 + u_2, u_1 - u_2, 0, u_1 - u_2, u_1 + u_2, 0) \tag{$*$}$$

consists in a torus sitting inside the unit sphere in $\mathbb{C}^3 \times \mathbb{C}^3$. Then consider the frame σ induced by the standard framing $e_1, \ldots, e_{10} \in \mathbb{R}^2 \oplus \mathbb{C}^4$ at m, an auxiliary subspace of $\mathbb{C}^3 \times \mathbb{C}^3$ namely E defined by $z_1 = w_2$, $z_2 = w_1$ and $z_3 = w_3 = 0$ as well as the unit sphere of E denoted $S^3(E)$ and an auxiliary frame ν constructed by Pontrjagin [20] and reordered.

$$\begin{array}{ll}
\nu_1 = (z_2, z_1, 0, z_1, z_2, 0) & = \text{normal vector of } M \text{ in } S^3(E) \text{ at } \zeta; \\
\nu_2 = (z_1, z_2, 0, z_2, z_1, 0) & = \text{the position vector;} \\
\nu_3 = (1, 0, 0, 0, -1, 0), & \nu_4 = (i, 0, 0, 0, -i, 0), \\
\nu_5 = (0, 1, 0, -1, 0, 0), & \nu_6 = (0, i, 0, -i, 0, 0), \\
\nu_7 = (0, 0, 1, 0, 0, 0), & \nu_8 = (0, 0, i, 0, 0, 0), \\
\nu_9 = (0, 0, 0, 0, 0, 1), & \nu_{10} = (0, 0, 0, 0, 0, i).
\end{array}$$

Recall from Chapter 3 the Kervaire invariant of a $2n$-dimensional framed manifold (M, σ) is computed as the Arf invariant of a quadratic form defined on $H_n(M; \mathbb{Z}_2)$. In this case, the quadratic form q_ν can be defined on $H_1(M; \mathbb{Z}_2)$. So, the axes of the torus we get a homology basis γ_1, γ_2 for the 1-dimensional cycles on M.

Consider γ_i be the one dimensional cycle on M defined by setting $u_i = 1$ in $(*)$ and the cycle $\gamma_1 + \gamma_2$ defined by $z_1 = 0$ (or $z_2 = 0$) on M. Let μ consist of the vector fields ν_1, ν_2 on M. We should notice that $q_\mu = q_\nu$ and this can be checked by examining the values of the quadratic map on the three cases for the cycles in M. The value $q_\mu(\gamma_1) = 0$ since $\gamma_1 = \nu_1 + \nu_2$ for the fixed parameter $z_1 = 1$ and by considering, in a quadratic map, the invariance of the framed manifold (M, μ) up to a sign change of the normal vectors and up to the involution $(z_1, z_2) \mapsto (z_1, -z_2)$, so we can write:

$$q_\mu(\gamma_1) = q_\mu(\nu_1 + \nu_2) = q_\mu(\nu_1) + q_\mu(\nu_2) + B(\nu_1, \nu_2) = 0.$$

Similarly is obtained $q_\mu(\gamma_2) = 0$ while $q_\mu(\gamma_1 + \gamma_2) = q_\mu(\gamma_1) + q_\mu(\gamma_2) + I(\gamma_1, \gamma_2) = 1$. This allows us to determine that $\mathrm{Arf}(q_\mu) = 0$. An alternative argument provided by Lam says that the homotopy class of the framed manifold (M, μ) in $\pi_4(S^2)$ represents the suspension of the class of (M, ν_1) in $\pi_3(S^1) = 0$ which implies that $\mathrm{Arf}(q) = 0$ and hence the values $q_\mu(\gamma_1) = q_\mu(\gamma_2) = 0$ while $q_\mu(\gamma_1 + \gamma_2) = 1$.

To calculate the Kervaire invariant of (M, σ) notice that σ and ν differ by the rotation $\phi : M \to GL(10, \mathbb{R})$ needed to get the a point in M between these frames. Given any 1-cycle γ in M, the rotation restricted to the cycle γ, $\phi|_\gamma$, is therefore a element of $\pi_1(GL(10, \mathbb{R})) = \mathbb{Z}_2$ that actually describes the difference between $Q_\mu(\gamma)$ and $Q_\nu(\gamma)$. Since σ is the framing induced from the standard frame $e_1, \ldots, e_{10}$ at m, the matrix associated to the rotation $\phi(\zeta)$ with $\zeta = (z_1, z_2, 0, z_2, z_1, 0) \in M$ can be obtained by writing out the vectors of the differential map $Dh_\zeta(\nu_i)$ running through the frame ν. Consider $z_1 = a + bi$ and $z_2 = c + di$.

$$\begin{array}{llllll}
Dh_\zeta(\nu_1) = 2(0 & 1 & 1 & 0 & 0 & 0) \\
Dh_\zeta(\nu_2) = 2(0 & 0 & 0 & 1 & 0 & 0\ ) \\
Dh_\zeta(\nu_3) = 4(a & c & -di & bi & 0 & 0\ ) \\
Dh_\zeta(\nu_4) = 4(b & d & ci & -ai & 0 & 0\ ) \\
Dh_\zeta(\nu_5) = 4(c & -a & -bi & -di & 0 & 0\ ) \\
Dh_\zeta(\nu_6) = 4(d & -b & ai & ci & 0 & 0\ ) \\
Dh_\zeta(\nu_7) = 2(0 & 0 & 0 & 0 & -z_1 & -z_2\ ) \\
Dh_\zeta(\nu_8) = 2(0 & 0 & 0 & 0 & -z_1 i & -z_2 i\ ) \\
Dh_\zeta(\nu_9) = 2(0 & 0 & 0 & 0 & z_2 & z_1\ ) \\
Dh_\zeta(\nu_{10}) = 2(0 & 0 & 0 & 0 & -z_2 i & -z_1 i\ ).
\end{array}$$

The first two columns are real and each of the other columns are complex i.e. equivalent to two real columns. We can check this matrix is nonsingular which actually proves that the point $m \in \mathbb{R}^2 \oplus \mathbb{C}^4$ is a regular value of h. We can also notice the upper diagonal 6×6 block is nonsingular so it contributes nothing to the value of ϕ_γ because any 1-cycle γ on M is homologous to zero in $S^3(E)$. On the other hand, the lower 4×4 diagonal block, shows $\phi|_{\gamma_1 + \gamma_2} = 0$, $\phi|_{\gamma_2} = 1$. In quadratic forms Q_σ takes the value 1 on γ_1 and γ_2 and has Arf invariant 1.

The Nonsingular Map [12, 12, 20]. A similar bilinear map can be constructed by taking the values of the z's and w's in $\mathbb{H}$ instead of $\mathbb{C}$ in the definition of f, we obtain a map $f : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R} \oplus \mathbb{H}^4$ which is again nonsingular. Furthermore, f is also a modified polynomial multiplication whose representation is recovered from the matrix 4.2 (this is the reason to keep the *priming* displayed). On the other hand, the Hopf construction map associated to f will be $h : \mathbb{R}^{24} \backslash \{0\} \to \mathbb{R}^{18} \backslash \{0\}$. The point $m = (0, 0, 0, 1, 0, 0)$ in $\mathbb{R}^2 \oplus \mathbb{H}^4$ is a regular value for h, where $M = h^{-1}(m)$ is diffeomorphic to $S^3 \times S^3$ with an induced framing σ and the class $[h]$ represents the generator ν^2 in $\pi_{23}(S^{17})$. In this case, ν^2 represent the composition $(\Sigma^{16}\nu) \circ (\Sigma^{13}\nu)$ as seen in note 4.1.

4.2 The Normed Map $\mathbb{R}^{10} \times \mathbb{R}^{10} \to \mathbb{R}^{16}$

Consider $\mathbb{R}^{10}$ as $\mathbb{K} \oplus \mathbb{C}$, and let (x_1, x_2) be a element in $\mathbb{R}^{10}$ where $x_1 \in \mathbb{K}$ and $x_2 \in \mathbb{C}$. The map f given by

$$f((x_1, x_2), (y_1, y_2)) = (x_1 y_1 - x_2 y_2, \bar{y_2} x_1 + x_2 \bar{y_1}) \in \mathbb{K}^2.$$

Again, as we have been suggested all the previous examples, this map represents a modified polynomial multiplication by means of the matrix:

$$\begin{bmatrix} X_1 & -X_2 \\ X_2 \kappa & X_1' \kappa \end{bmatrix}. \tag{4.2}$$

This map is a slight modification of the multiplication of Cayley numbers regarded as pair of quaternions and its Hopf construction map $h_f : S^{19} \to S^{16}$ represents the homotopy class 2ν.

The inverse image $h_f^{-1}(m)$ with $m = (0, 0, 1) \in \mathbb{R} \oplus \mathbb{K}^2$ is a subvariety:

$$M = \{(x_1, x_2, y_1, y_2) \in S^{19} / y_1 = x_2, y_2 = x_1, x_2, y_2 \in \mathbb{C}\},$$

diffeomorphic to S^3. Then we must check h is a regular map i.e. for any $\zeta \in M$, the derivative $Dh_\zeta : \mathbb{R}^{20} \to \mathbb{R}^{17}$ has maximal rank.

Proceeding as in the previous example, let us fix the standard frame $\{e_i\}_{i=1}^{16}$ and conveniently ordered: $e_1 - e_4$, $e_9 - e_{12}$, $e_5 - e_8$ and $e_{13} - e_{16}$ at the point m in $\mathbb{R} \oplus \mathbb{K}^2 \cong \mathbb{R}^{17}$ and call its induced frame ρ.

Also let σ be an auxiliary frame at the point $\zeta = (x_1, x_2, x_2, x_1) \in M$ given by the vectors:

$$\begin{array}{llll} \sigma_1 = (0, -1, 1, 0), & \sigma_2 = (0, -\varepsilon_1, \varepsilon_1, 0), & \sigma_3 = (-1, 0, 0, 1), & \sigma_4 = (-\varepsilon_1, 0, 0, \varepsilon_1), \\ \sigma_5 = (\varepsilon_2, 0, 0, 0), & \sigma_6 = (\varepsilon_3, 0, 0, 0), & \sigma_7 = (0, 0, \varepsilon_2, 0), & \sigma_8 = (0, 0, \varepsilon_3, 0), \\ \sigma_9 = (\varepsilon_6, 0, 0, 0), & \sigma_{10} = (\varepsilon_7, 0, 0, 0), & \sigma_{11} = (\varepsilon_4, 0, 0, 0), & \sigma_{12} = (\varepsilon_5, 0, 0, 0), \\ \sigma_{13} = (0, 0, \varepsilon_4, 0), & \sigma_{14} = (0, 0, \varepsilon_5, 0), & \sigma_{15} = (0, 0, \varepsilon_6, 0), & \sigma_{16} = (0, 0, \varepsilon_7, 0), \end{array}$$

Then we must measure the difference between the frames σ and ρ. This can be done by a map $\phi : M \to SO(16)$ and writing out the vectors of the Jacobian matrix of the Hopf construction map h at the point $\zeta = (x_1, x_2, x_2, x_1) \in M$ if $x_1 = a + b\varepsilon_1$ and $x_2 = c + d\varepsilon_1$ we obtain:

$$\begin{array}{lcccc} Dh_\zeta(\sigma_1) = 4(& x_1 & d\varepsilon_1 & 0 & 0) \\ Dh_\zeta(\sigma_2) = 4(& x_1\varepsilon_1 & -c\varepsilon_1 & 0 & 0) \\ Dh_\zeta(\sigma_3) = 4(& -x_2 & b\varepsilon_1 & 0 & 0) \\ Dh_\zeta(\sigma_4) = 4(& -x_2\varepsilon_1 & -a\varepsilon_1 & 0 & 0) \\ Dh_\zeta(\sigma_5) = 4(& -\overline{x_1}\varepsilon_2 & -\overline{x_2}\varepsilon_2 & 0 & 0\ ) \\ Dh_\zeta(\sigma_6) = 4(& -\overline{x_1}\varepsilon_3 & -\overline{x_2}\varepsilon_3 & 0 & 0\ ) \\ Dh_\zeta(\sigma_7) = 4(& -x_2\varepsilon_2 & x_1\varepsilon_2 & 0 & 0\ ) \\ Dh_\zeta(\sigma_8) = 4(& -x_2\varepsilon_3 & x_1\varepsilon_3 & 0 & 0\ ) \\ Dh_\zeta(\sigma_9) = 4(& 0 & 0 & -\overline{x_1}\varepsilon_4 & -\overline{x_2}\varepsilon_5\ ) \\ Dh_\zeta(\sigma_{10}) = 4(& 0 & 0 & -\overline{x_1}\varepsilon_5 & -\overline{x_2}\varepsilon_5\ ) \\ Dh_\zeta(\sigma_{11}) = 4(& 0 & 0 & -\overline{x_1}\varepsilon_6 & -\overline{x_2}\varepsilon_6\ ) \\ Dh_\zeta(\sigma_{12}) = 4(& 0 & 0 & -\overline{x_1}\varepsilon_7 & -\overline{x_2}\varepsilon_7\ ) \\ Dh_\zeta(\sigma_{13}) = 4(& 0 & 0 & -x_2\varepsilon_4 & x_1\varepsilon_4\ ) \\ Dh_\zeta(\sigma_{14}) = 4(& 0 & 0 & -x_2\varepsilon_5 & x_1\varepsilon_5\ ) \\ Dh_\zeta(\sigma_{15}) = 4(& 0 & 0 & -x_2\varepsilon_6 & x_1\varepsilon_6\ ) \\ Dh_\zeta(\sigma_{16}) = 4(& 0 & 0 & -x_2\varepsilon_7 & x_1\varepsilon_7\ ). \end{array}$$

The matrix displayed has quaternionic columns i.e. two complex columns and it is nonsingular which implies that m is a regular value of h. Moreover, the rearrange of the standard frame allows us to keep our focus on the lower 8×8 diagonal block which represents a not nullhomotopic rotation $\phi : S^3 \to SO(16)$ between the frames σ and ρ then the Kervaire invariant of the frame manifold (S^3, σ) is one. This rotation is more likely an element of the orthogonal group $O(16)$ of all rotations with determinant ± 1 while the special orthogonal group $SO(16)$ of rotations with determinant 1 consists of a double covering of $O(16)$ by composing the rotation ϕ twice to get it as an element in $SO(16)$. Therefore the class of h represent twice the generator of $\pi_3(SO(16))$.

Conclusions

The tools developed all along such as the theory on vector bundles related to framed cobordism, the invariant constructed by Pontrjagin on surfaces and the contribution of Lam with maps and the method to classify them were used to another of the nonsingular map $\phi : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}^3$ provided by Lam as the map $[16, 16, 23]$, which is explicitly defined for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbb{K}$ by the formulas:

$$\begin{aligned}\phi_1(x, y) &= x_1 y_1 - \overline{y_2} x_2, \\ \phi_2(x, y) &= y_2 x_1 + x_2 \overline{y_1}, \\ \phi_3(x, y) &= x_2 y_2 - y_2 x_2.\end{aligned}$$

However, there was a serious inconvenient showed up in order to obtain the pre–image manifold of the Hopf construction map associated to ϕ, $M = h_{\phi}^{-1}(m)$, where $m = (0, 0, 1, 0) \in \mathbb{R}^{24}$. Since the equation given by the commutator component ($\phi_3(x, y) = 0$) implies that its solution is a subspace isomorphic to $\mathbb{C}$ immersed in $\mathbb{K}$. Despite of the fact that it was proved that the condition for the coordinates x_2 and y_2 is not trivial and then x and y are not equal to zero. It was expected for the solution to be seen as elements $a + b\varepsilon_k$ for each coordinate i.e. octonions lying on a complex plane up to any rotation a 90° respect to the real axis which changes the imaginary part ε_k. Further calculations around this assumption lead us to say that M was a 2–torus in $\mathbb{K}^2 \times \mathbb{K}^2$ while it was found an e–invariant one which involves notions on surgery theory and so far it has not been part of the original topic of the present work.

At this moment the goals to be achieved are precise: to complete the classification of the map $[16, 16, 23]$ showed above by finding out the solution for $h(x, y) = m$, then characterise the resultant manifold and compute its Kervaire invariant. Then makes natural to attack the problem of classify the nonsingular maps constructed by Lam in [14] which consist of restrictions of the original map defined in $\mathbb{K}^2 \times \mathbb{K}^2$. Finally, the work in medium–long term to attempt will consist in the use these maps to construct new bilinear maps introducer them as a modified polynomial multiplication as well as classify them.

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