UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS
DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

TOPOLOGICAL COMPLEXITY OF CONFIGURATION SPACES FOR ROBOT MOTION PLANNING

PRESENTA

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PARA OBTENER EL GRADO DE MAESTRO EN CIENCIAS CON OPCIÓN A MATEMÁTICAS APLICADAS

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Aguascalientes, Ags. Enero de 2015

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Sin otro particular me permito saludarle muy afectuosamente.

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Resumen

En este trabajo, estudiaremos el concepto de Complejidad Topológica (TC) de espacios de configuraciones para el movimiento planificado de robots. TC es un número entero positivo que mide las discontinuidades en el proceso de movimiento planificado en el espacio de configuraciones X (el conjunto de posibles posiciones del robot). En [6], Michael Farber determinó cotas superiores para TC en términos de la dimensión y de la Categoría de Lusternik–Schnirelman del espacio de configuraciones X; además, proporcionó una cota inferior trabajando con la cohomología del espacio de configuraciones. Con estas herramientas, es posible calcular TC para esferas de cualquier dimensión n y, en general, de cualquier superficie bidimensional cerrada, compacta, orientable y de género g. En particular, el producto de m esferas de dimensión n, puede considerarse como un brazo de robot con m articulaciones, el cual puede moverse en n dimensiones; el cálculo de TC para ese caso, también es proporcionado en [6]. Posteriormente en [8], se estudia el caso de espacios reales proyectivos de dimensión n, $\mathbb{RP}^n$, donde se proporciona una clasificación de TC para tales espacios y, en ciertos casos particulares, el cálculo es explícito. Se introducen otros conceptos que son de gran utilidad. De hecho, uno de los resultados más importantes es que el cálculo de $\mathrm{TC}(\mathbb{RP}^n)$, coincide con el problema clásico de inmersión de espacios reales proyectivos. Proporcionaremos una sólida justificación de todos los resultados mencionados en [6] y [8], además de varios ejemplos de casos particulares, que permitirán al lector comprender con mayor precisión lo que determina la teoría.

Abstract

In this work, we study the concept of Topological Complexity (TC) of configuration spaces for robot motion planning. TC is a positive integer which measures discontinuity of the process of motion planning in the configuration space X (the set of possible positions of the robot). In [6], Michael Farber gave upper bounds for TC in terms of the dimension and of the Lusternik–Schinerlman Category of the configuration space X; he also provided a lower bound working with the cohomology of the configuration space. With these tools, it is possible to calculate TC for spheres of any dimension n and, in general, for any 2–dimensional closed, compact, orientable surface of genus g. In particular, the product of m spheres of dimension n, can be seen like a robot arm with m articulations which moves in n dimensions. The calculation of TC for this case, is also given in [6]. Later in [8], the case of real projective spaces of dimension n, $\mathbb{RP}^n$, is studied, and a classification of TC for these spaces is provided and, in particular cases, the calculation is explicit. Other useful concepts are introduced. In fact, one of the most important results is that the calculation of $\mathrm{TC}(\mathbb{RP}^n)$, coincides with the classical immersion problem of real projective spaces. We will provide a solid justification for all results mentioned in [6] and [8], as well as several examples of specific cases, which allow the reader to understand more precisely what the theory determines.

Introduction

Suppose we want to do a task in a dangerous environment for human conditions. One way to solve this problem would be with the help of a robot. The idea is that the robot receives the coordinates of the starting point and destination, and it is able to move between these two points. Also we want that if we make a small modification to the coordinates, the way forward for the robot also has a slight change.

Mathematically, let X be the space of all possible positions of a mechanical system, i.e. a **configuration space**. The **motion plannining problem** consists of constructing a program, which takes pairs of configurations $(A, B) \in X \times X$ as an input and produces as an output a continuous path in X, which stars at A and ends at B. For simplicity we will assume that the configuration space X is path–connected. Nevertheless, the same set of ideas applies to the various path–connected components.

Throughout this work, I represents the interval $[0, 1]$. Let PX the space of all continuous paths $\gamma : I \to X$ in X. We denote by $\pi : PX \to X \times X$ the map associating to any path $\gamma \in PX$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space PX with compact–open topology. **The problem of motion planning in X consists of finding a function $s : X \times X \to PX$ such that the composition $\pi \circ s$ is the identity map.**

An interesting question arising from this problem is: Whether it is possible to construct a motion planning in the configuration space X so that the continuous path $s(A, B)$ in X, depends continuously on the pair of points (A, B).

Continuity of motion planning is an important natural requirement, because absence of continuity will result in the instability of behavior:

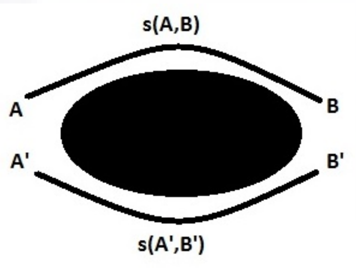


Figure 1: A non–continuous section

There will exist arbitrarily **close** pairs (A, B) and (A', B') of initial-desired configurations such that the corresponding paths $s(A, B)$ and $s(A', B')$ **are not close**.

But we have a serious problem, because a continuous motion planning exists only in very simple situations. The next best attempt would be to decompose the configuration space as the union of subspaces in each of which there exists a continuous motion planner. This situation is precisely measured by the Topological Complexity.

Chapter 1

Preliminaries

Before defining TC, recall some important concepts.

Definition 1.0.1. Let X and Y be two topological spaces.

- The maps $h_0 : X \to Y$ and $h_1 : X \to Y$ are **homotopic**, $h_0 \simeq h_1$, if there exists a map, a **homotopy**, $H : X \times I \to Y$ such that:

  $$h_0(x) = H(x, 0) \text{ and } h_1(x) = H(x, 1) \text{ for all } x \in X.$$

  If we want to emphasize the homotopy H between h_0 and h_1, we denote:

  $$h_0 \underset{H}{\simeq} h_1$$

- A map is **null–homotopic** if it is homotopic to a constant map.
- The space X **dominates** the space Y if there exist maps $f : X \to Y$, $g : Y \to X$, such that $f \circ g \simeq 1_Y$. We say that X and Y are **homotopy equivalent** if both spaces dominate each other. In such case, we write $X \simeq Y$.
- A space is **contractible** if it is homotopy equivalent to a one-point space.
- X is called **locally contractible** if any point of X has an open neighborhood U such that the inclusion $U \to X$ is null–homotopic.

Notice that if X is contractible $1_X \circ c_* \simeq 1_X$ and $c_* \circ 1_X \simeq 1_X$, where c_* is a constant map in X. Therefore $c_* \simeq 1_X$, i.e. the identity map of X is null–homotopic. With this in mind, we can prove our first result.

Theorem 1.0.2. A continuous motion planning $s : X \times X \to PX$ exists if and only if the configuation space is contractible.

Proof. Suppose that a continuous section $s : X \times X \to PX$ exists. Consider the fixed point $A_0 \in X$ and the homotopy $h_t : X \to X$, given by $h_t(B) = s(A_0, B)(t)$, where $B \in X$ and $t \in [0, 1]$. Notice that $h_1(B) = B$ and $h_0(B) = A_0$. Therefore h_t gives a contraction of the space X into the point $A_0 \in X$.

Conversely if X is contractible, there is a continuous homotopy $h_t : X \to X$ such that $h_0(A) = A$ and $h_1(A) = A_0$ for any $A \in X$. Given a pair $(A, B) \in X \times X$, we may compose the path $t \mapsto h_t(A)$ with the inverse of $t \mapsto h_t(B)$, which gives a continuous motion planning in X.

Thus, we get a motion planning in X by first moving A into the base point A_0 along the contraction, and the following inverse of the path, which brings B to A_0. $\square$

1.1 Definition of Topological Complexity

Now that we know more about the concept of continuous motion planning, we can define topological complexity.

Definition 1.1.1. Given a path–connected topological space X, we define the **topological complexity of the motion planning** in X as the minimal number $\mathrm{TC}(X) = k$, such that the Cartesian product $X \times X$ may be covered by k open subsets:

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

such that for any $i = 1, \ldots, k$ there exists a continuos motion planning $s_i : U_i \to PX$, $\pi \circ s_i = 1_{U_i}$. If no such k exists we will set $\mathrm{TC}(X) = \infty$.

According to **Theorem 1.0.2**, we have $\mathrm{TC}(X) = 1$ if and only if the space X is contractible. For example, any convex subset of $\mathbb{R}^n$. Explicitly: Given a pair (A, B), we may move along the straight line segment connecting A and B, which is a motion planning of one instruction.

Observe that the topological complexity $\mathrm{TC}(X)$ is the measure of the discontinuity of any motion planner in X. In other words, this number tells us how many different instructions our algorithm should have in order to ensure the continuity of our motion planner regardless of the start and end points.

Example 1.1.2. Topological Complexity of the circle S^1

With the tools that we have at the moment, we can do some calculations. Let's start with the circle. First, S^1 is not contractible, so $\mathrm{TC}(S^1) > 1$. Define $U_1 \subset S^1 \times S^1$ as $U_1 = \{(A, B) | A \neq -B\}$. A continuous motion planning over U_1 is given by the map $s_1 : U_1 \to PS^1$ which moves A towards B along the unique shortest arc connecting A to B. See **Fig. 1.1** (left). This map s_1 cannot be extended to a continuos map on the pairs of antipodal points $A = -B$, because we will have two arcs between $-B$ and B. See **Fig. 1.1** (right).

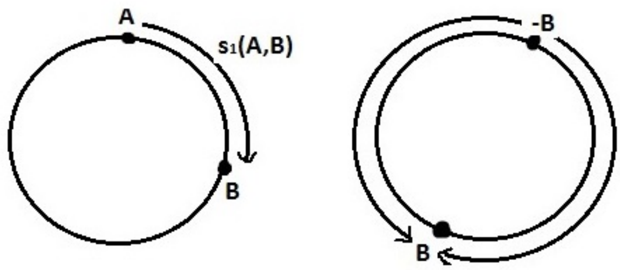


Figure 1.1: Motion planning over U_1

Now define $U_2 = \{(A, B) | A \neq B\}$. Fix an orientation of the circle S^1, for example the clockwise sense. A continuous motion planning over U_2 is given by the map $s_2 : U_2 \to PS^1$ which moves A towards B in the positive direction along the circle.

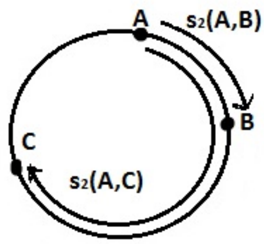


Figure 1.2: Motion planning over U_2

Again, s_2 cannot be extended to a continuous map on the whole $S^1 \times S^1$, because equal points could have different paths. One option is to not move (**Fig. 1.3 (left)**) and the other is a full turn (**Fig. 1.3 (right)**).

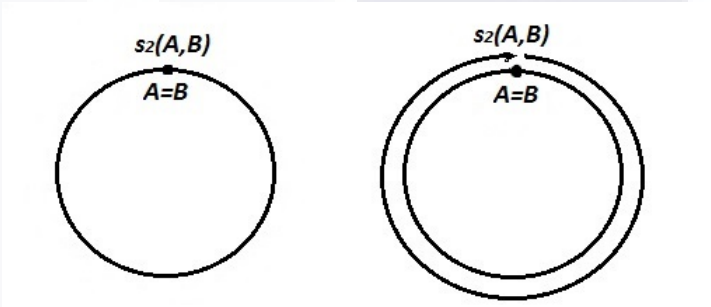


Figure 1.3: Lost of continuity for equal points

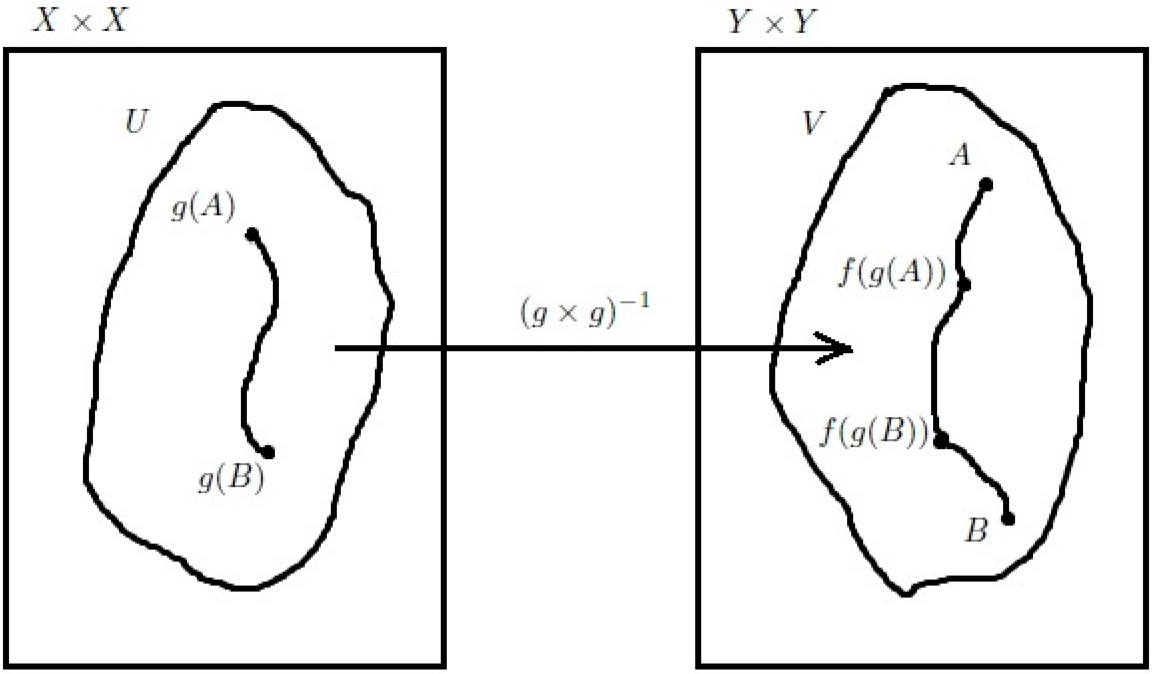
1.2 Homotopy Invariance

Often we have spaces whose Topological Complexity will be difficult to calculate. It would be desirable to obtain information about these spaces from other spaces simpler to handle.

If X dominates Y, there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$. Assume that $U \subset X \times X$ is an open subset such that there exists a continuous motion planning $s : U \to PX$. We will construct a motion planner for Y. Define $V = (g \times g)^{-1}(U) \subset Y \times Y$. We have the following diagram:

$$\begin{tikzcd}
& PX \arrow[d, "\pi_X"] & & PY \arrow[d, "\pi_Y"'] & \\
U \arrow[ur, bend left, "s"] \arrow[r, hook] & X \times X \arrow[rr, shift left, "f\times f"] & & Y \times Y \arrow[ll, shift left, "g\times g"] & V \arrow[l, hook'] \arrow[ul, bend right, "\sigma"']
\end{tikzcd}$$

Now, the question is: What is σ? Let $(A, B) \in V$. Fix a homotopy $h_t : Y \to Y$ with $h_0 = id_Y$ and $h_1 = f \circ g$. We know that in U it is possible connect $g(A)$ with $g(B)$ applying our motion planning s. By continuity we have a new path in V if we compose the path between $g(A)$ and $g(B)$ with the map f. Since $f \circ g$ is homotopic to the identity in Y, we can move the point A until $f(g(A))$ via h_t. Finally, we move $f(g(B))$ to B through h_{1-t}. Graphically:



Explicitly, if $\tau \in [0,1]$, the continuous motion planning $\sigma : V \to PY$ is given by:

$$\sigma(A,B)(\tau) = \begin{cases} h_{3\tau}(A) & 0 \leqslant \tau \leqslant \frac{1}{3} \\ f(s(g(A),g(B))(3\tau - 1)) & \frac{1}{3} \leqslant \tau \leqslant \frac{2}{3} \\ h_{3(1-\tau)}(B) & \frac{2}{3} \leqslant \tau \leqslant 1 \end{cases}$$

Notice that any open cover $\{U_1, \ldots, U_k\}$ of $X \times X$ with a continuous motion planning over each U_i defines an open cover $\{V_1, \ldots, V_k\}$ of $Y \times Y$ with similar properties. Therefore $\mathrm{TC}(Y) \leqslant \mathrm{TC}(X)$. Then, if $X \simeq Y$, we conclude that $\mathrm{TC}(X) = \mathrm{TC}(Y)$. In other words, $\mathrm{TC}(X)$ **depends only on the homotopy type of** X.

Chapter 2

An upper bound for $\mathrm{TC}(X)$

2.1 Lusternik-Schnirelman category

Since we have just seen, the Topological Complexity is a homotopic invariant. We would like to relate it with another one known invariant which allows us to obtain information about the TC. If it is not possible to calculate the TC of certain configuration spaces, a first approach would be to bound it. The invariant that we will use is a very strong tool in the homotopy theory. Such an invariant is very similar to the TC. In fact, both invariants are particular cases of a more general concept that will be studied later.

Definition 2.1.1. The **Lusternik-Schnirelman category** of the space X, $\mathrm{cat}(X)$, is defined as the smallest integer k, such that X may be covered by k open subsets $V_1, \ldots, V_k$, with each inclusion $V_i \to X$ null-homotopic. Such cover is called **categorical cover**.

Notice that the difference between TC and cat, are the properties of the elements of the covers, nevertheless, there seem to be similar characteristics. Thus, we might think that in effect, there is a relation between both invariants.

Theorem 2.1.2. For every topological space X, we have:

$$\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X).$$

Proof. Let $U \subset X \times X$ be an open subset such that there exists a continuous motion planning $s : U \to PX$ over U. Let $A_0 \in X$ be a fixed point. Define $V \subset X$ as $V = \{B \in X | (A_0, B) \in U\}$. Notice that V is open since $V = g^{-1}(U)$ where $g(B) = (A_0, B)$. Moreover, $H : V \times I \to X$ given by $H(B, t) = s(A_0, B)(t)$ is such that $\{A_0\} \underset{H}{\simeq} 1_V$, therefore $V \hookrightarrow X$ is null–homotopic.

If $\mathrm{TC}(X) = k$, and $U_1 \cup \cdots \cup U_k$ is a covering of $X \times X$ with a continuous motion planning over each U_i, then the sets V_i where $A_0 \times V_i = U_i \cap (A_0 \times X)$ form a categorical open cover of X. This shows that $\mathrm{cat}(X) \leqslant \mathrm{TC}(X)$. Now, if $\mathrm{cat}(X \times X) = l$, there is a categorical open cover $W_1, \ldots, W_l$ of $X \times X$. By **Theorem 1.0.2** there is a continuous section in each W_i, thus $k \leqslant l$. ☐

The above result is very general, however, in some cases it is difficult to work with the space $X \times X$. To obtain another upper bound in terms of X, remember the following:

Definition 2.1.3. Let be X a topological space:

1. A family $\{A_\alpha | \alpha \in \Omega\}$ of sets in X is called **locally finite**, if each point of X has a neighborhood V such that $V \cap A_\alpha \neq \emptyset$, for at most finitely many indices α.

2. If $\{A_\alpha|\alpha \in \Omega_\alpha\}$ and $\{B_\beta|\beta \in \Omega_\beta\}$ are two covers of X, $\{A_\alpha\}$ is a **refinement** of $\{B_\beta\}$, if for each A_α there is some B_β with $A_\alpha \subset B_\beta$.

3. X is **paracompact** if each open cover of X has an open locally finite refinement.

4. The **support** of a map $f : X \to \mathbb{R}$ is the closed set:

$$\text{support}(f) = \overline{\{x \in X|f(x) \neq 0\}}.$$

Naturally, we asked: What is the importance of paracompactness hypothesis in our configuration spaces? The reason is the following concept, which plays an important role in various topological problems.

Definition 2.1.4. Let X be a Hausdorff space. A family $\{\kappa_\alpha|\alpha \in \Omega\}$ of continuous maps $\kappa_\alpha : X \to I$ is called a **partition of unity** on X if:

1. The supports of the κ_α form a closed locally finite cover of X.

2. For each $x \in X$,

$$\sum_{\alpha\in\Omega} \kappa_\alpha(x) = 1$$

(this sum is finite because each x lies in the support of at most finitely many κ_α).

Definition 2.1.5. If $\{U_\alpha|\alpha \in \Omega\}$ is an open cover of X, we say that a partition $\{\kappa_\alpha|\alpha \in \Omega\}$ of unity is **subordinated** to $\{U_\alpha\}$, if the support of each κ_α lies in the corresponding U_α.

As the following theorem indicates, paracompact spaces always have a partition of unity subordinated to each open cover of the space.

Theorem 2.1.6. Let X be paracompact. Then for each open cover $\{U_\alpha|\alpha \in \Omega\}$ of X, there is a partition of unity subordinated to $\{U_\alpha\}$.

Proof. See [5, Theorem 4.2 p. 170] □

With this condition, it will be possible relate Lusternik–Schnirelman category with another invariant, and of course, this invariant will be another bound for topological complexity.

Definition 2.1.7. Let X be a space and $\mathcal{U} = \{U_\alpha\}$ any open cover of X.

- We say that the **order** of the cover $\{U_\alpha\}$ is k, if no point of X belongs to more than $k+1$ open sets of $\mathcal{U}$.

- The **dimension** of X, $\dim(X)$, is the least k, such that any open cover has a refinement of order k.

In the next result, we will exhibit importance of partitions of unity subordinated to a cover of our space.

Lemma 2.1.8. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X of order n with a partition of unity subordinate to the cover. Then, there is an open cover of X refining $\mathcal{U}$, $\mathcal{V} = \{V_{i\beta}\}$, $i = 1, \dots, n+1$, such that $V_{i\beta} \cap V_{i\beta'} = \varnothing$ for $\beta \neq \beta'$. In particular, there is such a refinement if X is a paracompact space with cover dimension n and $\mathcal{U} = \{U_\alpha\}$ is any open cover of X.

Proof. Since the order of the cover is n, no $x \in X$ can belong to more than $n+1$ of the U_α. Let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\mathcal{U}$. That is, support$(\phi_\alpha) \subseteq U_\alpha$. Now, let B_i denote the set of $i-$tuples obtained from the set $\{1, \ldots, n+1\}$. Given $\beta = (\alpha_1, \ldots, \alpha_i) \in B_i$, set

$$V_{i\beta} = \left\{ x \in X | \phi_{\alpha_j}(x) > 0 \text{ for } j = 1, \ldots, i \text{ and } \phi_\alpha(x) < \phi_{\alpha_j}(x) \text{ for } \alpha \notin \beta \right\}.$$

Since in a neighborhood of any $x \in X$, only a finite number of ϕ_α are not identically zero, each $V_{i\beta'}$ is open. If $\beta \neq \beta'$, then, by the second condition, $V_{i\beta} \cap V_{i\beta'} = \varnothing$. Also,

$$V_{i\beta} \subseteq \bigcap_{\alpha \in \beta} \text{support}(\phi_\alpha) \subseteq \bigcap_{\alpha \in \beta} U_\alpha$$

so that $\mathcal{V} = \{V_{i\beta}\}$ refines $\mathcal{U}$. Now, given $x \in X$ let $(\alpha_1, \ldots, \alpha_m)$ be all the indices such that $\phi_{\alpha_j}(x) > 0$. Then $x \in \cap_{i=1}^m U_{\alpha_i}$ and the order of the cover $\mathcal{U}$ is n, so we must have $m \leqslant n+1$. Without lost of generality, suppose

$$\phi_{\alpha_1}(x) = \phi_{\alpha_2}(x) = \cdots = \phi_{\alpha_j}(x) > \phi_{\alpha_{j+1}}(x) \geqslant \phi_{\alpha_{j+2}}(x) \geqslant \cdots \geqslant \phi_{\alpha_m}(x)$$

This means that $x \in V_{j(\alpha_1, \ldots, \alpha_j)}$ and $\mathcal{V}$ covers X. □

The following result relates the category with the dimension of the space X.

Theorem 2.1.9. If X is path–connected, locally contractible and paracompact, then:

$$\text{cat}(X) \leqslant \dim(X) + 1.$$

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_{n+1}\}$ be a categorical cover of X and suppose $\dim(X) = k$. Then, by **Lemma 2.1.8**, there is an open cover $\{V_i\}$, $i = 1, \ldots, k+1$ of X refining $\mathcal{U}$ with the property that each V_i is a union of disjoint open sets each of which lies in some U_j. Since each U_j is contractible in X and the open sets making up V_i are disjoint, then V_i is also contractible in X (each component of V_i is in a U_j). We have then found a categorical cover of X by $k+1$ sets and the definition of category therefore says that $\text{cat}(X) \leqslant k+1 = \dim(X)+1$. □

The following results, require the introduction of a very intuitive concept:

Definition 2.1.10. For any topological space X we describe a finite sequence $\varnothing = V_0 \subset V_1 \subset \cdots \subset V_n = X$ of open subsets of X as **categorical**, if each of the differences $V_i \backslash V_{i-1}$ (with $i = 1, \ldots, n$), is the union of a finite number of disjoint open sets each contained in an open categorical subset of X.

Lemma 2.1.11. Let X be a path–connected and paracompact space. Then, X admits a categorical sequence of length n if and only if $\text{cat}(X) \leqslant n$.

Proof. Since the sets $V_i \backslash V_{i-1}$ form a categorical cover of X, it is clear that $\text{cat}(X)$ is at most n. Now, if $\text{cat}(X) \leqslant n$, there is a categorical cover $\{V_1, \ldots, V_n\}$ of X, and we can define:

$$W_m = \bigcup_{i=1}^m V_i.$$

Notice that $W_i \backslash W_{i-1} \subseteq V_i$ is open and categorical. Hence, the W_m form a categorical sequence of length n. □

Categorical sequences are used for various "product inequalities". The most important for our purposes is:

Theorem 2.1.12. If X and Y are path-connected and paracompact, then:

$$\operatorname{cat}(X \times Y) < \operatorname{cat}(X) + \operatorname{cat}(Y).$$

Proof. Let $A_0 \subset \cdots \subset A_{m-1}$ and $B_0 \subset \cdots \subset B_{n-1}$ be categorical sequences for X and Y, respectively. Then $C_0 \subset \cdots \subset C_{m+n-2}$ is a categorical sequence for $X \times Y$, where $C_0 = \varnothing$ and:

$$C_j = \bigcup_{i=1}^{j} (A_i \times B_{j-i+1}).$$

Notice that $C_j - C_{j-1}$ is the union of the disjoint sets:

$$(A_i - A_{i-1}) \times (B_{j+1-i} - B_{j-i}).$$

Hence, **Lemma 2.1.11** implies the statement of the Theorem. □

In summary, we have shown this result:

Theorem 2.1.13. If X is path-connected and paracompact, then:

$$\operatorname{cat}(X) \leqslant \operatorname{TC}(X) \leqslant 2\operatorname{cat}(X) - 1 \leqslant 2\dim(X) + 1.$$

□

2.2 General upper bound

Let X and Y be two topological spaces. For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$.

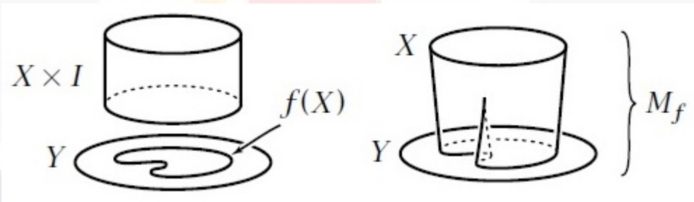


Figure 2.1: Mapping cylinder, [10, p. 2]

Let $c_f : (X \times I) \sqcup Y \to M_f$ be the identification map. Notice that the spaces X and Y are naturally embedded in the space M_f by the mappings $i(x) = c_f(x, 0)$ and $j(y) = c_f(y)$. In fact, M_f is also embedded in Y by the map $h : M_f \to Y$ given by:

$$h(z) = \begin{cases} f(x) & \text{if } x \in X, 0 \leqslant t \leqslant 1 \\ y & \text{if } y \in Y \end{cases}$$

Moreover, $h \circ j = 1_Y$ and $j \circ h \simeq 1_{M_f}$, which implies $Y \simeq M_f$.

Closely related to the mapping cylinder M_f is the **mapping cone** $C_f = Y \sqcup CX$ where CX is the cone $(X \times I)/(X \times \{0\})$ and we attach this to Y along $X \times \{1\}$ via the identifications $(x, 1) \sim f(x)$.

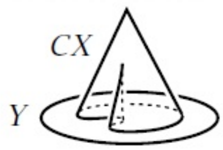


Figure 2.2: Mapping cone, [10, p. 13]

More generally, the **join** $X * Y$, is the quotient space of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Thus we are collapsing the subspace $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y.

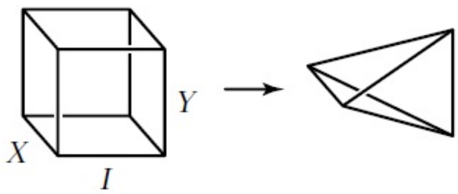


Figure 2.3: Join, [10, p. 9]

Definition 2.2.1. The space X with a base point x_0 is said to be n**-connected** if $\pi_q(X, x_0) = 0$ for $q \leqslant n$.

Thus 0–connected means path–connected and 1–connected means simply–connected. Since n–connected implies 0–connected, the choice of the base point x_0 is not significant.

In the next Theorem, we will prove the relation between the connectedness of the join of two spaces. We will use singular homology with integer coefficients (see [10]).

Theorem 2.2.2. Suppose that X is $(r-1)$–connected and Y is $(s-1)$–connected. Then $X * Y$ is $(r+s)$–connected.

Proof. Consider the long exact homology sequence of the pair $(CX \times CY, X * Y)$:

$$\cdots \to H_q(X * Y) \to H_q(CX \times CY) \to H_q(CX \times CY, X * Y) \to \dots$$

Since CX and CY are contractible (the cone is homotopically equivalent to a disk and this is homotopically equivalent to a point), then $H_q(CX \times CY) = 0$. Therefore:

$$H_q(CX \times CY, X * Y) \cong H_{q-1}(X * Y).$$

Notice that the map:

$$h : X * Y \to CX \times Y \cup X \times CY$$

which sends $[x, y, t] \in X * Y$ to $(x, [y, 2t]) \in X \times CY$ if $t \leqslant \frac{1}{2}$ and to $([x, 2t-2], y) \in CX \times Y$ if $t \geqslant \frac{1}{2}$ is an homeomorphism (see [15, p. 56-57]), therefore:

$$(CX \times CY, X * Y) \approx (CX \times CY, CX \times Y \cup X \times CY) = (CX, X) \times (CY, Y).$$

Now, for the pair (CX, X) we have:

$$\cdots \to H_q(X) \to H_q(CX) \to H_q(CX, X) \to \dots$$

Since $H_q(CX) = 0$, it follows that:

$$H_q(CX, X) \cong H_{q-1}(X)$$

By **Künneth's Theorem** (see [10, p. 276]) we have the exact sequence:

$$0 \longrightarrow \bigoplus_{i+j=q} H_i(CX, X) \otimes H_j(CY, Y) \longrightarrow$$
$$\longrightarrow H_q((CX, X) \times (CY, Y)) \longrightarrow$$
$$\longrightarrow \bigoplus_{i+j=q-1} Tor(H_{i-1}(CX, X), H_{j-1}(CY, Y)) \longrightarrow 0$$

By the above remarks, this sequence can be written as:

$$0 \longrightarrow \bigoplus_{i+j=q} H_{i-1}(X) \otimes H_{j-1}(Y) \longrightarrow H_{q-1}(X * Y) \longrightarrow$$
$$\longrightarrow \bigoplus_{i+j=q-1} Tor(H_{i-1}(X), H_{j-1}(Y)) \longrightarrow 0$$

Applying **Hurewicz's Theorem** and reduced homology (see [10, p. 366]), we have $\widetilde{H}_i(X) = 0 = \widetilde{H}_j(Y)$ for $i < r$ and $j < s$. Then, all factors on the sums are zeros, which implies that for $q < r + s + 1$ we have:

$$H_q(X * Y) = 0.$$

Moreover, $\pi_r(X) \cong H_r(X)$ and $\pi_s(Y) \cong H_s(Y)$. Therefore, if $i - 1 = r$ and $j - 1 = s$, the exact sequence is:

$$0 \longrightarrow H_r(X) \otimes H_s(Y) \longrightarrow H_{r+s+1}(X * Y) \longrightarrow 0.$$

Then:

$$H_{r+s+1}(X * Y) \cong H_r(X) \otimes H_s(Y) \cong \pi_r(X) \otimes \pi_s(Y) \neq 0.$$

Since the join of a path–connected space and nonempty space is simply–connected (see [13]), we can apply **Hurewicz's Theorem** again and conclude that $\pi_q(X * Y) = 0$ for $q \leqslant r + s$. In other words, $X * Y$ is $(r + s)$–connected. □

As mentioned at the beginning of this section, the Topological Complexity and Lusternik Schnirelman Category are particular cases of a more general concept which we introduce now.

Definition 2.2.3. A **fibre space** is an ordered quadruple (B, E, F, p) where B is the base space, E is the total space and F is the typical fibre of the map $p : E \to B$. We say that the map $\phi : A \subset B \to E$ is a **cross–section** of the fibration p on the set A if $\forall a \in A$

$$(p \circ \phi)(a) = a.$$

Definition 2.2.4. The **Sectional Category** of the fibration p (denoted secat(p)) is the cardinality of the smallest open cover of the base space B consisting of the sets on each of which there exists a cross–section.

Notice that the Topological Complexity $\mathrm{TC}(X)$ can be viewed as the secat of the path space fibration $\pi : PX \to X \times X$, which has the base of dimension:

$$\dim(X \times X) = 2 \dim(X).$$

Moreover:

$$\pi_q(X) = [S^q, X] \cong [\Sigma S^{q-1}, X] \cong [S^{q-1}, \Omega X] = \pi_{q-1}(\Omega X)$$

i.e. the fibre is homotopy equivalent to the space ΩX of based loops in X, and if X is r–connected, ΩX is $(r - 1)$–connected.

The next Theorem of [15] is the principal result to obtain the general upper bound.

Theorem 2.2.5. Let $F \to E \xrightarrow{p} B$ be a fibre space $\mathcal{B}$, whose fibre F is $(s-1)$–connected and whose base B is a k–dimensional CW-polyhedron. Then:

$$\operatorname{secat}(\mathcal{B}) < \frac{k+1}{s+1} + 1$$

Proof. By **Theorem 2.2.2**, the fibre $F_n = F * \cdots * F$ of the fibre space $F_n \to E_n \xrightarrow{p_n} B$, $\mathcal{B}_n$, is $(n(s+1)-1)$–connected; therefore on the $(n(s+1)-1)$–dimensional skeleton of the base B there exists a cross-section of the fibration $\mathcal{B}_n$. If $n(s+1)-1 \geqslant k$, then there exists a cross-section on all of the base B of $\mathcal{B}_n$ and hence, for n satisfying the condition $n \geqslant (k+1)/(s+1)$, we have:

$$\operatorname{secat}(\mathcal{B}) \leqslant n.$$

$\square$

In other words:

$$\mathrm{TC}(X) < \frac{2\dim X + 1}{r+1} + 1 \tag{2.1}$$

The bound given by (2.1.13) is for path–connected space X, this is, if X is 0–connected, then:

$$\mathrm{TC}(X) \leqslant 2\dim(X) + 1 < \frac{2\dim(X)+1}{0+1} + 1.$$

Chapter 3

A lower bound for $\mathrm{TC}(X)$

3.1 Zero-divisors-cup-lenght

Until now, we have only done the calculation of topological complexity for the circle. We would like to generalize it for any sphere. nevertheless, we need algebraic tools to obtain another bound that enables us to do the aforementioned calculation.

Let K be a field. Recall that The cohomology $H^*(X;K)$ is a graded K-algebra (see [10]) with the multiplication given by the cup-product:

$$\cup : H^*(X;K) \otimes H^*(X;K) \to H^*(X;K) \tag{3.1}$$

The tensor product $H^*(X;K) \otimes H^*(X;K)$ is also a graded K-algebra with the multiplication:

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|} u_1 u_2 \otimes v_1 v_2 \tag{3.2}$$

where $|v_1|$ and $|u_2|$ denote the degrees of cohomology classes v_1 and u_2 correspondingly. The cup-product (3.1) is then a graded algebra homomorphism.

Definition 3.1.1. The kernel of homomorphism (3.1) is called **the ideal of the zero-divisors** of $H^*(X;K)$. The **zero-divisors-cup-lenght** of $H^*(X;K)$, (denoted $\mathrm{zcl}(X)$) is the length of the longest nontrivial cup product in the ideal of the zero-divisors of $H^*(X;K)$.

Example 3.1.2. We now calculate $\mathrm{zcl}(S^n)$

Let $u \in H^*(S^n;K)$ be the fundamental class, and let $1 \in H^0(S^n;K)$ be the unit. Observe that

$$a = 1 \otimes u - u \otimes 1 \in H^*(S^n;K) \otimes H^*(S^n;K)$$

is a zero-divisor, since applying homomorphism (3.1) to a we obtain:

$$\cup a = \cup(1 \otimes u - u \otimes 1) = \cup(1 \otimes u) - \cup(u \otimes 1) = 1 \cdot u - u \cdot 1 = 0.$$

Another zero-divisor is $b = u \otimes u$, since $u^2 = 0$. Now, applying (3.2)

$$\begin{aligned}
a^2 &= (1 \otimes u - u \otimes 1) \cdot (1 \otimes u - u \otimes 1) \\
&= (1 \otimes u) \cdot (1 \otimes u) - (1 \otimes u) \cdot (u \otimes 1) - \\
&\quad -(u \otimes 1) \cdot (1 \otimes u) + (u \otimes 1) \cdot (u \otimes 1) \\
&= (-1)^{|u|\cdot|1|} 1 \otimes u^2 - (-1)^{|u|\cdot|u|} u \otimes u - \\
&\quad -(-1)^{|1|\cdot|1|} u \otimes u + (-1)^{|1|\cdot|u|} u^2 \otimes 1 \\
&= 1 \otimes u^2 - (-1)^{n^2} u \otimes u - u \otimes u + u^2 \otimes 1 \\
&= \left((-1)^{n^2+1} - 1\right) u \otimes u
\end{aligned}$$

Hence $a^2 = -2b$ for n even and $a^2 = 0$ for n odd; the product ab vanishes for any n. We conclude that:

$$\mathrm{zcl}(S^n) = \begin{cases} 1 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$$

If we observe the particular case of the circle, we have that:

$$\mathrm{zcl}(S^1) = 1 < 2 = \mathrm{TC}(S^1).$$

In the following section, we will prove that this inequality is true for any configuration space.

3.2 Fibre Spaces

Let M_p be the mapping cylinder of $p : E \to B$ and $i : E \hookrightarrow M_p$, $h : M_p \to B$, the natural maps (see 2.2). Define $M'_p = M_p \backslash i(E)$, and consider:

- $p_n : E_n \to B$ the fibration
- $h_n : M_p^n \to B^n$ the product of n copies of h
- $\lambda : E_n \to M_p^n$ the inclusion map
- $d : B \to B^n$ the diagonal embedding

where $E_n \subseteq M_p^n \backslash M_p'^n$. Thus we have the commutative diagram:

$$\begin{tikzcd} E_n \arrow[r, "p_n"] \arrow[d, "\lambda"'] & B \arrow[d, "d"] \\ M_p^n \arrow[r, "h_n"'] & B^n \end{tikzcd}$$

Notice that:

- $p_n^* : H^*(B; R) \to H^*(E_n; R)$ is a monomorphism.
- $d^* : H^*(B; R) \otimes \cdots \otimes H^*(B; R) \to H^*(B; R)$.

Here is our first result:

Lemma 3.2.1. If $\mathrm{secat}(p_n) \leqslant n$ and $\xi_1, \ldots, \xi_n \in H^*(B; R)$ are such that $d^*(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \cdots \xi_n \neq 0$, then $\lambda^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) \neq 0$.

Proof. Since p_n^* is monomorphism and by commutativity, it follows that:

$$\lambda^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) = p_n^* d^*(\xi_1 \otimes \cdots \otimes \xi_n) = p_n^*(\xi_1 \cdots \xi_n) \neq 0$$

□

Let $\mu : M_p^n - M_p'^n \to M_p^n$. Notice that $\ker(\lambda^* h^*) = \ker(\mu^* h^*)$.

Lemma 3.2.2. In order that the cohomology class $\xi_1 \otimes \cdots \otimes \xi_n \in H^*(B^n; R)$ be contained in $\ker(\mu^* h_n^*)$ it is sufficient that $p^*(\xi_i) = 0$ for all $i = 1, \ldots, n$.

Proof. If $p^*(\xi_i) = 0$, since $p^* = i^*h^*$ and by the exact sequence of the pair $(M_p, i(E))$

$$\cdots \longrightarrow H^*(M_p, i(E); R) \xrightarrow{\sigma^*} H^*(Z; R) \xrightarrow{i^*} H^*(E; R) \longrightarrow \cdots$$

there exists $\eta_i \in H^*(M_p, i(E); R)$ such that $\sigma^*(\eta_i) = h^*(\xi_i)$. Therefore $\sigma_n^*(\eta_1 \otimes \cdots \otimes \eta_n) = h_n^*(\xi_1 \otimes \cdots \otimes \xi_n)$, where σ_n^* is the homomorphism appearing in the exact sequence of the pair $(M_p^n, M_p^n \backslash M_p'^n)$. Then $\mu^*\sigma_n^* = 0$, and consequently $\mu^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) = \mu^*\sigma_n^*(\eta_1 \otimes \cdots \otimes \eta_n) = 0$. $\square$

Theorem 3.2.3. Let $p : E \to B$ be a fibration. If there exist cohomology classes $\xi_1, \ldots, \xi_n \in H^*(B; R)$ for which $p^*(\xi_i) = 0$ for all $i = 1, \ldots, n$, and the cohomology class $\xi_1 \cdots \xi_n \in H^*(B; R)$ is different from zero, then $\operatorname{secat}(p) \geqslant n + 1$.

Proof. By **Lemma 3.2.2**, $\mu^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) = 0$, then $\lambda^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) = 0$. If $\operatorname{secat}(p) \leqslant n$, from **Lemma 3.2.1**, $\lambda^* h_n^*(\xi_1 \otimes \cdots \otimes \xi_n) \neq 0$, which leads to a contradiction. $\square$

Now, we are ready to prove the next theorem of [6].

Theorem 3.2.4. The topological complexity $\mathrm{TC}(X)$ is greater than the zero–divisors–cup–length of $H^*(X)$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\Delta"'] & PX \arrow[d, "\pi"] \\ & X \times X \end{tikzcd}$$

Here α associates to any point $x \in X$ the constant path c_x at this point and Δ is the diagonal map:

$$\Delta(x) = (x, x)$$

Let $\beta : PX \to X$ given by $\beta(\gamma) = \gamma(0)$, and define $H : PX \times I \to PX$ as:

$$H(\gamma, t)(s) = \gamma(ts)$$

Then:

$$H(\gamma, 0) = c_{\gamma(0)} = \alpha \circ \beta(\gamma) \qquad \text{and} \qquad H(\gamma, 1) = \gamma = 1_{PX}(\gamma)$$

Equivalently $\alpha \circ \beta \underset{H}{\simeq} 1_{PX}$, which means that α is a homotopy equivalence.

Notice that the cup–product homomorphism (3.1), coincides with the following composition:

$$H^*(X; R) \otimes H^*(X; R) \overset{\kappa}{\simeq} H^*(X \times X; R) \xrightarrow{\pi^*} H^*(PX; R) \underset{\simeq}{\xrightarrow{\alpha^*}} H^*(X; R) \tag{3.3}$$

Here the homomorphism on the left κ is the **Künneth isomorphism** (see [10, Theorem 3.16, p. 219]), and the isomorphism on the right is due to the homotopy equivalence of α.

If $\operatorname{zcl}(X) = n$, there are $a_1, \ldots, a_n \in \ker \cup$ such that:

$$a_1 \cdots a_n \neq 0.$$

Then, by (3.3), $\forall i = 1, \ldots, n$

$$\alpha^* \pi^* \kappa(a_i) = \cup(a_i) = 0.$$

Since α^* is isomorphism and $\kappa(a_1) \cdots \kappa(a_n) = \kappa(a_1 \cdots a_n) \neq 0$, it follows that $\pi^*\kappa(a_i) = 0$. Therefore, by **Theorem 3.2.3**, $\operatorname{secat}(\pi) \geqslant n + 1$ which implies:

$$\mathrm{TC}(X) > \operatorname{zcl}(X).$$

$\square$

3.3 Basic Examples

Now we have a lower bound for Topological Complexity. Then, given a configuration space, the idea is calculate zcl of that space and estimate its TC using some property of the space. The following examples show the application of **Theorem 3.2.4**.

Example 3.3.1. The n–dimensional sphere S^n

Case n odd. Let $U_1 = \{(A,B)|A \neq -B\} \subset S^n \times S^n$ and $s_1 : U_1 \to PS^n$ where $s_1(A,B) \in PS^n$ is the shortest arc of S^n connecting A and B. The second set is $U_2 = \{(A,B)|A \neq B\} \subset S^n \times S^n$ and $s_2 : U_2 \to PS^n$ will be constructed in two steps. First we move the initial point A to the point $-B$ along the shortest arc as above. Now, fix a continuous tangent vector field v on S^n, which is nowhere zero. Then we may move $-B$ to B along the spherical arc:

$$-B\cos\pi t + \frac{v(B)}{|v(B)|}\sin\pi t$$

with $t \in [0,1]$. Since S^n is not contractible, we have $\mathrm{TC}(S^n) = 2$. Notice that this case is a generalization of **Example 1.1.2**, and in fact is not necessary use the lower bound.

Case n even. Let $U_1 \subset S^n \times S^n$ and $s_1 : U_1 \to PS^n$ as above. In this case we may construct a continuous tangent vector field w on S^n which vanishes only at a single point $B_0 \in S^n$. Then we define:

$$U_2 = \{(A,B)|A \neq B \& B \neq -B_0\} \subset S^n \times S^n$$

and $s_2 : U_2 \to PS^n$ as above. We see that $U_1 \cup U_2$ covers everything except the pair $(-B_0, B_0)$. Choose a point $C \in S^n$ such that $B_0 \neq C$ and $C \neq -B_0$. Notice that the set $Y = S^n - C$ is contractible, then there exists a continuous motion planning $s_3 : U_3 \to PS^n$ where $U_3 = Y \times Y$. Using **Theorem 3.2.4** and **Example 3.1.2**, we conclude that $\mathrm{TC}(S^n) = 3$.

Example 3.3.2. The n–dimensional complex projective space $\mathbb{CP}^n$

Remember that:

$$H^k(\mathbb{CP}^n; R) = \begin{cases} R & \text{for } k \leqslant 2n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, if $k \leqslant 2n$ is even, then the generator of $H^k(\mathbb{CP}^n;\mathbb{Q})$ is u^k, where $u \in H^2(\mathbb{CP}^n;\mathbb{Q})$ is the generator. Then, for $\mathbb{CP}^2$, we have:

$$\begin{aligned}
(1\otimes u - u\otimes 1)^2 &= 1\otimes u^2 - 2u\otimes u - u^2\otimes 1 \\
(1\otimes u - u\otimes 1)^3 &= (1\otimes u - u\otimes 1)(1\otimes u^2 - 2u\otimes u - u^2\otimes 1) \\
&= 3u^2\otimes u - 3u\otimes u^2 \\
(1\otimes u - u\otimes 1)^4 &= (1\otimes u - u\otimes 1)(3u^2\otimes u - 3u\otimes u^2) \\
&= 6u^2\otimes u^2 \\
(1\otimes u - u\otimes 1)^5 &= (1\otimes u - u\otimes 1)(6u^2\otimes u^2) \\
&= 0
\end{aligned}$$

In general, if $u \in H^2(\mathbb{CP}^n;\mathbb{Q})$ is a generator we have:

$$(1\otimes u - u\otimes 1)^{2n} = (-1)^n \binom{2n}{n} u^n \otimes u^n \neq 0$$

In other words:

$$\mathrm{zcl}(\mathbb{CP}^n) = 2n.$$

On the other hand:

$$2\dim(\mathbb{CP}^n) + 1 = 4n + 1,$$

but $\mathbb{CP}^n$ is simply—connected, then (2.1) implies that:

$$\mathrm{TC}(\mathbb{CP}^n) < 2n + \frac{3}{2}.$$

Hence, applying **Theorem 3.2.4**, we conclude:

$$\mathrm{TC}(\mathbb{CP}^n) = 2n + 1.$$

3.4 Orientable compact two–dimensional surfaces of genus g

Let Σ_g be an orientable compact two–dimensional surface of genus g. First, **Theorem 2.1.13** guarantees that:

$$\mathrm{TC}(\Sigma_g) \leqslant 2\dim(\Sigma_g) + 1 = 5 \tag{3.4}$$

Thus, we need to calculate another number that allows us to bring near the Topological Complexity of these surfaces on the left side of (3.4).

The orientability condition is vital for our objectives, because with that hypothesis, we can apply **Poincaré's Duality Theorem** [10, p. 241], which means that:

$$H^2(\Sigma_g; R) \cong H_0(\Sigma_g; R).$$

On the other hand, these surfaces are path–connected. Then [10, Prop 2.7, p. 109] implies that:

$$H_0(\Sigma_g; R) \cong R.$$

Therefore, $H^2(\Sigma_g; R)$ is a non–trivial cohomology group, and in it we can obtain information of use in order to compute $\mathrm{zcl}(\Sigma_g)$. Of course, to determine that number, it will be necessary to know which is the cup product structure on Σ_g .

Remember that:

$$H^q(T^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the generators for the non–trivial cohomology groups are:

$$1 \in H^0(T^2; \mathbb{Z}),$$
$$a, b \in H^1(T^2; \mathbb{Z}),$$
$$c \in H^2(T^2; \mathbb{Z}).$$

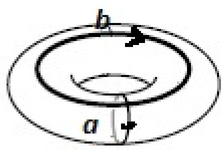


Figure 3.1: Torus [10, p. 232]

If $\gamma \in H_2(T^2;\mathbb{Z})$, then:

$$(a \cup b)(\gamma) = 1 = c \tag{3.5}$$

Based on the above analysis, we can infer what happens in Σ_g. Consider the quotient map q from Σ_g to a wedge sum of g tori, this is:

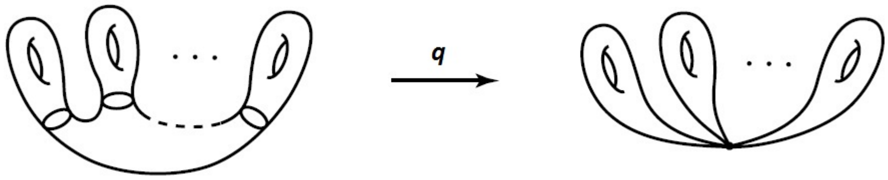


Figure 3.2: The map $q : \Sigma_g \to \bigvee_g T^2$ [10, Ex1, p. 228]

Let Y be the subspace which collapses to a point to obtain the wedge sum. Notice that Y is homeomorphic to an S^2 with g disks removed, which is homotopy equivalent to a wedge sum of $g-1$ circles.

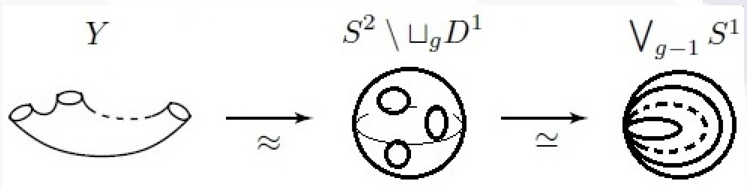


Figure 3.3: The subspace Y viewed as wedge sum of circles

Define $X = \Sigma_g/Y$ and consider the long exact sequence in cohomology of the pair (Σ_g, Y):

$$H^1(X) \xrightarrow{q^1} H^1(\Sigma_g) \xrightarrow{f^1} H^1(Y) \to H^2(X) \xrightarrow{q^2} H^2(\Sigma_g) \to 0$$

The zero on the right (which implies that q^2 is epimorphism) is due to $H^2(Y) \cong H^2(\bigvee_{g-1} S^1) = 0$. Now, by **Poincaré's Duality Theorem**, the above sequence can be written as:

$$\bigoplus_g \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{q^1} H^1(\Sigma_g) \xrightarrow{f^1} \bigoplus_{g-1} \mathbb{Z} \to \bigoplus_g \mathbb{Z} \xrightarrow{q^2} \mathbb{Z} \to 0$$

Since the map q is epimorphism, the induced map in homology $q_1 : H_1(\Sigma_g) \to H_1(X)$ is also epimorphism. Then, if $\alpha \in H^1(X)$ is such that $q^1(\alpha) = 0$, for any $b \in H_1(\Sigma_g)$, it follows that:

$$0 = q^1(\alpha)(b) = \alpha(q_1(b))$$

which implies that $\alpha = 0$ since q_1 is epimorphism. Hence, q^1 is monomorphism. This means $0 = \ker q_1 = \mathrm{Im} f_1$, so $f_1 = 0$. Now, if $\beta \in H^1(\Sigma_g)$, we have $f^1(\beta) \in H^1(Y) \cong \bigoplus_{g-1} \mathbb{Z}$ and it can be evaluated in each generator of $H^1(Y)$ and the result will be zero in all cases, then $\beta \in \ker f^1 = \mathrm{Im} q^1$. We conclude that q^1 is epimorphism. Therefore we have:

$$H^1(\Sigma_g) \cong H^1(X) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}}_{2g}$$

where each summand $\mathbb{Z} \oplus \mathbb{Z}$ represents one of the g tori in X. Each wedge summand has two cohomology classes $a_i, b_i \in H^1(X)$ with

$$a_i \cup b_i = c_i$$

where $c_i \in H^2(X)$ is the generator corresponding to the i–th torus (see 3.5). Moreover, if $i \neq j$

$$a_i \cup b_j = 0$$

because either a_i or b_j (or both) are zero when we apply the map induced in cohomology by including the k–th torus in X. Finally, the map f^* takes each generator c_i of the $H^2(X)$ to the unique generator T of the $H^2(\Sigma_g)$. Hence:

$$a_i \cup b_j = \delta^i_j T \tag{3.6}$$

Now, suppose that $g > 1$. From the above result (3.6), we may find cohomology classes:

$$u_1, v_1, u_2, v_2 \in H^1(\Sigma_g; \mathbb{Q})$$

such that for $i \neq j$,

$$u_i^2 = v_i^2 = u_i v_j = u_i u_j = v_i v_j = 0$$

and $u_1 v_1 = u_2 v_2 = A \neq 0$ where the fundamental class is precisely $A \in H^2(\Sigma_g; \mathbb{Q})$. Then, it holds in the algebra $H^*(\Sigma_g; \mathbb{Q}) \otimes H^*(\Sigma_g; \mathbb{Q})$ that:

$$\prod_{i=1}^{2} (1 \otimes u_i - u_i \otimes 1)(1 \otimes v_i - v_i \otimes 1) = 2A \otimes A \neq 0$$

Therefore $\mathrm{zcl}(\Sigma_g)$ is at least 4, which implies $\mathrm{TC}(\Sigma_g) \geqslant 5$. But, by (3.4), we have:

$$\mathrm{TC}(\Sigma_g) = 5 \tag{3.7}$$

Chapter 4

Motion Planning for a Robot Arm

4.1 Product Inequality

Previously we saw (**Theorem 2.1.12**) that if X and Y are path–connected and paracompact, then:

$$\mathrm{cat}(X \times Y) \leqslant \mathrm{cat}(X) + \mathrm{cat}(Y) - 1.$$

Under the same conditions and proceeding similarly, is it possible to obtain a similar result but with topological complexity? Fortunately the answer is yes.

Theorem 4.1.1. For any path–connected and paracompact spaces X and Y:

$$\mathrm{TC}(X \times Y) \leqslant \mathrm{TC}(X) + \mathrm{TC}(Y) - 1.$$

Proof. Suppose that $\mathrm{TC}(X) = n$, then, there is an open cover $\{U_i\}$ of $X \times X$ with a continuous motion planner $s_i : U_i \to PX$, for $i = 1, \ldots, n$. Analogously, if $TC(Y) = m$, there exist an open cover $\{V_j\}$ of $Y \times Y$ with a continuous motion planner $\sigma_j : V_j \to PY$, for $j = 1, \ldots, m$. Let $f_i : X \times X \to \mathbb{R}$ and $g_j : Y \times Y \to \mathbb{R}$ be partitions of unity subordinated to the covers $\{U_i\}$ and $\{V_j\}$, respectively. For any pair of nonempty subsets $S \subset \{1, \ldots, n\}$ and $T \subset \{1, \ldots, m\}$, define $W(S,T) \subset (X \times Y) \times (X \times Y)$ as the set of all 4–tuples $(A, B, C, D) \in (X \times Y) \times (X \times Y)$, such that for any $(i, j) \in S \times T$ and for any $(i', j') \notin S \times T$ it holds that:

$$f_i(A, C) \cdot g_j(B, D) > f_{i'}(A, C) \cdot g_{j'}(B, D).$$

Notice that each set $W(S,T)$ is open, and $W(S,T) \cap W(S',T') = \varnothing$ if neither $S \times T \subset S' \times T'$ nor $S' \times T' \subset S \times T$. Moreover, if $(A, B, C, D) \in W(S,T)$, we have $f_i(A, C) \neq 0$ and $g_j(B, D) \neq 0$, then, remembering **Definition 2.1.3(4)** and **Definition 2.1.5**, it follows that $(A, C) \in \mathrm{support}(f_i) \subset U_i$ and $(B, D) \in \mathrm{support}(g_j) \subset V_j$. Equivalently:

$$W(S,T) \subset U_i \times V_j \tag{4.1}$$

Then, (4.1) implies that there exists a continuous motion planning over each $W(S,T)$. Now, if $(A, B, C, D) \in (X \times Y) \times (X \times Y)$ we can define S as the set of all indices $i \in \{1, \ldots, n\}$ such that $f_i(A, C)$ is the maximum of $f_k(A, C)$, where $k = 1, 2, \ldots, n$. Similarly, let T be the set of all $j \in \{1, \ldots, m\}$ such that $g_j(B, D)$ equals the maximum of $g_l(B, D)$, where $l = 1, 2, \ldots, m$. Hence $(A, B, C, D) \in W(S,T)$ and the sets $W(S,T)$ cover $(X \times Y) \times (X \times Y)$.

Finally, let $W_k \subset (X \times Y) \times (X \times Y)$ denote the union of all sets $W(S,T)$, where $|S| + |T| = k$, with $k = 2, 3, \ldots, n+m$ ($|A|$ is the cardinality of the space A). Clearly, $\{W_2, \ldots, W_{n+m}\}$ is a cover of $(X \times Y) \times (X \times Y)$. Observe that if $|S| + |T| = |S'| + |T'| = k$, the corresponding sets $W(S,T)$ and $W(S',T')$ either coincide (if $S = S'$ and $T = T'$) or are disjoint. Therefore, there exists a continuous motion planning over each set W_k, and $\mathrm{TC}(X \times Y)$ is at most $n + m - 1$. $\square$

As a first consequence of **Theorem 4.1.1**, one easily checks that for a product of n copies of the space X:

$$\mathrm{TC}(X \times \cdots \times X) \leqslant n\,\mathrm{TC}(X) - (n-1) \tag{4.2}$$

Now, we are ready to prove the principal Theorem of this section.

Theorem 4.1.2. Let $X = S^m \times \cdots \times S^m$ be a Cartesian product of n copies of the m–dimensional sphere S^m, then:

$$\mathrm{TC}(X) = \begin{cases} n+1 & \text{for } m \text{ odd} \\ 2n+1 & \text{for } m \text{ even} \end{cases}$$

Proof. Using (4.2), we can see that:

$$\mathrm{TC}(S^m \times \cdots \times S^m) \leqslant n\,\mathrm{TC}(S^m) - (n-1).$$

Then, applying **Example 3.3.1**:

$$\mathrm{TC}(X) \leqslant \begin{cases} n+1 & \text{for } m \text{ odd} \\ 2n+1 & \text{for } m \text{ even} \end{cases} \tag{4.3}$$

Consider the projection $\pi_i : X \to S^m$ onto the i–th factor (π_i projects the Cartesian product onto the i–th sphere of the product). This projection induces a map:

$$\pi_i^* : H^*(S^m; \mathbb{Q}) \to H^*(X; \mathbb{Q})$$

If $u_i \in H^*(S^m; \mathbb{Q})$ is the fundamental class of the i–th factored sphere, let:

$$a_i = \pi_i^*(u_i) \in H^*(X; \mathbb{Q}).$$

Notice that for m odd:

$$\prod_{i=1}^{n} (1 \otimes a_i - a_i \otimes 1) \neq 0 \in H^*(X \times X; \mathbb{Q})$$

and for m even:

$$\prod_{i=1}^{n} (1 \otimes a_i - a_i \otimes 1)^2 \neq 0 \in H^*(X \times X; \mathbb{Q}).$$

In other words:

$$\mathrm{zcl}(X) \geqslant \begin{cases} n & \text{for } m \text{ odd} \\ 2n & \text{for } m \text{ even} \end{cases} \tag{4.4}$$

From **Theorem 3.2.4** and equations (4.3) and (4.4), we complete the proof. □

Remark. Applying **Theorem 4.1.2** to $T^2 = S^1 \times S^1$, one has:

$$\mathrm{TC}(T^2) = \mathrm{TC}(S^1 \times S^1) = 3 \tag{4.5}$$

Combining (4.5) and (3.7) with the fact that $\mathrm{TC}(S^2) = 3$, we have a general result for a compact orientable two–dimensional surfaces of genus g, Σ_g:

$$\mathrm{TC}(\Sigma_g) = \begin{cases} 3 & \text{if } g \leqslant 1 \\ 5 & \text{if } g > 1 \end{cases}$$

4.2 Planar robot arm

At the beginning of this work, we mentioned that we would like to solve the task of moving in a complicated environment for human. We proposed that a robot could help do the work. In the simplest case (planar case), we can imagine the robot, just like an arm consisting of n bars $L_1, \dots\ L_n$, such that L_i and L_{i+1} are connected by flexible joins. If the initial bar L_1 is fixed, the configuration of the arm is determined by n angles $\alpha_1, \dots, \alpha_n$, where α_i is the angle between L_i and the x–axis. Note that each of the n bars can move circularly, so our configuration space (without obstacles) is the Cartesian product of n circles.

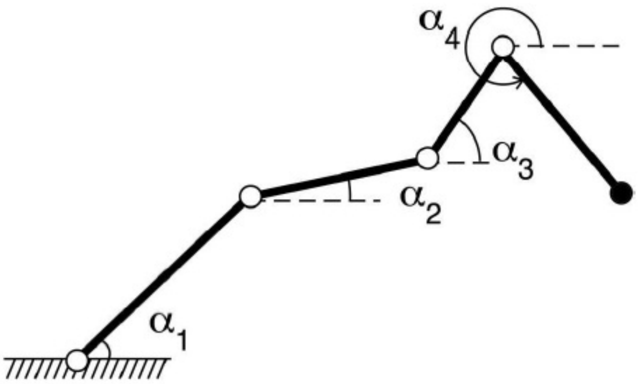


Figure 4.1: Planar robot arm, [6, p. 10]

Then, by **Theorem 4.1.2**, we conclude that the Topological Complexity of the motion planning problem of a plane n–bar robot arm is:

$$n + 1.$$

Similarly, the configuration space of a robot arm in $\mathbb{R}^3$ is the Cartesian product of n copies of the two–dimensional sphere:

$$S^2 \times \cdots \times S^2.$$

Therefore, for a spacial n–bar robot arm, the Topological Complexity is:

$$2n + 1.$$

Remark. For a n–bar robot arm in $\mathbb{R}^4$, the Topological Complexity is one more time $n + 1$ (the configuration space is the Cartesian product of n–copies of S^3, which are spheres of odd dimension). Therefore, in theory, it is easier to program a continuous motion planning of an arm that moves in space–time than in space.

Chapter 5

Motion Planning in Projective Spaces

5.1 A covering space for $\mathbb{RP}^n \times \mathbb{RP}^n$

In this last section, we study a very specific configuration space; the n–dimensional real projective space $\mathbb{RP}^n$. Recall that $\mathbb{RP}^n$ can be thought of as the set of lines in $\mathbb{R}^{n+1}$ passing through the origin. In particular, we will give an algorithm to move such a line in $\mathbb{R}^3$, which is equivalent to constructing a motion planner over $\mathbb{RP}^2$. To begin our study, it is necessary to recall some concepts.

Definition 5.1.1. Let X be a topological space:

1. A **covering space** of X, is a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: There exists an open cover $\mathcal{U} = \{U_\alpha\}$ of X such that for all α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\widetilde{X}$, each of which is mapped homeomorphically onto U_α by p.

2. A **lift** of a point $x \in X$, is a point $\widetilde{x} \in p^{-1}(x)$, and a **lift** of a map $f : Y \to X$, is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$.

3. For a covering space $p : \widetilde{X} \to X$ the homeomorphisms $g : \widetilde{X} \to \widetilde{X}$ such that $pg = p$, are called **deck transformations**. These form a group G under composition.

4. A covering space $p : \widetilde{X} \to X$ is called **regular** if for each $x \in X$ and each pair of lifts $\widetilde{x_1}, \widetilde{x_2}$ of x, there is a deck transformation $g \in G$ such that

$$g\widetilde{x_1} = \widetilde{x_2}$$

In the context of TC, we work with the Cartesian product of our configuration space, then it is expected to study covering spaces of Cartesian products. Let $p : \widetilde{X} \to X$ be a regular covering map with the group of covering transformations G. Let $\widetilde{X} \times_G \widetilde{X}$ be obtained from the product $\widetilde{X} \times \widetilde{X}$ by factorizing with respect to the diagonal action of G, i.e. $g(\widetilde{x_1}, \widetilde{x_2}) = (g\widetilde{x_1}, g\widetilde{x_2})$. Define the map $q : \widetilde{X} \times_G \widetilde{X} \to X \times X$ which maps an equivalence class $[\widetilde{x_1}, \widetilde{x_2}]$ of a pair $(\widetilde{x_1}, \widetilde{x_2})$ to (x_1, x_2). Notice that the map q is well defined (there is $g \in G$ which connects any pair of lifts) and continuous. Consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \times \widetilde{X} \arrow[r, "h"] \arrow[dr, "p \times p"'] & \widetilde{X} \times_G \widetilde{X} \arrow[d, "q"] \\
& X \times X
\end{tikzcd}$$

Here $h(\widetilde{x_1}, \widetilde{x_2}) = [\widetilde{x_1}, \widetilde{x_2}]$. If $\mathcal{U}$ is an open cover of X satisfying **Definition 5.1.1(1)**, we can define an open cover $\mathcal{V}$ of $X \times X$ consisting of open sets $U_{\alpha_1} \times U_{\alpha_2}$ with $U_{\alpha_1}, U_{\alpha_2} \in \mathcal{U}$. Moreover, since p is covering, $p^{-1}(U_\alpha) = \sqcup W_\alpha$, where W_α is an open subset of $\widetilde{X}$. Let $(x_1, x_2) \in U_1 \times U_2 \in \mathcal{V}$. Notice that:

$$\begin{aligned} q^{-1}(U_1 \times U_2) &= \{[\widetilde{x_1}, \widetilde{x_2}] \mid q([\widetilde{x_1}, \widetilde{x_2}]) \subset U_1 \times U_2\} \\ &= \left\{[\widetilde{x_1}, \widetilde{x_2}] \mid \widetilde{x_1} \in p^{-1}(U_1), \widetilde{x_2} \in p^{-1}(U_2)\right\} \\ &= h\left(p^{-1}(U_1) \times p^{-1}(U_2)\right) \\ &= h\left((\sqcup_i W_{1i}) \times (\sqcup_j W_{2j})\right) \\ &= h\left(\sqcup_{i,j} (W_{1i} \times W_{2j})\right) \end{aligned}$$

Therefore, $q : \widetilde{X} \times_G \widetilde{X} \to X \times X$ is a covering space. Now, define the map $f : PX \to \widetilde{X} \times_G \widetilde{X}$ as follows: given a continuous path $\gamma : [0,1] \to X$, let $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ be any lift of γ, and set:

$$f(\gamma) = (\widetilde{\gamma}(0), \widetilde{\gamma}(1)) \in \widetilde{X} \times_G \widetilde{X}$$

So we have this commutative diagram:

$$\begin{tikzcd} PX \arrow[r, "f"] \arrow[dr, "\pi"'] & \widetilde{X} \times_G \widetilde{X} \arrow[r, equal] \arrow[d, "q"] & \widetilde{X} \times_G \widetilde{X} \arrow[dl, "q"] \\ & X \times X & \end{tikzcd}$$

For any path $\gamma : [0,1] \to X$, and for each lift $\widetilde{x_0}$ of the starting point $\gamma(0) = x_0$, there is a unique path $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ lifting γ starting at $\widetilde{x_0}$. Therefore, the lift $\widetilde{\gamma}$ of γ depends on the choice of the initial point $\widetilde{\gamma}(0) \in \widetilde{X}$, but nevertheless the map f is well defined by the same reason of the covering space q (there is $g \in G$ which connects any pair of lifts), and moreover, f is continuous (f is a lift of π).

This is when TC appears again. The next result is **Theorem 4.1** of [8].

Theorem 5.1.2. The Sectional Category of the covering space $q : \widetilde{X} \times_G \widetilde{X} \to X \times X$ is less than or equal to $\mathrm{TC}(X)$.

Proof. Let $U \subset X \times X$ an open subset and $s : U \to PX$ a continuous section of the fibration π over U. By the above diagram:

$$q \circ (f \circ s) = (q \circ f) \circ s = \pi \circ s = 1_U$$

i.e. $f \circ s$ is a continuous section of q over U. If $\mathrm{TC}(X) = k$, there is an open cover $U_1 \cup \cdots \cup U_k$ of $X \times X$ with a continuous section s_i of π over U_i, then $f \circ s_i$ is a continuous section of q over U_i. Hence $\mathrm{secat}(q)$ is at most k. $\square$

Remark. By **Theorem 5.1.2**, we know that $\mathrm{TC}(\mathbb{RP}^n)$ is greater than or equal to the Sectional Category of the two–fold covering:

$$S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{RP}^n \times \mathbb{RP}^n$$

Let ξ be the canonical real line bundle over $\mathbb{RP}^n$, this is:

$$\xi = \{(L, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in L\}.$$

The sphere bundle associated to ξ, $S(\xi)$, corresponds to the inclusion $S^n \hookrightarrow \xi$ given by $x \mapsto ([x], x)$ and, in these terms, the bundle projection $S(\xi) \to \mathbb{RP}^n$ is the canonical projection $S^n \to \mathbb{RP}^n$. In particular, ξ is recovered as the Borel construction $S^n \times_{\mathbb{Z}_2} \mathbb{R}$ where $\mathbb{Z}_2$ acts on $\mathbb{R}$ by change of signs. If $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is the generator, the first Stiefel–Whitney class of ξ is:

$$w_1(\xi) = \alpha.$$

Since we want information about $\mathbb{RP}^n \times \mathbb{RP}^n$, it is necessary to construct a new bundle in that base space, starting from ξ.

The exterior tensor product $\xi_1 \otimes \xi_2$ is a real line bundle over $\mathbb{RP}^n \times \mathbb{RP}^n$:

$$\begin{tikzcd}
 & \xi_1 \otimes \xi_2 \arrow[dl] \arrow[d] \arrow[dr] & \\
\xi \arrow[d] & \mathbb{RP}^n \times \mathbb{RP}^n \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \xi \arrow[d] \\
\mathbb{RP}^n & & \mathbb{RP}^n
\end{tikzcd}$$

where ξ_1 and ξ_2 are the pullbacks of ξ under the projections π_1 and π_2 respectively. Define $\eta = \xi_1 \otimes \xi_2$. Since $w_1(\eta) = w_1(\xi_1) + w_1(\xi_2)$ (see [13, p. 87]), it follows that:

$$w_1(\eta) = w_1(\pi_1^*(\xi)) + w_1(\pi_2^*(\xi)) = \pi_1^*(w_1((\xi)) + \pi_2^* w_1((\xi)) = \pi_1^*(\alpha) + \pi_2^*(\alpha) \tag{5.1}$$

Now, consider this commutative diagram:

$$\begin{tikzcd}
\mathbb{RP}^n \arrow[r, hook, "i_1"] \arrow[d, equal] & \mathbb{RP}^n \times \mathbb{RP}^n \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathbb{RP}^n \arrow[l, hook', "i_2"'] \arrow[d, equal] \\
\mathbb{RP}^n & & \mathbb{RP}^n
\end{tikzcd}$$

By **Künneth** (see [10, Theorem 3.16, p. 219]), we know that:

$$H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2) \cong H^1(\mathbb{RP}^n; \mathbb{Z}_2) \otimes H^0(\mathbb{RP}^n; \mathbb{Z}_2) \oplus H^0(\mathbb{RP}^n; \mathbb{Z}_2) \otimes H^1(\mathbb{RP}^n; \mathbb{Z}_2)$$

where $H^1(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ is generated by α and $H^0(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ is generated by 1. Then, $H^1(\mathbb{RP}^n) \otimes H^0(\mathbb{RP}^n) = \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2$ is generated by $\alpha \otimes 1$ and by the same reason, $H^0(\mathbb{RP}^n) \otimes H^1(\mathbb{RP}^n)$ is generated by $1 \otimes \alpha$. Therefore:

$$H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2) \text{ is generated by } \alpha \otimes 1 \text{ and } 1 \otimes \alpha \tag{5.2}$$

Finally, remember that, in general we have the product:

$$\times : H^*(X; R) \otimes H^*(Y; R) \longrightarrow H^*(X \times Y; R) \tag{5.3}$$

given by:

$$u \otimes v \mapsto u \times v = \pi_1^*(u)\pi_2^*(v)$$

where π_1 and π_2 are the canonical projections. Hence, we apply (5.3):

$$\pi_1^*(\alpha) + \pi_2^*(\alpha) = \alpha \times 1 + 1 \times \alpha \in H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2)$$

and we conclude that (5.1) is:

$$w_1(\eta) = \alpha \times 1 + 1 \times \alpha. \tag{5.4}$$

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The above constructions can be interpreted in topological terms as follows: Let the generator of $\mathbb{Z}_2$ act on the Borel construction $S^n \times_{\mathbb{Z}_2} S^n$ by the rule

$$[(x, y)] \to [(-x, y)] = [(x, -y)]. \tag{5.5}$$

The canonical projection

$$S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{RP}^n \times \mathbb{RP}^n \tag{5.6}$$

is a regular 2–fold covering.

Lemma 5.1.3. The associated line bundle $(S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} \mathbb{R} \to \mathbb{RP}^n \times \mathbb{RP}^n$ is isomorphic to $\xi \otimes \xi$. In particular, (5.6) is the sphere bundle of $\xi \otimes \xi$.

Proof. This follows from (5.4) in view of the noted isomorphism $\xi \cong S^n \times_{\mathbb{Z}_2} \mathbb{R}$ and the pull-back diagrams of double covers:

$$\begin{tikzcd} S^n \arrow[r, equal] \arrow[d] & S^n \times_{\mathbb{Z}_2} \mathbb{Z}_2 \arrow[r, hook] \arrow[d] & S^n \times_{\mathbb{Z}_2} S^n \arrow[d] \\ \mathbb{RP}^n \arrow[r, equal] & \mathbb{RP}^n \times * \arrow[r, hook] & \mathbb{RP}^n \times \mathbb{RP}^n, \end{tikzcd} \qquad \begin{tikzcd} S^n \arrow[r, equal] \arrow[d] & \mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n \arrow[r, hook] \arrow[d] & S^n \times_{\mathbb{Z}_2} S^n \arrow[d] \\ \mathbb{RP}^n \arrow[r, equal] & * \times \mathbb{RP}^n \arrow[r, hook] & \mathbb{RP}^n \times \mathbb{RP}^n. \end{tikzcd}$$

□

Corollary 5.1.4. $\mathrm{TC}(\mathbb{RP}^n) \geqslant k$ where k is the minimal positive integer such that $k(\xi \otimes \xi)$ admits a nowhere vanishing section.

Proof. As remarked right after the statement of **Theorem 5.1.2**, $\mathrm{TC}(\mathbb{RP}^n)$ is bounded from below by the Sectional Category of (5.6). The result then follows from **Lemma 5.1.3** and the following result. □

Lemma 5.1.5. Let $\lambda \to B$ be a vector bundle over a paracompact base B. If k stands for the Sectional Category of the sphere bundle $S(\lambda)$, then $k\lambda$ admits a nowhere vanishing section.

Proof. Let $U_1, \ldots, U_k$ be an open covering of B so that the restriction $S(\lambda)_{|U_i}$ of $S(\lambda)$ to each U_i admits a section $s_i \colon U_i \to S(\lambda)_{|U_i}$. Think of each s_i as a nowhere zero section of λ defined on U_i. By a standard partition of unit argument, it is possible to refine the covering $\{U_i\}$ to an open covering $\{V_i\}$, $\overline{V_i} \subseteq U_i$, so that each restriction $s_{i|V_i}$ can be extended to a (continuous) section σ_i of λ defined over all of over B. Note that, although each σ_i may vanish at some points of B, it never vanishes in V_i. Since the map

$$(\sigma_1, \ldots, \sigma_k) \colon B \to \underbrace{\lambda \times \cdots \times \lambda}_{k}$$

is a lift of the diagonal, it determines a section of $k\lambda$ which, by construction, is evidently nowhere zero. □

It is well known (see [4, p. 4]) that $\mathrm{cat}(\mathbb{RP}^n) = n + 1$, then $\mathrm{TC}(\mathbb{RP}^n) \geqslant n + 1$. In particular, we have another bound in terms of powers of 2, given by **Theorem 4.5** of [8].

Theorem 5.1.6. If $2^r > n \geqslant 2^{r-1}$, then $\mathrm{TC}(\mathbb{RP}^n) \geqslant 2^r$.

Proof. Let $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ be the generator. Notice that $1 \otimes \alpha + \alpha \otimes 1$ is a zero–divisor since $2\alpha = 0$. Consider its power:

$$\begin{aligned}(1\otimes\alpha+\alpha\otimes 1)^{2^r-1} &= \sum_{i=0}^{2^r-1}\binom{2^r-1}{i}(1\otimes\alpha^{2^r-1-i})(\alpha^i\otimes 1)\\ &= \sum_{i=0}^{2^r-1}\binom{2^r-1}{i}(\alpha^i\otimes\alpha^{2^r-1-i})\\ &= 1\otimes\alpha^{2^r-1}+\alpha\otimes\alpha^{2^r-2}+\cdots+\\ &\quad+\alpha^{2^r-2}\otimes 1+\alpha^{2^r-2}\otimes\alpha\end{aligned}$$

Remember that:

$$H^q(\mathbb{RP}^n)=\begin{cases}\mathbb{Z}_2 & \text{if } q\leqslant n\\ 0 & \text{if } q>n\end{cases}$$

Since for all i, $\binom{2^r-1}{i}$ is odd, the only factor which is certainly nonzero is:

$$\binom{2^r-1}{n}\alpha^k\otimes\alpha^n$$

where $k=2^r-1-n$. Applying **Theorem 3.2.4** we have $\mathrm{TC}(\mathbb{RP}^n)\geqslant 2^r$.

□

5.2 Nonsingular and Axial maps

Definition 5.2.1. A continuous map $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^k$ is called **nonsingular** if it has the following properties:

1. $f(\lambda u,\mu v)=\lambda\mu f(u,v)$ for all $u,v\in\mathbb{R}^n,\lambda,\mu\in\mathbb{R}$
2. $f(u,v)=0$ implies that either $u=0$ or $v=0$.

Consider the vectors $e_1=(1,0),e_2=(0,1),e_3=(1,1)$. The respectively functionals are $\alpha_1(x,y)=x,\alpha_2(x,y)=y,\alpha_3(x,y)=x-y$. Notice that any two of them are linearly independent. In general, for any n, we can fix a sequence $\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}:\mathbb{R}^n\to\mathbb{R}$ of linear functionals such that any n of them are linearly independent. Let $u,v\in\mathbb{R}^n$, we define:

$$f(u,v)=(\alpha_1(u)\alpha_1(v),\ldots,\alpha_{2n-1}(u)\alpha_{2n-1}(v))$$

If $u\neq 0$, then at least n among the numbers $\alpha_1(u),\ldots,\alpha_{2n-1}(u)$ are nonzero. Therefore, if $u\neq 0$ and $v\neq 0$, there exists $i\in\{1,\ldots,2n-1\}$ such that $\alpha_i(u)\alpha_i(v)\neq 0$, and thus $f(u,v)\neq 0\in\mathbb{R}^{2n-1}$. Hence, for any n, the maximal dimension in which we can guarantee the existence of a nonsingular map is $2n-1$.

Lemma 5.2.2. There are not nonsingular maps $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^k$ with $k<n$.

Proof. Suppose that there exists such map. If $v\neq 0$ is fixed and $u\in S^{n-1}\subset\mathbb{R}^n$, we can consider the map:

$$u\mapsto f(u,v)\in\mathbb{R}^k\subseteq\mathbb{R}^{n-1}$$

By **Borsuk–Ulam Theorem** (see [10, p. 32]), there exists $w\in S^{n-1}$ such that $f(w,v)=f(-w,v)$. But if f is nonsingular, $f(-w,v)=-f(w,v)$ which implies that $f(w,v)=0$, and clearly this is a contradiction. □

Lemma 5.2.3. For $n = 1, 2, 4, 8$, there exists a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with the property that for any $u \in \mathbb{R}^n$, $u \neq 0$, the first coordinate of $f(u, u)$ is positive.

Proof. For $n = 1$, define $f(u, v) = uv$, the usual product of real numbers. For $n = 2$, we take $f(u, v) = u\overline{v}$, the product of u and the conjugate of v viewed as complex numbers. Similarly, for $n = 4$, we may define $f(u, v) = u\overline{v}$, where $\overline{v} = x_1 - x_2 i - x_3 j - x_4 k$ is the conjugation of the quaternion $v = x_1 + x_2 i + x_3 j + x_4 k$. In all cases, $f(u, u) = |u|^2$. Finally, a Cayley number can be uniquely written in the form $q + Qe$, where q and Q are quaternions and e is a formal symbol. The multiplication is defined by the formula:

$$(q + Qe) \cdot (r + Re) = \big(qr - \overline{R}Q\big) + (Rq + Q\overline{r})\, e$$

If we define:

$$f(q + Qe, r + Re) = (q + Qe) \cdot (\overline{r} - Re)$$

Then, $f(q + Qe, q + Qe) = q\overline{q} + Q\overline{Q}$ is real and positive, as long as $q + Qe \neq 0$ is nonzero. □

Lemma 5.2.4. There are not nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, for $n \neq 1, 2, 4, 8$.

Proof. Suppose that f as above exists, where $n > 2$. Consider the map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ given by:

$$g(x, y) = \frac{f(x, y)}{|f(x, y)|}$$

where $x, y \in S^{n-1}$. The map g is such that $g(-x, y) = -g(x, y) = g(x, -y)$. Now restricting g onto one factor $S^{n-1} \times *$ (where $*$ is a base point) we have a self map h of S^{n-1}, which commutes with the antipodal map a.

$$\begin{tikzcd} S^{n-1} \arrow[r, "a"] \arrow[d, "h"'] & S^{n-1} \arrow[d, "h"] \\ S^{n-1} \arrow[r, "a"'] & S^{n-1} \end{tikzcd}$$

Therefore, by **Proposition 2B.6** of [10], the degree of h is odd. Analogously, $g|_{* \times S^{n-1}}$ has an odd degree. Hence, the bidegree of g is (k, l), where both integers k and l are odd. If $\iota \in \pi_{n-1}(S^{n-1})$ is the generator, [16, Theorem 7.7] implies that the Whitehead product:

$$[k\iota, l\iota] = kl\, [\iota, \iota] \in \pi_{2n-3}(S^{n-1})$$

should be zero. Nevertheless, if n is odd, then $[\iota, \iota] \in \pi_{2n-3}(S^{n-1})$ is of infinite order, hence $[k\iota, l\iota]$ cannot vanish. Now, if $n \neq 1, 2, 4, 8$ is even, $[\iota, \iota] \in \pi_{2n-3}(S^{n-1})$ is nonzero (see [1]) and has order two, which again implies that $[k\iota, l\iota]$ is nonzero as kl is odd. We conclude that the map f cannot exist. □

Definition 5.2.5. Let n and k be two positive integers with $n < k$. A Continuous map $g : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ is called axial of type (n, k) if its restrictions to $* \times \mathbb{RP}^n$ and $\mathbb{RP}^n \times *$ ($*$ is a base point of $\mathbb{RP}^n$) are homotopic to the inclusion maps $\mathbb{RP}^n \to \mathbb{RP}^k$.

We will denote by $[X, Y]$ the set of all homotopy classes $[f]$ between X and Y.

Lemma 5.2.6. If $n < k$, there is a bijection between $[\mathbb{RP}^n, \mathbb{RP}^k]$ and $[\mathbb{RP}^n, \mathbb{RP}^\infty]$.

Proof. Notice that any $f \in [\mathbb{RP}^n, \mathbb{RP}^\infty]$ can be factorized through the n–dimensional skeleton by another map $f' \in [\mathbb{RP}^n, \mathbb{RP}^n]$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathbb{RP}^n \arrow[r, "f"] \arrow[d, "f'"'] & \mathbb{RP}^\infty \\
\mathbb{RP}^n \arrow[r, hook, "i"'] & \mathbb{RP}^k \arrow[u, hook, "j"']
\end{tikzcd}$$

Then, $i \circ f' \mapsto f$ determines a surjective map.

Now, if $H : \mathbb{RP}^n \times I \to \mathbb{RP}^\infty$ is the homotopy between the compositions:

$$\begin{tikzcd}
\mathbb{RP}^n \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & \mathbb{RP}^k \arrow[r, hook] & \mathbb{RP}^\infty
\end{tikzcd}$$

by the **Cellular Aproximation Theorem** (see [10, p. 349]) there is an homotopy $H' : \mathbb{RP}^n \times I \to \mathbb{RP}^k$ such that initial and final branches of H' in $\mathbb{RP}^k$, are f_0 and f_1, i.e. $f_0 \simeq f_1$.

□

Definition 5.2.7. Let G be a group a $n > 0$. A connected topological space X is called **Eilenberg–MacLane** space of type $K(G, n)$, if it has n–th homotopy group $\pi_n(X)$ isomorphic to G and all other homotopy groups trivial.

Remark. Since $\mathbb{RP}^\infty$ is $K(\mathbb{Z}_2, 1)$, applying **Lemma 5.2.6** and **Brown Representability Theorem** (see [10, p. 448]), we have:

$$[\mathbb{RP}^n, \mathbb{RP}^k] = [\mathbb{RP}^n, \mathbb{RP}^\infty] = [\mathbb{RP}^n, K(\mathbb{Z}_2, 1)] = H^1(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$$

In other words, any continuous map $h : \mathbb{RP}^n \to \mathbb{RP}^k$ with $n < k$ is either homotopically trivial or it is homotopic to the inclusion map. Therefore, if $\alpha_k \in H^1(\mathbb{RP}^k; \mathbb{Z}_2)$ denotes the generator, $h^*\alpha_k \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is either zero or equal to α_n, the generator of $H^1(\mathbb{RP}^n; \mathbb{Z}_2)$. The map h is homotopically trivial if and only if $h^*\alpha_k = 0$.

If g is an axial map, the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathbb{RP}^n \times * \arrow[dr, hook, "i_1"] \arrow[drr, bend left, "\simeq"] & & \\
& \mathbb{RP}^n \times \mathbb{RP}^n \arrow[r, "g"] & \mathbb{RP}^k \\
* \times \mathbb{RP}^n \arrow[ur, hook, "i_2"'] \arrow[urr, bend right, "\simeq"'] & &
\end{tikzcd}$$

Therefore, in cohomology with coefficients in $\mathbb{Z}_2$, we have:

$$H^1(\mathbb{RP}^k) \longrightarrow H^1(\mathbb{RP}^n \times \mathbb{RP}^n) \cong H^1(\mathbb{RP}^n) \otimes H^0(\mathbb{RP}^n) \oplus H^0(\mathbb{RP}^n) \otimes H^1(\mathbb{RP}^n)$$

Applying (5.2), it follows that:

$$\alpha_k \longmapsto g^*\alpha_k = c_1 \cdot \alpha_n \otimes 1 + c_2 \cdot 1 \otimes \alpha_n$$

Notice that $c_1 = 1$ due to:

$$\begin{aligned} i_1^*(\alpha_n \otimes 1) &= (1_{\mathbb{RP}^n} \times i)^*(\alpha_n \otimes 1) \\ &= 1_{\mathbb{RP}^n}^* \alpha_n \otimes i^* 1 \\ &= \alpha_n \otimes 1 \end{aligned}$$

and

$$\begin{aligned} i_2^*(\alpha_n \otimes 1) &= (i \times 1_{\mathbb{RP}^n})^*(\alpha_n \otimes 1) \\ &= i^* \alpha_n \otimes 1_{\mathbb{RP}^n}^* 1 \\ &= 0 \end{aligned}$$

Similarly, applying i_1^* and i_2^* to $1 \otimes \alpha_n$, is easy to see that $c_2 = 1$. So, g^* maps α_k to $\alpha_n \otimes 1 + 1 \otimes \alpha_n$. Again, applying (5.3) we have the formula:

$$g^* \alpha_k = \alpha_n \times 1 + 1 \times \alpha_n \in H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2)$$

This last condition fixes the homotopy type of a map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^\infty$ since the inclusions:

$$\cdots \hookrightarrow \mathbb{RP}^m \hookrightarrow \mathbb{RP}^{m+1} \hookrightarrow \cdots \hookrightarrow \mathbb{RP}^\infty$$

gives in cohomology (with coefficients in $\mathbb{Z}_2$):

$$\begin{array}{ccccccccc} H^1(\mathbb{RP}^\infty) & \hookrightarrow & \cdots & \hookrightarrow & H^1(\mathbb{RP}^{m+1}) & \hookrightarrow & H^1(\mathbb{RP}^m) & \hookrightarrow & \cdots \\ \alpha & \longmapsto & \cdots & \longmapsto & \alpha_{m+1} & \longmapsto & \alpha_m & \longmapsto & \cdots \end{array}$$

where $\alpha \in H^1(\mathbb{RP}^\infty)$ is the universal class given by **Brown Representability Theorem** (see [10, p. 448]), and $\alpha_m \in H^1(\mathbb{RP}^m)$ is the class α restricted by the inclusion $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^\infty$. Therefore, if $ax : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^\infty$ is an axial map, we want to find the smallest k such that this map can be factorized through the inclusion $\mathbb{RP}^k \to \mathbb{RP}^\infty$. This is:

$$\begin{tikzcd} \mathbb{RP}^k \arrow[r, hook] & \cdots \arrow[r, hook] & \mathbb{RP}^m \arrow[r, hook] & \mathbb{RP}^\infty \\ & & & \mathbb{RP}^n \times \mathbb{RP}^n \arrow[u, "ax"'] \arrow[ul, "g"'] \arrow[ulll, dashed, "?"] \end{tikzcd}$$

We will prove that k is precisely $\mathrm{TC}(\mathbb{RP}^n)$.

Remark. Since $\mathbb{RP}^n \times \mathbb{RP}^n$ has dimension $2n$, by **Cellular Aproximation Theorem** (see [10, p. 349]), there always exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{2n}$. In fact, it is possible to show that there always exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{2n-1}$.

Lemma 5.2.8. For $n < k$, the map $\overline{h} \colon \mathbb{RP}^n \to \mathbb{RP}^k$ induced by an odd map $h \colon S^n \to S^k$ (i.e. one satisfying $h(-x) = -h(x)$ for all x) is homotopic to the equatorial inclusion.

Proof. Assume that $\overline{h}$ is null–homotopic. Since $S^k \to \mathbb{RP}^k$ is the universal cover, $\overline{h}$ admits a lifting $H \colon \mathbb{RP}^n \to S^k$ so the bottom triangle in the diagram:

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d, "\pi"'] & S^k \arrow[d, "\pi"] \\ \mathbb{RP}^n \arrow[r, "\overline{h}"'] \arrow[ur, "H"] & \mathbb{RP}^k \end{tikzcd}$$

commutes on the nose. In particular $h(x) = \pm H(\pi(x))$ for all $x \in S^k$. Consequently S^n is covered by the two subsets $P = \{x \in S^n \mid h(x) = H(\pi(x))\}$ and $N = \{x \in S^n \mid h(x) = -H(\pi(x))\}$. Since these are closed and disjoint, one of them is empty and the other agrees with S^n. But this is incompatible with h been odd. $\square$

Lemma 5.2.9. Assume that $1 < n < k$. There exists a bijection between nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ (viewed up to multiplication by a nonzero scalar) and axial maps $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$.

Proof. Let $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ be nonsingular. Consider the map $g : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$, where for $u, v \in S^n \subset \mathbb{R}^{n+1}$, the value $g([u],[v])$ is the line through the origin containing the point $f(u,v) \in \mathbb{R}^{k+1}$. If we fix $v \in S^n$ and vary only $u \in S^n$, the resulting map $\mathbb{RP}^n \to \mathbb{RP}^k$ lifts to a map $S^n \to S^k$ given by:

$$u \mapsto \frac{f(u,v)}{|f(u,v)|}$$

Since $f(-u,v) = -f(u,v)$, the map $S^n \to S^k$ is also odd, therefore by **Lemma 5.2.8**, the map $\mathbb{RP}^n \to \mathbb{RP}^k$ is not null–homotopic. Similarly, using $f(u,-v) = -f(u,v)$, we find that the restriction of g onto $* \times \mathbb{RP}^n$ is not null–homotopic. Then, both maps are homotopic to the inclusion maps. Hence, g is an axial map.

Now, given an axial map g, and passing to the universal covers, we obtain a map $\overline{g} : S^n \times S^n \to S^k$ (defined up to a sign), such that the following diagram is commutative:

$$\begin{tikzcd}
S^n \times S^n \arrow[r, "\overline{g}"] \arrow[d] & S^k \arrow[d] \\
\mathbb{RP}^n \times \mathbb{RP}^n \arrow[r, "g"'] & \mathbb{RP}^k
\end{tikzcd}$$

Therefore, for all $u, v \in S^n$:

$$\overline{g}(-u,v) = -\overline{g}(u,v) = \overline{g}(u,-v)$$

Then, we may define a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ given by:

$$f(u,v) = |u| \cdot |v| \cdot \overline{g}\left(\frac{u}{|u|}, \frac{v}{|v|}\right)$$

where $u, v \in \mathbb{RP}^{n+1} - \{0\}$. □

Lemma 5.2.10. Suppose that for a pair of integers $1 < n < k$, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$. Then, there exists a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, such that for any $u \in \mathbb{R}^{n+1} - \{0\}$, the first coordinate of $f(u,u) \in \mathbb{R}^{k+1}$ is positive.

Proof. Given a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, consider the corresponding axial map $g : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. If we restricted g to the diagonal $\Delta : \mathbb{RP}^n \to \mathbb{RP}^n \times \mathbb{RP}^n$ then:

$$\begin{array}{ccccccc}
H^1(\mathbb{RP}^k;\mathbb{Z}_2) & \xrightarrow{g^*} & H^1(\mathbb{RP}^n \times \mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\Delta^*} & H^1(\mathbb{RP}^n;\mathbb{Z}_2) & \cong & \mathbb{Z}_2 \\
\alpha_k & \mapsto & \alpha_n \times 1 + 1 \times \alpha_n & \mapsto & 2\alpha_n & = & 0
\end{array}$$

which implies that the restriction is null–homotopic. Hence, there exists $g' \simeq g$ such that $g' : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ is constant along the diagonal. Now, consider the nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ corresponding to the axial map g'. Notice that for all $u \in \mathbb{R}^{n+1}$, $u \neq 0$, we can take the map n which sends u to $\frac{u}{|u|}$. Then:

$$\begin{tikzcd}
\mathbb{R}^{n+1} - \{0\} \times \mathbb{R}^{n+1} - \{0\} \arrow[r, "f_0"] \arrow[d, "n \times n"'] & \mathbb{R}^{k+1} - \{0\} \arrow[d, "n"] \\
S^n \times S^n \arrow[r] \arrow[d] & S^k \arrow[d] \\
\mathbb{RP}^n \times \mathbb{RP}^n \arrow[r, "g'"'] & \mathbb{RP}^k
\end{tikzcd}$$

where f_0 is the restriction of the map f. By construction any pair $([u_n], [u_n])$ is mapped to the same line L_* in $\mathbb{RP}^k$. Since the above diagram is commutative, the values $f(u, u) \in \mathbb{R}^{k+1}$ lie on L_*. In fact, since $n > 1$, the image $f_0((\mathbb{R}^{n+1} - \{0\}) \times (\mathbb{R}^{n+1} - \{0\}))$ is connected, so that it is completely contained in one of the two rays determined by L_*. By performing an orthogonal rotation, we may assume that all nonzero vectors of this ray have positive first coordinates. $\square$

5.3 The main theorem

Now, we want to relate the topological complexity with the dimensions of the above maps.

Proposition 5.3.1. For $n > 1$, let k be an integer such that the vector bundle $k(\xi \otimes \xi)$ over $\mathbb{RP}^n \times \mathbb{RP}^n$ admits a nowhere vanishing section. Then, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$

Proof. By **Lemma 5.1.3**, there is an isomorphism of vector bundles $k(\xi \otimes \xi) \cong (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} \mathbb{R}^k$ where $\mathbb{Z}_2$ acts on $\mathbb{R}^k$ by change of signs, and on $S^n \times_{\mathbb{Z}_2} S^n$ as described in (5.5). In particular, we get an isomorphism of associated sphere bundles $S\,(k(\xi \otimes \xi)) \cong (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} S^{k-1}$. Then, a nowhere zero section for $k(\xi \otimes \xi)$ can be normalized to a section $s\colon \mathbb{RP}^n \times \mathbb{RP}^n \to (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} S^{k-1}$. Let S denote the composition:

$$S^n \times S^n \to S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{RP}^n \times \mathbb{RP}^n \xrightarrow{s} (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} S^{k-1}$$

where unlabeled maps stand for canonical projections. Note that:

$$S(-x, y) = S(x, y) = S(x, -y) \quad \text{for all} \quad x, y \in S^n. \tag{5.7}$$

Since $n > 1$, $S^n \times S^n$ is simply connected, so that [10, Prop 1.33 p. 61], implies that S admits a lifting $\sigma\colon S^n \times S^n \to (S^n \times S^n) \times S^{k-1}$ through the composition of 2–fold covering projections:

$$(S^n \times S^n) \times S^{k-1} \to (S^n \times_{\mathbb{Z}_2} S^n) \times S^{k-1} \to (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} S^{k-1}. \tag{5.8}$$

Actually, since the composite (5.8) is a 4-sheeted covering projection, there are four choices of such liftings. Explicitely, fixing a lifting $\sigma_0\colon S^n \times S^n \to (S^n \times S^n) \times S^{k-1}$, the other three liftings are $\sigma_1 = \tau_1 \circ \sigma_0$, $\sigma_2 = \tau_2 \circ \sigma_0$, and $\sigma_3 = \tau_2 \circ \tau_1 \circ \sigma_0$ where

$$\tau_1(x, y, z) = (-x, -y, z),\ \tau_2(x, y, z) = (-x, y, -z) \text{ and, so, } \tau_2 \circ \tau_1(x, y, z) = (x, -y, -z). \tag{5.9}$$

Since each such lifting gives the commutative diagram:

$$\begin{tikzcd}
 & & & (S^n \times S^n) \times S^{k-1} \arrow[r, "\mathrm{proj}"] \arrow[d] & S^n \times S^n \arrow[dddl, bend left] \\
 & & & (S^n \times_{\mathbb{Z}_2} S^n) \times S^{k-1} \arrow[d] & \\
 & & & (S^n \times_{\mathbb{Z}_2} S^n) \times_{\mathbb{Z}_2} S^{k-1} \arrow[d] & \\
S^n \times S^n \arrow[r] \arrow[uuurrr, "\sigma_i"] & S^n \times_{\mathbb{Z}_2} S^n \arrow[r] & \mathbb{RP}^n \times \mathbb{RP}^n \arrow[r, equal] \arrow[ur, "s"] & \mathbb{RP}^n \times \mathbb{RP}^n &
\end{tikzcd}$$

where the rightmost map and the horizontal composition are the standard 4-sheeted covering, it follows that one (and only one) of the composites $\mathrm{proj} \circ \sigma_i$ is the identity. Without loss of generality assume this holds for $i = 0$, and let $f\colon S^n \times S^n \to S^{k-1}$ denote the third component of σ_0. Note that (5.7) and (5.9) imply that f is $\mathbb{Z}_2$-biequivariant, that is, it satisfies:

$$f(-x, y) = -f(x, y) = f(x, -y) \text{ for } x, y \in S^n.$$

Consequently, the required nonsingular map $F\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ is given by setting:

$$F(\lambda x, \mu y) = \lambda\mu f(x, y) \text{ for } x, y \in S^n \text{ and } \lambda, \mu \in \mathbb{R}.$$

$\square$

Proposition 5.3.2. If there exists a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ with $n + 1 < k$, then $\mathrm{TC}(\mathbb{RP}^n) \leqslant k$.

Proof. Let $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a scalar continuous map such that $\phi(\lambda u, \mu v) = \lambda\mu\phi(u, v)$ for all $u, v \in \mathbb{R}^{n+1}$ and $\lambda, \mu \in \mathbb{R}$. Define $U_\phi \subset \mathbb{RP}^n \times \mathbb{RP}^n$ as the set of all pairs of lines (L_1, L_2) in $\mathbb{R}^{n+1}$ such that $L_1 \neq L_2$ and $\phi(u, v) \neq 0$ for some points $u \in L_1$ and $v \in L_2$. Notice that U_ϕ is open. Moreover, we may find unit vectors $u \in L_1$ and $v \in L_2$ such that $\phi(u, v) > 0$. Instead of (u, v), we may take $(-u, -v)$, and both pairs determine the same orientation of the plane spanned by L_1, L_2. Then, there exists a continuous motion planning map $s : U_\phi \to P(\mathbb{RP}^n)$ consists in rotating L_1 toward L_2 in this plane, in the positive direction determined by the orientation.

Assume in addition that $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is **positive**, i.e. for any $u \in \mathbb{R}^{n+1}, u \neq 0$, $\phi(u, u) > 0$. Therefore, instead of U_ϕ, we may take a slightly larger set $U'_\phi \subset \mathbb{RP}^n \times \mathbb{RP}^n$ which consists of all pairs of lines (L_1, L_2) in $\mathbb{RP}^{n+1}$ such that $\phi(u, v) \neq 0$ for some $u \in L_1$ and $v \in L_2$. Now, all pairs of lines (L, L) belong to U'_ϕ. Thereby, if $L_1 \neq L_2$, the path from L_1 to L_2 is defined as above, and if $L_1 = L_2$, we choose the constant path at L_1.

Our map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ determines k scalar maps $\phi_1, \ldots, \phi_k : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ (the coordinates) with its respective neighbourhoods U_{ϕ_i} which cover the product $\mathbb{RP}^n \times \mathbb{RP}^n$ minus the diagonal. Since $n + 1 < k$, by **Lemma 5.2.10**, we may replace the initial nonsingular map f by such an f' that for any nonzero $u \in \mathbb{R}^{n+1}$, the first coordinate $\phi'_1(u, u)$ of $f'(u, u)$ is positive. The open sets $U'_{\phi_1}, U_{\phi_2}, \ldots, U_{\phi_k}$ cover $\mathbb{RP}^n \times \mathbb{RP}^n$. We have described explicit motion planning rules over each of these sets. Hence, $\mathrm{TC}(\mathbb{RP}^n) \leqslant k$. $\square$

Proposition 5.3.3. For $n = 1, 3, 7$, $\mathrm{TC}(\mathbb{RP}^n) = n + 1$.

Proof. Proceeding as in the proof of **Proposition 5.3.2** with the nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by **Lemma 5.2.3**, it follows that $\mathrm{TC}(\mathbb{RP}^n) \leqslant n + 1$. On the other hand, we have $\mathrm{TC}(\mathbb{RP}^n) \geqslant \mathrm{cat}(\mathbb{RP}^n) = n + 1$. $\square$

With all the tools ready, we can state the main theorem.

Theorem 5.3.4. The number $\mathrm{TC}(\mathbb{RP}^n)$ coincides with the smallest integer k such that there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$.

Proof. If $n \neq 1, 3, 7$, **Lemma 5.2.4** implies that there are not nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and therefore $n + 1 < k$. By **Proposition 5.3.2**, $\mathrm{TC}(\mathbb{RP}^n) \leqslant k$. Let l be the smallest integer such that the vector bundle $l(\xi \otimes \xi)$ over $\mathbb{RP}^n \times \mathbb{RP}^n$ admits a nowhere vanishing section, then, by **Proposition 5.3.1**, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^l$. Notice that necessarily $k \leqslant l$. Nevertheless, by **Corollary 5.1.4**, $l \leqslant \mathrm{TC}(\mathbb{RP}^n)$. The cases $n = 1, 3, 7$ are covered by **Proposition 5.3.3**. $\square$

Remark. The power of **Theorem 5.3.4**, is that the computation $\mathrm{TC}(\mathbb{RP}^n)$ is the same as finding nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$, which at first, seem to be different problems.

Chapter 6

Discussion of Results

6.1 Motion Planner in $\mathbb{RP}^1$ and $\mathbb{RP}^2$

In this section, we will solve the problem of moving a line through the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 6.1.1. Topological Complexity of $\mathbb{RP}^1$.

First, notice that **Proposition 5.3.3** states that $\mathrm{TC}(\mathbb{RP}^1) = 2$, then there exist exactly two open subsets of $\mathbb{RP}^1 \times \mathbb{RP}^1$ with continuous motion strategy. Define $U_1 = \{(L_1, L_2) \mid L_1 \text{ and } L_2 \text{ are not perpendicular } \}$ and consider the map $s_1 : U_1 \to P(\mathbb{RP}^1)$ which moves L_1 towards L_2 sweeping the smallest angle, see **Fig. 6.1** (left). The problem is when L_1 and L_2 are perpendicular, since we have two right angles, as shown in **Fig. 6.1** (right):

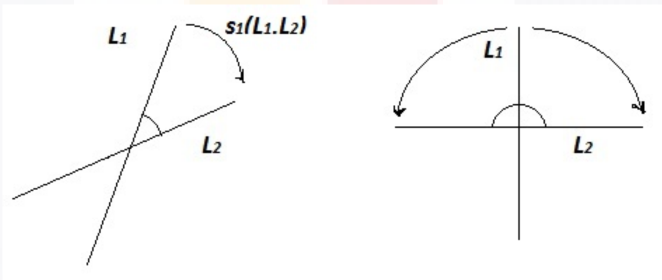

Figure 6.1: Motion planning over U_1

The second open set is $U_2 = \{(L_1, L_2) \mid L_1 \nparallel L_2\}$ and the continuous map is $s_2 : U_2 \to P(\mathbb{RP}^1)$ which moves L_1 towards L_2 in the clockwise sense, see **Fig. 6.2**. The problem with parallel (equal) lines (see **Fig. 6.3**), is that one option is to rotate half a revolution and the other is not to rotate.

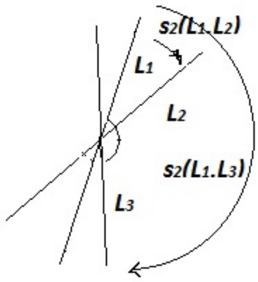

Figure 6.2: Motion planning over U_2

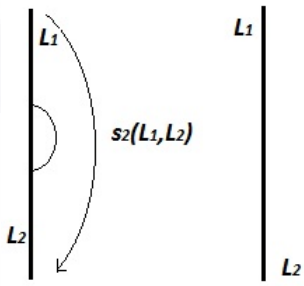

Figure 6.3: Case of parallel lines

Example 6.1.2. Topological Complexity of $\mathbb{RP}^2$.

By **Lemma 5.2.4**, there are not nonsingular maps $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, but notice that if $i, j, k \in \mathbb{R}^4$ are the imaginary units, the map:

$$(x_1, x_2, x_3) \mapsto x_1 + x_2 i + x_3 j$$

is an embedding of $\mathbb{R}^3$ into $\mathbb{R}^4$. Consider the nonsingular map $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ of **Lemma 5.2.3**. Restricting it onto $\mathbb{R}^3 \subset \mathbb{R}^4$, we have a nonsingular map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ given by:

$$f(x, y) = \langle x, y \rangle - \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j - \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} k$$

Therefore, **Theorem 5.3.4** implies that $\mathrm{TC}(\mathbb{RP}^2) = 4$. Then, proceeding as in the proof of **Proposition 5.3.2**, we have four open subsets U_1, U_2, U_3, U_4 covering $\mathbb{RP}^2 \times \mathbb{RP}^2$, where each U_i corresponds to the scalar map ϕ_i obtained from f by considering only the i–th coordinate:

$$f(x, y) = \phi_1(x, y) + \phi_2(x, y)i + \phi_3(x, y)j + \phi_4(x, y)k$$

The subset U_1 consists of the pairs of lines in $\mathbb{R}^3$ making an acute angle. The subset U_2 consists of pairs of lines in $\mathbb{R}^3$ such that their projections onto the x_1x_2–plane span this plane. The subsets U_3 and U_4 are defined analogously replacing the x_1x_2–plane by the x_2x_3–plane and x_1x_3–plane respectively. Again, by the proof of **Proposition 5.3.2**, we know that each functional ϕ_i defines a continuous motion planning strategy over the subset U_i. In U_1, if lines L_1 and L_2 make an acute angle, we rotate L_1 towards L_2 in the 2–plane spanned by L_1 and L_2 so that L_1 sweeps the acute angle. If $L_1 = L_2$, the L_2 stays fixed. Now, in U_2 fix an orientation of the x_1x_2–plane. For any pair $(L_1, L_2) \in U_2$, we obtain an orientation of the 2–plane spanned by L_1 and L_2, and we rotate L_1 towards L_2 in this 2–plane in the direction of the orientation. The motion planning strategies over U_3 and U_4 are similarly.

6.2 Immersions

In this section, we show that the problem of computing the topological complexity of the motion planning problem in $\mathbb{RP}^n$ is equivalent to the immersion problem for the real projective spaces. The next theorem of [2], will be fundamental for our purposes.

Theorem 6.2.1 (Adem, Gitler, James)**.** There exists an immersion $\mathbb{RP}^n \looparrowright \mathbb{R}^k$ (where $k > n$) if and only if there exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. $\square$

By **Lemma 5.2.9**, the existence of a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, is equivalent to the existence of an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. As a consequence of **Theorem 6.2.1** and **Theorem 5.3.4**, it follows that:

Theorem 6.2.2. For any $n \neq 1, 3, 7$, the number $\mathrm{TC}(\mathbb{RP}^n)$ equals to the smallest k such that the projective space $\mathbb{RP}^n$ admits an immersion into $\mathbb{R}^{k-1}$. $\square$

Now, in the next Theorem, we give a direct construction.

Theorem 6.2.3. If $\mathbb{RP}^n$ can be immersed into $\mathbb{R}^k$, then, $\mathrm{TC}(\mathbb{RP}^n) \leqslant k + 1$.

Proof. By definition, an immersion $f\colon \mathbb{RP}^n \looparrowright \mathbb{R}^k$ induces a monomorphism $T(\mathbb{RP}^n) \hookrightarrow f^*(T(\mathbb{R}^k))$ of tangent bundles. Since $\mathbb{R}^k$ is parallelizable (i.e. $T(\mathbb{R}^k)$ is trivial) and $\mathbb{RP}^n$ is compact, we get a Whitney sum decomposition[1] $k\epsilon \cong T(\mathbb{RP}^n) \oplus \nu$ where ϵ is the trivial line bundle over $\mathbb{RP}^n$, and $\nu \cong k\epsilon / T(\mathbb{RP}^n)$ is the normal bundle of the immersion f. In these terms, the k canonical sections of $k\epsilon$ are mapped under the canonical epimorphism $k\epsilon \cong T(\mathbb{RP}^n) \oplus \nu \to T(\mathbb{RP}^n)$ (an epimorphism of vector bundles) onto k tangent vector fields $v_1, v_2, \ldots, v_k$ on $\mathbb{RP}^n$ (i.e. sections of $T(\mathbb{RP}^n)$).

Define $U_0 \subset \mathbb{RP}^n \times \mathbb{RP}^n$ as the set of pairs of lines (L_1, L_2) in $\mathbb{R}^{n+1}$ making an acute angle. A nonzero tangent vector v to the projective space $\mathbb{RP}^n$ at a point L_1 (a line in $\mathbb{R}^{n+1}$) determines a line $\widehat{v}$ in $\mathbb{R}^{n+1}$, which is orthogonal to L_1. This vector v also determines an orientation of the two–dimensional plane spanned by L_1 and $\widehat{v}$.

For $i = 1, 2, \ldots, k$, define $U_i \subset \mathbb{RP}^n \times \mathbb{RP}^n$ as the open set of all pairs of lines (L_1, L_2) in $\mathbb{R}^{n+1}$ such that the vector $v_i(L_1)$ is nonzero and the line L_2 makes an acute angle with the line $\widehat{v_i(L_1)}$. Notice that the sets $U_0, U_1, \ldots, U_k$ cover $\mathbb{RP}^n \times \mathbb{RP}^n$. Indeed, given a pair (L_1, L_2), there exists indices $1 \leqslant i_1 < \cdots < i_n \leqslant k$ such that the vectos $v_{i_j}(L_1)$ $(j = 1, 2, \ldots, n)$, span the tangent space $T_{L_1}(\mathbb{RP}^n)$. Then, the lines:

$$L_1, \widehat{v_{i_1}(L_1)}, \ldots, \widehat{v_{i_n}(L_1)}$$

span $\mathbb{R}^{n+1}$ and therefore the line L_2 makes an acute angle with one of these lines. Hence, (L_1, L_2) belongs to one of the sets $U_0, U_{i_1}, \ldots, U_{i_n}$.

Now, if $(L_1, L_2) \in U_0$, we rotate L_1 towards L_2 with constant velocity in the two–dimensional plane spanned by L_1 and L_2 so that L_1 sweeps the acute angle. This is a continuous motion planning section $s_0 : U_0 \to P(\mathbb{RP}^n)$. Our continuous motion planning strategy $s_i : U_i \to P(\mathbb{RP}^n)$ where $i = 1, 2, \ldots, k$ is a composition of two motions. First we rotate line L_1 toward the line $\widehat{v_i(L_1)}$ in the two–dimensional plane spanned by both lines in the direction determined by the orientation of this plane. On the second step, we rotate the line $\widehat{v_i(L_1)}$ towards L_2 along the acute angle similarly to the action of s_0.

We found $k + 1$ continuous motion planning strategies s_i over each U_i, which proves the statement. $\square$

[1] Actually, Hirsch's Theorem on immersions of manifolds [11] asserts that an n-dimensional manifold M admits an immersion in $\mathbb{R}^k$ *precisely* when the tangent bundle τ_M admits a k-dimensional complement (realized as the normal bundle of the immersion).

Remark. Notice that **Theorem 5.3.4** can be written as:

$$\mathrm{TC}(\mathbb{RP}^n) \leqslant l \iff \exists \text{ nonsingular map } \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^l$$

By definition, if we have a motion planner for $\mathbb{RP}^n$ given by l rules, it means that:

$$\mathrm{TC}(\mathbb{RP}^n) \leqslant l.$$

Then there exists a non–singular map:

$$\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^l,$$

and by **Lemma 5.2.9** we have an axial map:

$$\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{l-1}$$

which (applying **Theorem 6.2.1**) gives an immersion:

$$\mathbb{RP}^n \looparrowright \mathbb{R}^{l-1}.$$

Hence, starting from a motion planner for $\mathbb{RP}^n$, it is possible to construct an immersion:

$$\mathbb{RP}^n \looparrowright \mathbb{R}^{k-1}.$$

Conclusions

In this work we reviewed the work done by Farber et. al. toward the study of Topological Complexity, offering more detailed explanations than those included in the original works. We also utilized the tools developed by several authors in order to establish results for other possible configuration spaces.

Looking at the future, there exist many configuration spaces for which TC is unknown or there exist some bounds. One of the most remarkable open examples is the **Klein Bottle** (denoted as K), which is the configuration space of a robot having a twist attached to every rotation in two different directions.

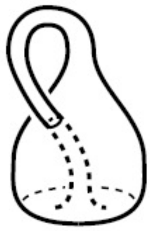

Figure 6.4: Klein Bottle, [10, p. 53]

It is not hard to see that $\mathrm{zcl}(K) = 3$. In fact, any non–orientable surface of genus g (N_g), is such that $\mathrm{zcl}(N_g) = 3$. On the other hand, by (2.1.13), we have $\mathrm{TC}(K) \leqslant 5$. Therefore, the Topological Complexity of **Klein Bottle** is 4 or 5.

This problem has been attacked by many people, even Michael Farber.

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