CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

On parabolic Dirac operators and its applications

PRESENTA

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PARA OBTENER EL GRADO DE MAESTRO EN CIENCIAS

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Aguascalientes, Ags., 04 de Octubre de 2023

UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES

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M. en C. Jorge Martín Alférez Chávez
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TIPO DE TRABAJO: (X) Tesis ( ) Trabajo Práctico

TITULO: ON PARABOLIC DIRAC OPERATORS AND ITS APPLICATIONS

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Acknowledgments

I want to thank my family, especially my parents for all their support, their encouragement and for always being there for me. This thesis and the work carried out during these years is devoted to them, thank you very much.

Also I want to thank my thesis tutor Phd. Briceyda B. Delgado for guiding me during this moment of my life, for giving me the opportunity to receive her advice, knowledge and lessons not only in academic matters but also in life, for giving me the opportunity to work and learn with her and giving me the encouragement to continue studying.

Also I am grateful to UAA for the support received throughout these two years as well as to CONACYT for the economic support given in order to carry out the masters, without which it wouldn't have been impossible.

Aarón Guillén Villalobos.

A mis padres, Celia y Jesús...

Contents

Resumen

En esta tesis se caracterizan las soluciones del operador de Dirac parabolico para el caso 4-dimensional dentro de un dominio acotado Ω. Más precisamente, probaremos que las soluciones del operador de Dirac parabolico $D^{+}_{x,t}$ satisfacen un sistema divergencia rotacional generalizado y que es suficiente que satisfagan este sistema para ser soluciones del operador de Dirac parabolico. Además, demostramos que dadas cualesquiera cuatro soluciones del operador de calor es posible hacer una construcción una solución del operador de Dirac parabolico de 16 componentes, donde cada una de las componentes siguen siendo soluciones del operador de calor. Por último, mostramos que las soluciones de dicho operador de Dirac parabólico heredan algunas propiedades del operador de calor y que además si consideramos un problema con valor en la frontera, entonces se tiene unicidad de las solución.

Abstract

In this thesis, we characterize the solutions of the parabolic Dirac operator for the case 4-dimensional for a bounded domain Ω. More precisely, we prove that the solutions of the parabolic Dirac operator $D^{+}_{x,t}$ satisfy a generalized div-rot system. Also, we will prove that given any four solutions of the heat operator, it is possible to construct solutions to the parabolic Dirac operator with 16 components, where each component function is a solution to the heat operator. Lastly, we will show that the solutions of the parabolic Dirac operator inherit some properties of the solutions of the heat operator and if we consider an initial value problem, then there is uniqueness of the solutions.

Introduction

In 1935, in [1] R. Fueter proposed a definition for monogenic functions defined on the quaternions, which has analogous properties to the Cauchy-Riemann equations in complex variable, some properties that this definition shares with the definition of the Cauchy-Riemann equations is the Cauchy's integral formula and the Laurent expansion [2]. Fueter's definition of a regular function followed the same idea as the Cauchy-Riemann equation for a complex function $f : \mathbb{C} \to \mathbb{C}$ which is

$$\overline{\partial} f = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0. \tag{1}$$

For the quaternionic case, that is, for functions defined in $\mathbb{H}$ and taking values in $\mathbb{H}$, $f : \mathbb{H} \to \mathbb{H}$, the Cauchy-Riemann-Fueter equations where defined as follows

$$\overline{\partial}_l f = \frac{\partial f}{\partial t} + \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 0, \tag{2}$$

and the functions belonging to the kernel of the operator $\overline{\partial}_l$ are usually called monogenic, regular, or hyper-analytic functions. Some of the most important properties are that if f is a quaternion-valued function that satisfies the Cauchy-Riemann-Fueter equations and is continuously differentiable, then for any 3-closed manifold M the integral over the frontier of the manifold M is equal to zero.

In 1979, A. Sudbery generalized the harmonic conjugation procedure of complex analysis in [3]. More precisely, if we have a scalar harmonic function $u : \mathbb{H} \to \mathbb{R}$ then there exists a quaternionic function $f : \mathbb{H} \to \mathbb{H}$ such that the scalar part of f coincides with the function u. The monogenic or hyper-analytic function is given by

$$f(q) = u(q) + \text{Vec}\,\left(\int_0^1 s^2\, \overline{\partial}_l u(sq)\, q\, ds\right), \tag{3}$$

where $\overline{\partial}_l$ is the operator defined in (2) and $q = t + x\hat{\imath} + y\hat{\jmath} + z\hat{k} \in \mathbb{H}$.

In this work, we are not going to consider domains in $\mathbb{H}$, we will work with 3-dimensional bounded domains. In fact, we will identify $\mathbb{R}^3$ with the vector part of $\mathbb{H}$ and we will work with the Dirac operator

$$D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3. \tag{4}$$

More recently, in [4] the monogenic completion process given by Sudbery was modified for the construct monogenic functions defined in $\mathbb{R}^3$ and taking values in $\mathbb{H}$ to give an explicit solution of the div-curl system, see Subsection 2.3. In this work, we are going to analyze the *parabolic Dirac operator*

$$D^+_{x,t} = D + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger, \tag{5}$$

where $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ are nilpotent vectors in the Clifford algebra $C\ell_{1,4}$. In this work, we will use some techniques of quaternionic analysis to construct solutions in the kernel of the parabolic Dirac operator when we know 4 scalar solution of the heat equation, see Theorem 5.1.2.

In the last century, there have been many advances in the area of quaternionic analysis as well as its applications in Mathematical Physics. We cannot mention them all, but we refer the reader to the following monographs [5–7].

The most famous factorization in Clifford analysis is the one of the Laplacian

$$-\Delta = D^2,$$

which tells us that the monogenic functions are harmonic component-wise. Over the years, other factorizations have been developed and have been the key to the analysis of different partial differential equations in Mathematical Physics. For instance, the factorization of the Schrödinger operator, the Helmholtz operator, and the heat operator. Next, we will see them in detail.

Let ν be a continuous complex-valued function, we define the Schrödinger operator as $(-\Delta + \nu)$, so this operator can be factorized as follows

$$(-\Delta + \nu) = (D + M^{\frac{Df}{f}})(D - M^{\frac{Df}{f}}), \tag{6}$$

where M^γ is the multiplication operator define as $M^\gamma z = z\gamma$ and the potential of the Schrödinger equation is $\nu = \frac{\Delta f}{f}$.

The persons who were first in a factorization of this operator were Swanhild Berstein and Klaus Gürlebeck [8, 9], they established a factorization of the Shcrödinger operator

in terms of the operators $D + M^{\gamma}$ in a way that requires a solution of an associated Riccati equation in the three-dimensional case. Later, in 2001 Kravchenko showed that the factorization could be reduced as it is shown in (6), see [10]. In the case of the Helmholtz operator, we can factor it as

$$(\Delta + \lambda^2) = -(D + \lambda)(D - \lambda) = -(D - \lambda)(D + \lambda), \tag{7}$$

where λ is a complex number, non-zero. Some applications of this factorization (7) can be seen in [11], [12], [13]. In the articles [14, 15] the authors use the above factorization 7 to find explicit solutions of the in-homogeneous equation (curl $+ \lambda)\vec{w} = \vec{g}$, where $\vec{g}$ is an integrable function whose divergence is also integrable.

In this thesis, we are interested in the analysis of the solution of the heat equation as well as in some factorizations of the heat operator. A factorization of the heat operator using fractional derivatives is as follows

$$-\Delta + \partial_t = \left(D + e_+ \partial_t^{1/2}\right)^2, \tag{8}$$

or it can be factorized as follows

$$-\Delta + \partial_t = (D + i\partial_t^{1/2})(D - i\partial_t^{1/2}). \tag{9}$$

where D is the Dirac operator, e_+ is an element of a Clifford algebra such that $e_+^2 = 1$, and in the second case i is an imaginary unit of -1, and the $\partial_t^{1/2}$ is taken in the sense of the Caputo fractional derivative, which was introduced by Michele Caputo in 1967 in [16]:

$$\partial_t^{1/2} F(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{F'(s)}{(t-s)^{1/2}} ds.$$

Precisely to avoid the of fractional operators in the factorization of the heat operator like in (8) and (9) we will use the tools provided by Clifford analysis.

Levy-Leblond was one of the first to investigate the factorization of this class of differential operators. In 1967, she was interested in the Schrödinger operator to find analogs of the Dirac operator in 4-dimensional space [17].

Later, in 2005 in the works of Cerejeiras, Kähler and Sommen [18] the authors showed some properties of the parabolic Dirac operator and give the following factorization

$$-\Delta + \partial_t = (D + \mathfrak{f}\partial_t + \mathfrak{f}^{\dagger})^2. \tag{10}$$

It is worth mentioning that the above-mentioned factorization (10) is the key for the present thesis. Among the important results obtained by the authors, we can mention an

orthogonal decomposition for the space $L^2(\mathscr{C})$, where $\mathscr{C} = \Omega \times (0,\infty)$ is the space-time domain and Ω is a bounded domain, see [18, Th. 3.3]. The orthogonal decomposition is

$$L^2(\mathscr{C}) = \mathrm{Ker} D^{\pm}_{x,t} \cap L^2(\mathscr{C}) \oplus D^{\mp}_{x,t} W^{1,2}_{0,\infty}(\mathscr{C}), \tag{11}$$

where $W^{1,2}_{0,\infty}(\mathscr{C})$ is the Sobolev space with vanishing trace and vanishing at infinity. Nevertheless, we present here a different proof based on the weak definition of solutions of the parabolic Dirac operator, see Theorem 4.2.4.

Also in the works Cerejeiras, Sommen, and Vieira [19] the authors proposed a way of factorizing the heat operator and its relatives. It was proven that if a function $F = F^{[0]} + \mathfrak{f} F^{[1]} + \mathfrak{f}^\dagger F^{[2]} + \mathfrak{f}\mathfrak{f}^\dagger F^{[3]}$ is a solution to the parabolic Dirac operator, where $F^{[i]} \in C\ell_{0,3}$, then it satisfies the following system [19, Th. 4.1]

$$\begin{cases} D_x F^{[0]} = -F^{[1]} \\ D_x F^{[1]} = \partial_t F^{[0]} \\ D_x F^{[2]} = F^{[0]} + F^{[3]} \\ D_x F^{[3]} = F^{[1]} - \partial_t F^{[2]} \end{cases} \tag{12}$$

Nevertheless, the construction of solutions of the parabolic Dirac operator using this system (12) requires to known at least half of data. In other words, they needed to star with 8 known solutions of the heat equation. The main result if this work is provided in Theorem 5.1.1, which characterized the kernel of the parabolic Dirac operator $D^+_{x,t}$ through a generalized div-curl system. In fact, $G = g^0 + g^1 e_+ + g^2 e_- + g^3 e_+ e_- \in \mathrm{Ker} D^+_{x,t}$, where $g^{[i]} \in C\ell_{0,3}$, if and only if

$$\begin{cases} -\mathrm{div}\,(\vec{g_0}) + \dfrac{1}{2}\left(\partial_t g_1 + \partial_t g_2 + g_1 - g_2\right) = 0, \\ \nabla g_0 + \dfrac{1}{2}\left(-\partial_t \vec{g_1} - \partial_t \vec{g_2} - \vec{g_1} + \vec{g_2}\right) = 0, \\ -\mathrm{div}\,(\vec{g_1}) + \dfrac{1}{2}\left(\partial_t g_0 - \partial_t g_3 + g_0 + g_3\right) = 0, \\ \nabla g_1 + \dfrac{1}{2}\left(-\partial_t \vec{g_0} + \partial_t \vec{g_3} - \vec{g_0} - \vec{g_3}\right) = 0, \\ -\mathrm{div}\,(\vec{g_2}) + \dfrac{1}{2}\left(-\partial_t g_0 + \partial_t g_3 + g_0 + g_3\right) = 0, \\ \nabla g_2 + \dfrac{1}{2}\left(\partial_t \vec{g_0} - \partial_t \vec{g_3} - \vec{g_0} - \vec{g_3}\right) = 0, \\ -\mathrm{div}\,(\vec{g_3}) + \dfrac{1}{2}\left(\partial_t g_1 + \partial_t g_2 - g_1 + g_2\right) = 0, \\ \nabla g_3 + \dfrac{1}{2}\left(-\partial_t \vec{g_1} - \partial_t \vec{g_2} + \vec{g_1} - \vec{g_2}\right) = 0. \\ \mathrm{curl}\,(\vec{g_0}) = \mathrm{curl}\,(\vec{g_1}) = \mathrm{curl}\,(\vec{g_2}) = \mathrm{curl}\,(\vec{g_3}) = 0. \end{cases} \tag{13}$$

The main difference between systems (12) and (13) is that in our case we only need to know 4 solutions of the heat equation in order to construct solution of the parabolic Dirac operator, see Theorem 5.1.2.

This thesis is structured as follows, in Chapter 1 we introduce L^p spaces and Sobolev spaces, also we give some basic results of operator theory.

Then in Chapter 2 we introduce the definition of a Clifford Algebra and we will see some examples of classic Clifford algebras. Moreover, we define the Dirac operator and introduce the concept of monogenic functions.

In Chapter 3, we introduce the heat operator and we recall some important features about a fundamental solution to the heat equation. Moreover, we introduce properties of the solutions of the heat operator, such as mean value properties and Cauchy-Schwartz type properties. All this chapter is based on the book [20].

In Chapter 4, we introduce the parabolic Dirac operator $D^{+}_{x,t}$ and we show that it factors the heat operator, this is $-\Delta \pm \partial_t = (D^{\pm}_{x,t})^2$, where the parabolic Dirac operator is defined over $C\ell_{1,4}$-valued functions, then later we define the surface integral over a space-time domain for functions $C\ell_{1,4}$-valued and we prove a relation between the surface integral and the volume integrals. Later, we introduce an integral operator $T^{+}_{x,t}$ which is an inverse of the parabolic Dirac operator up to a constant, see Theorem 4.3.2.

Finally, in Chapter 5 we will show an equivalent system using div-rot systems that are satisfied by the solutions of the parabolic Dirac operator and show a different construction from [19] where we use four solutions instead of the prescribed eight data employed in that article. We conclude this chapter by showing some properties that are inherited from the solutions of the heat equation.

The results obtained in this work were presented in two conferences, in October 2022 in the special session called "Clifford Analysis" at the 55 national congress of the Mexican Mathematical Society and in the International Conference on Clifford Algebras and Their Applications in Mathematical Physics in June 2023.

1. Preliminaries

In this Chapter, we are going to define the standard L^p spaces, C^∞ spaces, C_c^∞ spaces, and the classical Sobolev spaces $W^{1,p}$ as well as some relations between these spaces. Also in Theorem 1.1.1 we will talk about the existence of the trace of functions in $W^{1,p}(\Omega)$.

Later, in Section 1.2 we show some typical properties of linear operators such as continuity, the adjoint of an operator, properties of the adjoint operator, and the orthogonal complement of a closed subset in Hilbert spaces.

1.1 Sobolev spaces

Here $\Omega \subset \mathbb{R}^n$ is an open bounded set unless it is said otherwise, and the elements of $\mathbb{R}^n$ will be denoted by $x = (x_1, x_2, \cdots, x_n)$. Here we are going to define the classical sets for the space of integrable functions and Sobolev spaces.

Definition 1.1.1. Let $f : \Omega \to \mathbb{R}$ be a function, then we say f is infinite differentiable if for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}$

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f,$$

exists, the set of infinitely differentiable functions is denoted by $\mathcal{C}^\infty(\Omega)$, and $\mathcal{C}_c^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support. More precisely, when the following set is compact

$$\operatorname{supp} f = \overline{\{x \in \Omega | f(x) \neq 0\}}.$$

Definition 1.1.2. Let $p \geq 1$. Then we define the following

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \int_\Omega |f(x)|^p dx < \infty \right\}.$$

Then if $p = 2$ we have a natural inner product, which is

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f g \, dx$$

Definition 1.1.3 (Connectivity of a domain [7]). Here are a few definitions of special connected sets.

1. A compact and connected set is called a *continuum*.
2. A domain Ω is called *k-times connected* if its boundary $\partial\Omega$ consists of precisely n disjoint continua. For the *1-time* or *simple connectivity* $\partial\Omega = \emptyset$ is possible. If k is not specified one speaks also of *finite connectivity*.
3. If no k exists as assumed before, the domain is of *infinite connectivity*.

The classical theory of Sobolev spaces can be found in [21, 22]. Here, we will give a summary of some classical results.

Definition 1.1.4. Let k be a nonnegative integer and $p \geq 1$. Then we define the *Sobolev spaces* as follows

$$W^{k,p}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f, \partial^\alpha f \in L^p(\Omega), \alpha \in (\mathbb{Z}^{\geq 0})^{n+1}, |\alpha| \leq k \right\}.$$

It is well-known that $W^{k,p}(\Omega)$ is a Banach space under the norm

$$||f||^p_{W^{k,p}(\Omega)} = ||f||^p_{L^p(\Omega)} + \sum_{j=1}^{k} \sum_{|\alpha|=j} ||\partial^\alpha f||^p_{L^p(\Omega)}. \tag{1.1}$$

In particular, when $k = 1$, the norm (1.1) reduces to

$$||f||^p_{W^{1,p}(\Omega)} = ||f||^p_{L^p(\Omega)} + \sum_{k=0}^{n} ||\partial_k f||^p_{L^p(\Omega)}. \tag{1.2}$$

Then if $p = 2$ we have that $W^{1,2}(\Omega)$ is a Hilber space endowed with the natural inner product

$$\langle f, g \rangle_{W^{1,2}(\Omega)} = \langle f, g \rangle_{L^2(\Omega)} + \sum_{k=1}^{n} \langle \partial_k f, \partial_k g \rangle_{L^2(\Omega)}.$$

Definition 1.1.5. Let Ω be a domain of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. Then Ω is called a Lipschitz domain if for every point $\vec{p} \in \partial\Omega$ there exists a hyperplane H of dimension $n-1$ through $\vec{p}$, a Lipschitz-continuous function $g : H \to \mathbb{R}$ over that hyperplane, and real numbers $r, h > 0$ such that

$$\Omega \cap C = \{\vec{x} + \alpha\vec{\mathbf{n}} \,|\, \vec{x} \in B_r(\vec{p}) \cap H, -h < \alpha < g(\vec{x})\},$$
$$(\partial\Omega) \cap C = \{\vec{x} + \alpha\vec{\mathbf{n}} \,|\, \vec{x} \in B_r(\vec{p}) \cap H, g(\vec{x}) = \alpha\},$$

where

- $\vec{\mathbf{n}}$ is a unit vector that is normal to H.

- $B_r(\vec{p}) := \{\vec{x} \in \mathbb{R}^n \big| \, ||\vec{x} - \vec{p}|| < r\}$ is the open ball of radius r.
- $C := \{\vec{x} + \alpha\vec{\mathbf{n}} \big| x \in B_r(\vec{p}) \cap H, -h < \alpha < h\}$.

Now with these definitions, we can define the following set using the Trace theorem which establishes the existence of the boundary values of functions in $W^{1,p}(\Omega)$.

Theorem 1.1.1. [20, p. 258] (**Trace theorem**) Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator:

$$\mathrm{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega),$$

such that

$$\mathrm{tr}\, u = u\big|_{\partial\Omega}, \quad \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}),$$

and

$$||\mathrm{tr}\, u||_{L^p(\partial\Omega)} \leq C \, ||u||_{W^{1,p}(\Omega)} \,,$$

for each $u \in W^{1,p}(\Omega)$, with the constant C only depending on p and Ω.

The importance of this theorem is that allows us to define the boundary values of functions in the Sobolev space $W^{1,p}(\Omega)$.

Definition 1.1.6. Let us define

$$W_0^{k,p}(\Omega) = \Big\{ f \in W^{k,p}(\Omega) \big| \, \mathrm{tr}(f) = 0 \Big\} \,,$$

where tr is the trace operator.

We also have some obvious contentions such as

$$\mathcal{C}_c^\infty(\Omega) \subset W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega) \subset L^p(\Omega).$$

Also, we have the following propositions.

Proposition 1.1.2. [22, p. 64] The set $\mathcal{C}_c^\infty(\Omega)$ with the metric,

$$||f||_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{\frac{1}{p}},$$

is dense on $L^p(\Omega)$.

Proposition 1.1.3. [22, p. 149] The set $\mathcal{C}_c^\infty(\Omega)$ with the metric,

$$||f||^p_{W^{1,p}(\Omega)} = ||f||^p_{L^p(\Omega)} + \sum_{k=1}^{n} ||\partial_k f||^p_{L^p(\Omega)} \,.$$

is dense on $W_0^{1,p}(\Omega)$.

1.2 Operators

In this section, we are going to talk about Banach spaces and linear operators acting on Banach spaces. This section is based on the definitions presented in [23].

Definition 1.2.1. If $(V, ||\cdot||)$ is a complete normed vector space, then we say V is a Banach space.

Is obvious that any closed subspace of a Banach space is a Banach space. Some examples of Banach spaces are the following:

- $(\mathbb{R}^n, ||\cdot||)$.
- $L^1(\mathbb{R})$, the set of real-valued integrable functions, with the norm

$$\int_{\mathbb{R}} |f(x)|dx.$$

- $L^p(\mathbb{R}^n)$, the set of complex-valued integrable functions, with the norm

$$\int_{\mathbb{R}^n} |f(x)|^p dx.$$

The proof that $L^p(\mathbb{R}^n)$ is a Banach space can be found in [22].

Now we are going to define a linear operator between normed vector spaces.

Definition 1.2.2. Let $(V, ||\cdot||)$ and $(W, ||\cdot||')$ be normed vector spaces over a field $\mathbb{K}$. A function $T : V \to W$ is called linear if satisfies

- $T(x + y) = T(x) + T(y)$.
- $T(\alpha x) = \alpha T(x)$.

where $x, y \in V$ and $\alpha \in \mathbb{K}$.

If $T : V \to \mathbb{K}$ is a linear operator, we say T is a functional. Also, we say that a linear operator $T : V \to W$ is an isometry if $||T(x)||' = ||x||$ for all $x \in V$.

Some properties of the linear operators are the following:

Proposition 1.2.1. ([24, Prop. 5.45]) Let $(V, ||\cdot||)$ and $(W, ||\cdot||')$ be normed vector spaces over a field $\mathbb{K}$. Let $T : V \to W$ be a linear operator, then the following is equivalent:

- T is continuous.
- T is bounded.
- If $\{x_n\} \subset V$ is such that $x_n \to 0$ as $n \to \infty$, then $T(x_n) \to 0$ as $n \to \infty$.

Consequently, we can define the norm of continuous operators as follows

Definition 1.2.3. Let $(V, ||\cdot||)$ and $(W, ||\cdot||')$ be normed vector spaces over a field $\mathbb{K}$. Let $T : V \to W$ be a continuous linear operator. We define the norm of T as follows:

$$||T|| = \inf\{C : ||Tx||' \leq C \, ||x|| \, , x \in V\}.$$

This norm has also the following property for composition. More precisely, if $T : (V_1, ||\cdot||_1) \to (V_2, ||\cdot||_2)$ and $R : (V_2, ||\cdot||_2) \to (V_3, ||\cdot||_3)$, then

$$||T \circ R|| \leq ||T|| \, ||R|| \, .$$

We denote the set of continuous linear operators from V to W as $\mathcal{B}(V, W)$. If $W = V$ then we simply write $\mathcal{B}(V)$. If $W = \mathbb{K}$ then we simply write $\mathcal{B}(V, \mathbb{K}) = V^*$ and we call it the dual space of V.

We also have the following property of Banach spaces:

Proposition 1.2.2. ([24, Prop. 4.16]) Let $(V, ||\cdot||)$ and $(W, ||\cdot||')$ be normed vector spaces over a field $\mathbb{K}$. Then $\mathcal{B}(V, W)$ is a Banach space.

Another property is the following:

Proposition 1.2.3. ([24, Prop. 4.46]) Let $(V, ||\cdot||)$ and $(W, ||\cdot||')$ be Banach spaces over a field $\mathbb{K}$. Let $T : V \to W$ be a biyective bounded operator, then T^{-1} is a bounded operator and

$$\left|\left|T^{-1}\right|\right|^{-1} ||x|| \leq ||T(x)||' \leq ||T|| \, ||x|| \, .$$

Now we are going to talk about adjoint operators, for this, we need to recall the concept of an inner product $\langle \cdot, \cdot \rangle$ defined on the vector space V.

Definition 1.2.4. Let V be a vector space over a field $\mathbb{C}$, then V is called a hilbert space if there exists a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
- $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ for all $x, y \in V$ and $\lambda \in \mathbb{C}$.
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$.
- V is a complete metric space with the metric induced by $||x|| = \sqrt{\langle x, x \rangle}$.

So we have the following definition

Definition 1.2.5. Let V be a Banach space with an inner product. Let $T : V \to V$ be a bounded linear operator, then we say that $T^* : V \to V$ is the adjoint operator of T if it satisfies:

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle\,,$$

for all $x, y \in V$.

Some properties of these operators are the following:

Proposition 1.2.4. ([24, Prop. 3.51]) Let V be a Banach space with an inner product. Let $T : V \to V$ be a bounded linear operator, then we have:

- $T^{**} = T$.
- if T has an inverse, then $(T^*)^{-1} = (T^{-1})^*$.
- $(T + S)^* = T^* + S^*$.
- $(\lambda T)^* = \overline{\lambda} T^*$
- $(T \circ S)^* = S^* \circ T^*$.

An important result in functional analysis is the orthogonal decomposition of a vector space between a closed set and their orthogonal set, for example in [25, p. 133] they proved that if M is a closed subspace of a Hilbert space, then the Hilbert space is the direct sum of M and its orthogonal complement $M^\perp$, that is

$$H = M \oplus M^\perp,$$

where $M^\perp = \{g \in H : \langle g, h\rangle = 0, \forall h \in M\}$. For a bounded operator $T\colon H \to H$, we know that $\operatorname{Ker} T$ is always a closed subspace of H. Moreover, the following decomposition holds

$$H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^\perp = \operatorname{Ker} T \oplus \operatorname{Im} T^*. \tag{1.3}$$

For unbounded operators, the above orthogonal decomposition (1.3) is not straightforward, we do not even know if the kernel of unbounded operators is a closed subspace in the Hilbert space H. Later in Proposition 4.2.3 and Theorem 4.2.4, we will prove that the kernel of the unbounded parabolic Dirac operator is closed in L^2 and we will analyze its orthogonal complement.

On the other hand, for unbounded operators between Hilbert spaces, the definition of adjoint operators is similar.

Definition 1.2.6. Let V be a Banach space with the inner product over the field $\mathbb{C}$. Let $T : V \to V$ be a linear operator, with a domain $D(T)$. We define the domain $D(T^*)$ of its adjoint T^* as the set of all $y \in V$ such that there exists $z \in V$ that satisfies:

$$\langle T(x), y\rangle = \langle x, z\rangle \,,$$

for all $x \in D(T)$, and z is uniquely defined, and, by definition $T^*(y) = z$.

2. Background on Clifford Analysis

In this Chapter, we are going to introduce some definitions of Clifford algebras, then we enunciate some typical properties of the Clifford conjugation as well as when the elements in the Clifford algebra have an inverse. In Section 2.2, we introduce the Dirac operator, the class of monogenic functions and, we will show that the Dirac operator factors the Laplacian. Later, in Section 2.3.2 we will explain a reason why we should not define it using the standard definition of limit.

2.1 Clifford algebras

The concept of Clifford algebras was first introduced by H. Graßmann in his paper of 1899, but Graßmann's work remained in his time widely unnoticed and poorly understood.

This concept was popularized by H. Hankel's book (1867) "*Theory of the Complex number systems, in particular of the common imaginary numbers and the Hamiltonian quaternions with their geometrical representation*". This influenced the development of several theories such as tensor calculus, vector analysis, and Clifford analysis.

In modern notation Graßmann required his elementary quantities $e_1, \ldots, e_n$ to satisfy the following algebraic properties:

$$e_i e_j + e_j e_i = 0, \quad e_i^2 = 0 \qquad \forall i, j = 1, 2, \ldots, n,$$

$$e_i(e_j + e_k) = e_i e_j + e_i e_k.$$

The ideas of Graßmann and Hamilton's quaternions led W.K. Clifford in 1876-1878 to a structure of a "*geometrical algebra*", as Clifford called it. In his famous work published in 1878, called "*Applications of Graßmann's Extensive Algebra*", he built a new algebra, made of scalars, vectors, and general k-vectors, the elements of which are today called "*Clifford numbers*".

Another significant mathematician of his time was the German, Rudolf Lipschitz (1832-1903), who discovered again the "*geometric algebra*" in 1880 [26] while studying sums of squares. Also, K. Theodor Vahlen (1869-1945) introduced in 1902 [27] a multiplication rule between two basis elements of a Clifford algebra.

Definition 2.1.1. Let $\mathbb{R}^{n+1}$ be Euclidean space with basis $\{e_0, e_1, \ldots, e_n\}$. For multiplication let us consider the following rules. Let $p \in \{0, \ldots, n\}$, $q = n - p$, define

$$\begin{aligned} e_0 e_i = e_i e_0 = e_i, \qquad & i = 1, \ldots, n, \\ e_i e_j = -e_j e_i, \qquad i \neq j, \quad & i, j = 1, \ldots, n, \\ e_0^2 = e_1^2 = \cdots = e_p^2 = 1, \quad & e_{p+1}^2 = \cdots = e_{p+q}^2 = -1. \end{aligned} \tag{2.1}$$

Thus, we obtain a basis of an algebra $\mathcal{A} =: C\ell_{p,q}$:

$$\{e_0 = 1, e_A = e_{i_1} e_{i_2} \cdots e_{i_k} : A = \{i_1, i_2, \ldots, i_k\}, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Addition and multiplication by a real number are defined coordinate-wise. Further, let the condition holds

$$e_1 e_2 \cdots e_n \neq \pm 1, \text{ if } p - q \equiv 1 \mod 4.$$

Notice that $C\ell_{p,q}$ is a vector space with dimension 2^n over the field $\mathbb{R}$. We are interested in the Clifford algebras $C\ell_{0,2}$ and $C\ell_{1,4}$, we are going to see them more in detail later. First, we are going to see some examples of Clifford Algebras that are normally used, like the algebra of the complex numbers and the quaternions

Example 2.1.1 (Algebra of the complex numbers). Let $n = 1$ and $p = 0$. Then we have that the vector space $C\ell_{0,1}$ is generated by $\{1, e_1\}$ where $e_1^2 = -1$. Therefore $C\ell_{0,1}$ is isomorphic to the set of the complex numbers $\mathbb{C}$.

Example 2.1.2 (Algebra of the quaternions). Let $n = 2$ and $p = 0$. We have the basis $\{1, e_1, e_2, e_1 e_2\}$ which correspond to the basis of the quaternions $\mathbb{H}$.

Example 2.1.3 (Algebra $C\ell_{0,3}$). Let us now take $n = 3$ and $p = 0$. Then $C\ell_{0,3}$ becomes the algebra generated by the basis

$$\{1, e_1, e_2, e_3, e_1 e_2, e_1 e_3, e_2 e_3, e_1 e_2 e_3\},$$

where

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

and we have that the vector $e_1e_2e_3$ is a pseudoscalar, which means

$$(e_1e_2e_3)^2 = 1.$$

Here we introduce a definition that allows us to write the elements of the Clifford algebra as a linear combination of projections.

Definition 2.1.2. The following vector space

$$C\ell_{p,q}^k := \text{Span}\left\{e_A : |A| = k\right\},$$

is the space of the k-vectors over the field $\mathbb{R}$.

Remark 2.1.4. Notice that $C\ell_{p,q}^k$ is a vector space of dimension $\binom{n}{k}$. Therefore using the Dimension Theorem [28, p. 50] we can define the following linear projections over the Clifford algebra. Let $[\cdot]_k : C\ell_{p,q} \to C\ell_{p,q}^k$ be as follows

$$[x]_k = \sum_{|A|=k} x_A e_A.$$

Therefore every vector in a Clifford algebra can be written in the following way

$$x = \sum_{k=0}^{n} [x]_k.$$

The elements of the form $x = [x]_0$ are called *scalars*, the elements of the form $x = [x]_1$ are called *vectors*, the elements of the form $x = [x]_0 + [x]_1$ are called *paravectors*, and finally every other elements in the Clifford algebra are called *non-paravectors*.

Definition 2.1.3. Let x be an element of $C\ell_{p,q}$, then we define its conjugated as follows

$$\overline{x} = \sum_A x_A \overline{e_A},$$

where

$$\overline{e_A} = (-1)^{\frac{1}{2}|A|(|A|+1)} e_A,$$

and $|A|$ means the cardinality of the set A.

Although we have a way to conjugate elements in $C\ell_{p,q}$ it is not true that $x \cdot \overline{x}$ is a real number, but we can still define the modulus for an element in $C\ell_{p,q}$ by

$$|x| := \left(\sum_{A \in \mathscr{P}_n} x_A^2 \right)^{1/2}.$$

This is because we can regard the Clifford algebra as an Euclidean space of dimension 2^n with Euclidean metric.

Some relations that hold are the following

Proposition 2.1.5. [7, Prop. 3.11] Let $x \in C\ell_{p,q}, \lambda \in \mathbb{R}$ and let $Sc(x) = [x]_0 = x_0$. Then we have the following:

1. In case x is a paravector $Sc(x) = \frac{1}{2}(x + \overline{x})$.
2. $\overline{x + y} = \overline{x} + \overline{y}$.
3. $\overline{\overline{x}} = x$.
4. $\overline{xy} = \overline{yx}$.
5. $|\overline{x}| = |-x| = |x|$.
6. $|\lambda x| = |\lambda||x|$.
7. $||x| - |y|| \leq |x - y| \leq |x| + |y|$.

The proofs are analogous to those in $\mathbb{C}$, they can also be found in [7, Prop. 3.11].

Something unfortunate is the existence of the inverse of the multiplication, which in $C\ell_{p,q}$ is not always given, as seen in the following proposition.

Proposition 2.1.6. [7, Prop. 3.12] The following statements hold

(1) For $p > 0$ or for $p = 0$ and $q = n \geq 3, C\ell_{p,q}$ contains zero divisors.

(2) For $p = 0$ and for all n the paravectors, which are different from zero and only consist of scalars and vectors, have an inverse which is

$$x^{-1} = \frac{\overline{x}}{x\overline{x}}, \qquad x \neq 0.$$

(3) The positive and negative basis elements constitute a group.

An important Clifford algebra is $C\ell_{0,2} \cong \mathbb{H}$, this is the set of quaternions and is one that we are going to use in future sections. Let $x, y \in \mathbb{H}$ here we denote $e_3 = e_1e_2$, then we have a rule for multiplication, but first if

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

$$y = y_0 + y_1e_1 + y_2e_2 + y_3e_3$$

then we denote $Sc(x) = x_0$ and $Vec(x) = x_1e_1 + x_2e_2 + x_3e_3 := \vec{x}$. Then the rule for multiplication is as follows

$$xy = x_0y_0 - \vec{x} \cdot \vec{y} + x_0\vec{y} + y_0\vec{x} + \vec{x} \times \vec{y}$$

where $\vec{x} \cdot \vec{y}$ is the dot product and $\vec{x} \times \vec{y}$ is the cross product in $\mathbb{R}^3$.

Definition 2.1.4. Let $C\ell_{1,4}$ be the Clifford algebra generated by the basis

$$\{1, e_1, e_2, e_3, e_+, e_-\},$$

where

$$e_1^2 = e_2^2 = e_3^2 = e_-^2 = -1, \quad \text{and} \quad e_+^2 = 1.$$

In this way, we can consider the following elements of the algebra $C\ell_{1,4}$

$$\mathfrak{f} = \frac{e_+ - e_-}{2} \qquad , \qquad \mathfrak{f}^\dagger = \frac{e_+ + e_-}{2}.$$

We have that these elements satisfy the following equations

$$\begin{cases} \mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} &= 1, \\ \mathfrak{f}^2 = (\mathfrak{f}^\dagger)^2 &= 0, \\ \mathfrak{f}e_j + e_j\mathfrak{f} &= 0, \\ \mathfrak{f}^\dagger e_j + e_j\mathfrak{f}^\dagger &= 0. \end{cases} \tag{2.2}$$

Note that from these properties, $C\ell_{1,4}$ has zero divisors. Also, we have that

$$\bar{\mathfrak{f}} = \frac{\overline{e_+ - e_-}}{2} = \frac{\overline{e_+} - \overline{e_-}}{2} = \frac{-e_+ + e_-}{2} = -\mathfrak{f},$$

and

$$\overline{\mathfrak{f}^\dagger} = \frac{\overline{e_+ + e_-}}{2} = \frac{\overline{e_+} + \overline{e_-}}{2} = \frac{-e_+ - e_-}{2} = -\mathfrak{f}^\dagger.$$

The relations in (2.2) were key to providing a factorization of the heat operator in Theorem 4.1.1. But before that, we will introduce the functional framework that we are going to use.

2.2 Clifford analysis

Throughout this section $\Omega \subset \mathbb{R}^n$ will be a bounded domain and we will identify $\mathbb{R}^n$ with the vector part of a paravector $x = x_0 + \sum_{i=1}^n x_i e_i$ in $C\ell_{0,n}$. The following operator

$$D = \sum_{i=1}^{n} e_i \partial_i, \tag{2.3}$$

where $\partial_i = \partial/\partial x_i$, $i = 1, 2, \ldots n$ is called the *Moisil-Teodorescu operator*, and acts on functions in $C^1(\Omega)$, and due to the non-commutativity of $C\ell_{0,n}$, we need to specify in which order we are applying the operator.

Definition 2.2.1. Let $u : \Omega \subset \mathbb{R}^n \to C\ell_{0,n}$ then we define

$$Du = \sum_{k=1}^{n} e_k \partial_k u \tag{2.4}$$

and

$$uD = \sum_{k=1}^{n} (\partial_k u) e_k \tag{2.5}$$

An important property of the Dirac Operator is that

$$D^2 = -\Delta. \tag{2.6}$$

Here, we present the proof for completeness:

$$D^2 = \left(\sum_{i=1}^{n} e_i \partial_i \right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i e_j \partial_i \partial_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (e_i e_j + e_j e_i) \partial_i \partial_j = \sum_{i=1}^{n} e_i^2 \partial_i^2 . = -\Delta$$

The more important class of functions analyzed in Clifford analysis is the so-called class of monogenic functions, sometimes also called hyper-holomorphic functions or regular functions, see for instance the monographs [7, 29].

Definition 2.2.2. Let $u : \Omega \to C\ell_{0,n}$, then we say that u is a *left-monogenic* function in Ω if

$$Du = \sum_{i=1}^{n} e_i \partial_i u = 0, \qquad \text{in } \Omega.$$

We also say that u is a *right-monogenic* function if

$$uD = \sum_{i=1}^{n} (\partial_i u) e_i = 0.$$

And if u is left-monogenic, then we write $u \in \mathrm{Ker}(D)$.

An important property of the monogenic functions is that if u is right monogenic, then $\overline{u}$ is left monogenic. This allows us to focus our attention on the set of left monogenic functions.

Here we are going to define some classic operators used for Clifford analysis, these operators were taken from [7, p. 131].

Definition 2.2.3 (**Cauchy-Bitsadze operator**). Let $G \in C^1(\overline{\Omega})$ and $\partial\Omega$ be smooth. Then we define the *Cauchy-Bitsadze operator* as follows

$$(F_{\partial\Omega}G)(x) = \int_{\partial\Omega} E_n(y-x)dy^* f(y),$$

where

$$E_n(x) = \frac{1}{\sigma_n}\frac{\overline{x}}{|x|^n}, \tag{2.7}$$

and $dy^* = \eta dS_y$, where η denotes the outward pointing normal vector to $\partial\Omega$ and σ_n is the surface area of the unit ball in $\mathbb{R}^n$.

Definition 2.2.4 (**Teodorescu transform**). Let $G \in C^1(\overline{\Omega})$ and $\partial\Omega$ be smooth. Then we define the *Teodorescu transform* as follows

$$(T_\Omega G)(x) = -\int_\Omega E_n(y-x)f(y)dy. \tag{2.8}$$

An important property of the Teodorescu transform is that T_Ω is a right inverse of the Dirac operator as in the following Theorem.

Theorem 2.2.1. [7, p. 153] Let u be a continuously differentiable function defined on Ω. Then $T_\Omega[u]$ is also differentiable for all $x \in \Omega$ and

$$\partial_i(T_\Omega[u])(x) = -\frac{1}{\sigma_n}\int_\Omega \partial_{i,x}E_n(y-x)u(y)dy + \overline{e_i}\frac{u(x)}{n}.$$

In particular, we have the identity

$$DT_\Omega[u] = u, \qquad \text{in} \qquad \Omega. \tag{2.9}$$

Later in Section 4.3, we will generalize this result for the parabolic Dirac operator up to a constant. Now with these definitions, we have that there exists an interrelation between these operators. In fact, the relation is called "**Formula of Borel-Pompeiu in** $C\ell_{0,n}$":

Theorem 2.2.2. [7, p. 129] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary and an outwards pointing normal. Then we have for any function $G \in C^1(\overline{\Omega})$,

$$(T_\Omega DG)(x) + (F_{\partial\Omega}G)(x) = \begin{cases} G(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

Here the Teodorescu operator is not a left inverse for the Dirac operator, but it is true if the function is zero on the boundary of Ω.

One property that we get from this theorem is a generalized Cauchy integral formula.

Corollary 2.2.3. [7, p. 131] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary and an outwards pointing normal. Then we have for any function G which is left-monogenic we have

$$(F_{\partial\Omega}G)(x) = G(x),$$

whenever $x \in \Omega$.

Another property that we have as shown in [30] is the following result, see also [7, A.2.23.]

Theorem 2.2.4. (**Gauss's Theorem**)
Let $\Omega \subset \mathbb{R}^n$ be a domain of finite connectivity with sufficiently smooth boundary $\partial\Omega$. Then

$$\int_{\partial\Omega} u dy^* v = \int_\Omega (uD)v + u(Dv)dy,$$

where $dy^* = \eta dy$ and $u, v \in C^1(\overline{\Omega})$

2.3 Quaternionic analysis

In this Section we will restrict ourselves to the three-dimensional case, $\Omega \subset \mathbb{R}^3$ with $\mathbb{R}^3$ the vector part of $\mathbb{H}$. Thus, the Dirac operator reduces to $D = \sum_{i=1}^3 e_i \partial_i$. For quaternionic valued functions usually denoted by $u = u_0 + \vec{u}$, we have the following decomposition rule when applying the Dirac operator

$$Du = -\mathrm{div}\,(\vec{u}) + \nabla u_0 + \mathrm{curl}\,(\vec{u}), \tag{2.10}$$

where div, ∇, curl stands for the classical divergence, gradient and rotational operators of vector calculus. The proof consists of applying the quaternionic multiplication. Also, we have a Leibniz-like formula for the product of two quaternionic functions

$$D[uv] = (D[u])v + (\overline{u})D[v] + 2(\mathrm{Sc}\ uD)[v],$$

where

$$\text{Sc } uD[v] = -\sum_{i=1}^{3} u_i \partial_i v.$$

Observe that if $u : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$, then the quaternionic-valued function $u = u_0 + \vec{u}$ is left monogenic if and only if u satisfies the following div-curl system

$$Du = 0 \iff \begin{array}{rl} \operatorname{div}(\vec{u}) = & 0 \\ \operatorname{curl}(\vec{u}) = & -\nabla u_0. \end{array}$$

If we have a harmonic function u_0 and we want to construct a function w monogenic such that the scalar part of w be exactly u_0 we can consider the following operator, called *monogenic completion operator* in [4]

$$U_\Omega[u_0](x) = \int_0^1 tx \times \nabla u_0(tx) dt, \tag{2.11}$$

where $x \in \Omega$ and u_0 is harmonic in Ω. The first person who suggested this type of functions was Sudbery in [3].

To ensure that this operator is well-defined we require that the domain Ω is star-shaped with respect to the origin.

Definition 2.3.1. Let $\Omega \subset \mathbb{R}^3$ is star shaped with respect to $y \in \Omega$ if for all $x \in \Omega$ and for all $\lambda \in [0, 1]$, the vector $\lambda x + (1 - \lambda)y$ belongs to Ω.

Now we consider the function $w = u_0 + U_\Omega[u_0]$, it is a way to get a monogenic function that we were looking for. This function is unique up to the gradient of a harmonic function. That is,

Proposition 2.3.1. [4, Prop. 2.3] If h is a harmonic in Ω, then

$$\tilde{w} = u_0 + U_\Omega[u_0] + \nabla h = u_0 + \int_0^1 tx \times \nabla u_0(tx) dt + \nabla h \tag{2.12}$$

where h is an arbitrary monogenic function is monogenic in Ω with scalar part equal to u_0.

Now we are going to show an example of the monogenic completion process.

Example 2.3.2. Let $u_0 = x_1^2 - 2x_2^2 + x_3^2$ be a harmonic function, then we have that the monogenic part $\vec{u}$ need to satisfy the following:

$$D(u_0 + \vec{u}) = 0 \iff \begin{array}{rl} \operatorname{div}(\vec{u}) = & 0 \\ \operatorname{curl}(\vec{u}) = & (2x_1, -4x_2, 2x_3), \end{array}$$

so a solution is

$$\begin{aligned} U_\Omega[u_0] &= \int_0^1 t\vec{x} \times \nabla u_0(t\vec{x})dt \\ &= \int_0^1 t(x_1, x_2, x_3) \times (2tx_1, -4tx_2, 2tx_3)dt \\ &= \int_0^1 (6t^2x_2x_3, 0, -6t^2x_1x_2)dt \\ &= (2x_2x_3, 0, -2x_1x_2). \end{aligned}$$

Therefore, a monogenic function in Ω that has scalar part $u_0 = x_1^2 - 2x_2^2 + x_3^2$ is

$$u = (x_1^2 - 2x_2^2 + x_3^2)e_0 + 2x_2x_3e_1 - 2x_2x_3e_3.$$

2.3.1 Div-curl system

So now if we want the following more general div-curl system,

$$\begin{aligned} \operatorname{div}(\vec{w}) &= f_0, \\ \operatorname{curl}(\vec{w}) &= \vec{f}, \end{aligned} \tag{2.13}$$

Following [4, 31], we will decompose the three-dimensional Teodorescu transform previously defined in (2.8) as follows

$$T_\Omega[w_0 + \vec{w}] = T_{0,\Omega}[\vec{w}] + \overrightarrow{T_{1,\Omega}}[w_0] + \overrightarrow{T_{2,\Omega}}[\vec{w}],$$

where

$$\begin{aligned} T_{0,\Omega}[\vec{w}] &= \int_\Omega E_3(\vec{y} - \vec{x}) \cdot \vec{w}(\vec{y})d\vec{y}, \\ \overrightarrow{T_{0,\Omega}}[w_0] &= -\int_\Omega w_0(\vec{y})E_3(\vec{y} - \vec{x})d\vec{y}, \\ \overrightarrow{T_{0,\Omega}}[\vec{w}] &= -\int_\Omega E_3(\vec{y} - \vec{x}) \times \vec{w}(\vec{y})d\vec{y} \end{aligned}$$

where E_3 is the three-dimensional Cauchy kernel, recall (2.7). Notice that $T_{0,\Omega}[\vec{w}]$ is harmonic if and only if $\operatorname{div}(\vec{w}) = 0$ [4, Prop. 3.1], this fact was the key to the application of the monogenic completion process. More precisely, in [4] were used the components operators of the Teodorescu transform as well as the monogenic completion operator applied to $T_{0,\Omega}$ to provide a general solution of the div-curl system (2.13).

Theorem 2.3.3. [4, Th. 4.4] Let Ω be a star-shaped open set, and suppose that $f = f_0 + \vec{f} \in L^p(\Omega)$ and $\operatorname{div}\vec{f} = 0$ in Ω. Then a general weak solution of the div-curl system (2.13) is given by

$$\vec{w} = T_\Omega[f_0 + \vec{f}] - (\operatorname{Sc}(T_\Omega[f_0 + \vec{f}]) + U_\Omega[\operatorname{Sc}(T_\Omega[f_0 + \vec{f}])]) + \nabla h, \tag{2.14}$$

where $h \in \operatorname{Har}(\Omega, \mathbb{R})$ is arbitrary.

2.3.2 Monogenic functions in $\mathbb{H}$

Firstly we are going to focus our attention on the algebra $C\ell_{0,2}$. Now that we have a division algebra $\mathbb{H}$ just as $\mathbb{C}$. Therefore we may want to introduce monogenic functions using the classical difference quotient:

$$[f(x+h) - f(x)]h^{-1},$$

or

$$h^{-1}[f(x+h) - f(x)],$$

with $h \in \mathbb{H}$ and to define all functions which have such a limit for $h \to 0$. Note that when h varies in $\mathbb{C}$ for complex functions, h can approach 0 from all plane directions, and this causes severe restrictions for the existence of the limit, namely the Cauchy Riemann differential equations.

In $\mathbb{H}$, h can have even more freedom, so we have to expect a more restrictive system of differential equations for the existence of a difference quotient.

But in [7] was proved that a definition via difference quotient is too restrictive.

Proposition 2.3.4. [7, p. 96] Let $G \subset \mathbb{H}$ be a domain. Then if $f \in C^1(G)$ with values in $\mathbb{H}$ and if for all points in G the limit

$$\lim_{h \to 0} h^{-1}[f(x+h) - f(x)] =' f(x),$$

exists for every $x \in G$, then in G the function f has the following form:

$$f(x) = a + xb,$$

for some $a, b \in \mathbb{H}$.

Proof. First, we take the derivate in the directions $h_0, h_1e_1, h_2e_2, h_3e_3$ with real $h_i, i = 0, 1, 2, 3$. Then we have

$$'f = \partial_0 f = -e_1\partial_1 f = -e_2\partial_2 f = -e_3\partial_3 f.$$

Now notice that:

$$f(x) = f_0(x) + e_1f_1(x) + e_2f_2(x) + e_3f_3(x) = f_0(x) + e_1f_1(x) + e_2(f_2(x) - e_1f_3(x)),$$

now identifying e_1 with the complex unit i, we can split f by $F_1(x) := f_0(x)+if_1(x), F_2(x) = f_2(x) - if_3(x)$. Then by the previous equalities, we have the following

$$\partial_0(F_1 + e_2F_2) = -i\partial_1 f(F_1 + e_2F_2) = -e_2\partial_2(F_1 + e_2F_2) = -e_3\partial_3(F_1 + e_2F_2),$$

As $1, e_2$ are linearly independent (over the field $\mathbb{C}$). Thus we have the following equations

$$\partial_0 F_1 = -i\partial_1 F_1 = \partial_2 F_2 = i\partial_3 F_2,$$
$$\partial_0 F_2 = i\partial_1 F_2 = -\partial_2 F_1 = i\partial_3 F_1,$$

then

$$(\partial_0 + i\partial_1)F_1 = (\partial_2 - i\partial_3)F_2 = 0,$$
$$(\partial_0 - i\partial_1)F_2 = (\partial_2 + i\partial_3)F_1 = 0.$$

This implies that F_1 is a holomorphic function using the variables $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$, while for F_2 is true for $\overline{z_1}$ and $\overline{z_2}$. We also can see that

$$(\partial_0 - i\partial_1)F_1 = -2i\partial_1 F_1 = 2\partial_2 F_2 = (\partial_2 + i\partial_3)F_2,$$
$$(\partial_2 - i\partial_3)F_1 = -(\partial_0 + i\partial_1)F_2.$$

Then $(\partial_0 - i\partial_1)F_1$ is also holomorphic relative to z_2, so that the mixed derivative $\partial_{\overline{z_2}}\partial_{z_1}F_1$ exists and is continuous. Then we have

$$(\partial_0 - i\partial_1)^2 F_1 = (\partial_0 - i\partial_1)(\partial_2 + i\partial_3)F_2 = (\partial_2 + i\partial_3)(\partial_0 - i\partial_1)F_2 = 0$$

and same with $(\partial_2 - i\partial_3)^2 F_1 = 0$ and $(\partial_0 - i\partial_1)(\partial_2 - i\partial_3)F_1 = 0$. Thus $F_1(z_1 + e_2z_2) = a_0 + a_1z_1 + a_2e_2z_2$, analogously for F_2, so we can express f as follows

$$f(x) = a + \sum_{k=0}^{3} b_k x_k$$

with a, b_k quaternions. From the first expression, we can see that

$$b_0 = -ib_1 = -e_2b_2 = -e_3b_3.$$

Thus, $f(x) = a + xb_0$ as we desired. □

Then a definition of monogenic via the quotient difference is too restrictive, for that reason the definition of monogenic functions is given through the Dirac operator D in $\mathbb{R}^n$, recall Definition 2.2.2.

3. The Heat Equation

In this Chapter, we are going to introduce the heat operator and some properties of the solutions to the heat operator, such as smoothness and mean value type properties, see Theorem 3.3.2 and Theorem 3.3.3. Also, we are going to introduce the fundamental solution of the heat operator $e(x,t)$, and we will see somehow can be used to solve initial valued problems.

3.1 Introduction to the Heat Equation

Let $u : \mathscr{C} \to \mathbb{R}$, and u is continuous twice differentiable, where $\mathscr{C} = \Omega \times (0,\infty)$, and let $g : \overline{\Omega} \to \mathbb{R}$, then u is a solution to the homogeneous heat equation if u satisfies:

$$\partial_t u - \Delta u = 0 \qquad \text{in } \Omega \times \mathbb{R}^+,$$

and we say that u satisfies the initial condition g if

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0) \qquad \text{for all} \quad x_0 \in \overline{\Omega}.$$

Also, we say that u satisfies the non-homogeneous heat equation, with initial condition g if

$$\begin{aligned} \partial_t u - \Delta u &= f \qquad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \lim_{(x,t)\to(x_0,0)} u(x,t) &= g(x_0) \qquad \text{for all} \quad x_0 \in \overline{\Omega}, \end{aligned}$$

here f is a given continuous function.

The following physical interpretation of the heat equations was taken from [20]. Let $V \subset \Omega$ be a smooth domain, then the rate of change within V is the negative of the net flux through ∂V:

$$\partial_t \int_V u dx = -\int_{\partial V} \boldsymbol{F} \cdot \boldsymbol{\nu} dS,$$

where $\boldsymbol{F}$ being the flux density. Thus:

$$\partial_t u = -\mathrm{div}\left(\boldsymbol{F}\right),$$

as V was arbitrary. Normally $\boldsymbol{F}$ is proportional to the gradient of u, but with opposite direction, thus we have:

$$\boldsymbol{F} = \alpha \nabla u,$$

where α is the proportionality constant. Thus we have

$$u_t = -\mathrm{div}\left(\boldsymbol{F}\right) = \alpha \mathrm{div}\left(\nabla u\right) = \alpha \Delta u.$$

Here, without loss of generality we will consider the thermal coefficient $\alpha = 1$. Thus, we will say that u is a solution of the homogeneous *heat equation* if

$$-\Delta u + \partial_t u = 0. \tag{3.1}$$

3.2 The fundamental solution of the heat equation

The importance of knowing an explicit form of the solution to the heat equation is that we can compute solutions to non-homogeneous equations. Let say, we want to solve the following problem

$$\begin{aligned} (-\Delta + \partial_t)u(x,t) = 0 \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = f \qquad x \in \Omega. \end{aligned} \tag{3.2}$$

We can solve (3.2) by using a convolution

$$\tilde{u}(x) = (e * f)(x,t) = \int_{\mathbb{R}^n} e(y-x,t) f(y) dy,$$

so the function $\tilde{u}$ is going to be a solution to the problem, here the function $e(x,t)$ is a fundamental solution of the heat operator $-\Delta + \partial_t$

$$e(x,t) = \frac{\exp\left(\frac{-|x|^2}{4t}\right)}{(4\pi t)^{3/2}}.$$

That is, $(-\Delta + \partial_t)e(x,t) = \delta(x - x_0, t - t_0)$. So given this we have the following:

Proposition 3.2.1. The following function:

$$e(x,t) = \frac{\exp\left(\frac{-|x|^2}{4t}\right)}{(4\pi t)^{3/2}} \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{3.3}$$

is a solution of the heat equation (3.1).

Proof. Notice that the derivative with respect to x_i $i = 1, 2, 3$, is as follows

$$\partial_i^2 e(x,t) = \partial_i \left(\frac{\exp\left(\frac{-|x|^2}{4t}\right)}{(4\pi t)^{3/2}} \cdot \frac{-x_i}{2t} \right) = \frac{\exp\left(\frac{-|x|^2}{4t}\right)}{(4\pi t)^{3/2}} \left(\frac{-1}{2t} + \frac{x_i^2}{4t^2} \right).$$

The derivative with respect to t is as follows:

$$\begin{aligned}
\partial_t e(x,t) &= (4\pi)^{-3/2} \partial_t \left(t^{-3/2} \exp\left(\frac{-|x|^2}{4t} \right) \right) \\
&= (4\pi)^{-3/2} \left[\frac{-3}{2} t^{-5/2} \exp\left(\frac{-|x|^2}{4t} \right) + t^{-3/2} \exp\left(\frac{-|x|^2}{4t} \right) \cdot \frac{|x|^2}{4t^2} \right] \\
&= \frac{\exp\left(\frac{-|x|^2}{4t}\right)}{(4\pi t)^{3/2}} \left(\frac{-3}{2t} + \frac{|x|^2}{4t^2} \right).
\end{aligned}$$

Thus, $(-\Delta + \partial_t) e(x,t) = 0$. □

The fundamental solution e also has the following properties:

Lemma 3.2.2. [20, p. 46] For every $t > 0$,

$$\int_{\mathbb{R}^n} e(x,t) dx = 1.$$

Proof. The proof is straightforward. Indeed,

$$\int_{\mathbb{R}^n} e(x,t) dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz = 1.$$

□

Proposition 3.2.3. The following limits exist for every $t > 0$ and all $x \in \mathbb{R}^n$ such that $x_0 \neq 0$.

$$\lim_{|x| \to \infty} e(x,t) = \lim_{t \to \infty} e(x,t) = \lim_{t \to 0^+} e(x,t) = 0.$$

Proof. The first two limits are easy to prove, we are only going to prove the third one. Let $x_0 \in \mathbb{R}^n$ be fixed and not zero, then we have the following:

$$\frac{\exp\left(-\frac{|x_0|^2}{4t}\right)}{(4\pi t)^{3/2}} = \frac{1}{(\pi)^{3/2}|x_0|^2} \frac{\exp\left(\frac{-1}{u}\right)}{u^{3/2}},$$

where

$$u = \frac{4t}{|x_0|^2}.$$

Now without loss of generality let $0 < u < 1$. Then we have the following:

$$0 < u^2 < u^{3/2} < 1.$$

On the other hand, if $\alpha > 0$, then

$$1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} < e^{\alpha}.$$

Now notice that:

$$\lim_{u\to 0^+} \frac{\exp\left(\frac{-1}{u}\right)}{u^{3/2}} \leq \lim_{u\to 0^+} \frac{\exp\left(\frac{-1}{u}\right)}{u^2} = \lim_{u\to 0^+} \frac{1}{u^2 \exp\left(\frac{1}{u}\right)}.$$

Observe that

$$\underbrace{u^2 + u + \frac{1}{2} + \frac{1}{6u}}_{\xrightarrow[u\to 0^+]{}\infty} < u^2 \exp\left(\frac{1}{u}\right),$$

implies that

$$\lim_{u\to 0^+} \frac{1}{u^2 \exp\left(\frac{1}{u}\right)} = 0.$$

□

Now consider the following homogeneous heat equation problem, with initial value g, where $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty). \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{3.4}$$

We know that $e(x,t)$ solves the homogeneous heat equation (3.4) as well as its translation $e(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Consequently the convolution

$$u(x,t) = \int_{\mathbb{R}^n} e(x-y,t)g(y)dy,$$

also is a solution of (3.4). The proof of this fact can be found in [20], and the result is established as follows:

Theorem 3.2.4. [20, p. 47] Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define

$$u(x,t) = \int_{\mathbb{R}^n} e(x-y,t)g(y)dy,$$

Then

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

2. $u_t - \Delta u = 0$ on $x \in \mathbb{R}^n$ and $t > 0$,

   and

3. $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$ for every point $x_0 \in \mathbb{R}^n$

3.3 Mean Value Formula

We should recall that a very important result for harmonic functions is the mean value property, which says that if $u : \Omega \to \mathbb{R}$ is a harmonic function, and $\Omega \subset \mathbb{R}^n$ open, then for every $x \in \Omega$ and every $r > 0$ such that $B(x,r) \subset \Omega$, then

$$u(x) = \frac{1}{\sigma_n r^n} \int_{B(x,r)} u(x)dx = \frac{1}{n\sigma_n r^{n-1}} \int_{\partial B(x,r)} u(x)dS_x,$$

where σ_n is the volume of the unit ball in n dimensions.

Then for this Subsection, we are going to show some similar properties for the solutions to the heat operator, these results were taken from [32].

Definition 3.3.1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $T > 0$. Then

1. The **parabolic cylinder** is the following set

$$\Omega_T = \Omega \times (0, T],$$

2. and the **parabolic boundary** of Ω_T is:

$$\Gamma_T = \overline{\Omega_T} - \Omega_T.$$

   which comprises the bottom and vertical sides of Ω_T, but not the top (meaning the set $\Omega \times \{T\}$).

Definition 3.3.2. The **heat ball** at $(x,t) \in \mathbb{R}^{n+1}$ with radius $r > 0$ is

$$Hb(x,t;r) = \left\{(y,s) \in \mathbb{R}^{n+1} : e(x-y,t-s) \geq \frac{1}{r^n}\right\}.$$

Note that the heat ball is a region in space-time and the boundary is the level set

$$e(x-y,t-s) = \frac{1}{r^n},$$

And also notice the following,

$$e(x-y,t-s) \geq \frac{1}{r^n},$$

then

$$r^n \exp\left(\frac{-|x-y|^2}{4(t-s)}\right) \geq (4\pi(t-s))^{n/2},$$

Applying the logarithmic we obtain

$$n\ln(r) - \frac{|x-y|^2}{4(t-s)} \geq \frac{n}{2}\ln(4\pi(t-s)),$$

therefore

$$|x-y|^2 \leq 2n(t-s)\ln\left(\frac{r^2}{4\pi(t-s)}\right),$$

One verifies that the above inequality is zero if

$$s = t - \frac{r^2}{4\pi} \qquad \text{or} \qquad s = t.$$

Now, let us consider the following function

$$\psi(x,t) = -\frac{n}{2}\ln(4\pi t) - \frac{|x|^2}{4t} + n\ln(r),$$

this function is going to be useful to prove Theorem 3.3.2.

Lemma 3.3.1. Let $Hb(1) = Hb((0,0);1) \subset \mathbb{R}^3 \times \mathbb{R}$, then

$$\iint_{Hb(1)} \frac{|y|^2}{s^2} dyds = 4$$

Proof. Note the following:

$$\begin{aligned}\int_{|y|^2 \leq -6s\ln\left(\frac{-1}{4\pi s}\right)} \frac{|y|^2}{s^2} dy &= \int_0^{\left(6s\ln\left(\frac{1}{4\pi s}\right)\right)^{1/2}} \int_0^{\pi} \int_0^{2\pi} r^4 \sin(\theta) d\vartheta d\theta dr \\ &= 4\pi \int_0^{\left(6s\ln\left(\frac{1}{4\pi s}\right)\right)^{1/2}} r^4 dr \\ &= \frac{4\pi}{5}\left[6s\ln\left(\frac{1}{4\pi s}\right)\right]^{5/2}\end{aligned}$$

Now, we are going to compute the integral over the unitary ball centered at the origin,

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2} dyds &= \int_{-\frac{1}{4\pi}}^{0} \int_{|y|^2 \leq -6s\ln\left(\frac{-1}{4\pi s}\right)} \frac{|y|^2}{s^2} dyds \\
&= \int_{0}^{\frac{1}{4\pi}} \int_{|y|^2 \leq 6s\ln\left(\frac{1}{4\pi s}\right)} \frac{|y|^2}{s^2} dyds \\
&= \frac{4\pi}{5} \int_{0}^{\frac{1}{4\pi}} \frac{(-6s\ln(4\pi s))^{5/2}}{s^2} ds \\
&= \frac{4\pi}{5} 6^{5/2} \int_{0}^{\frac{1}{4\pi}} \sqrt{s}(ln(4\pi s))^{5/2} ds \\
&= \frac{4\pi}{5} 6^{5/2} \int_{0}^{1} \sqrt{\frac{u}{4\pi}}(ln(u))^{5/2} \frac{1}{4\pi} du \\
&= \frac{6^{5/2}}{5\sqrt{4\pi}} \cdot \frac{5\sqrt{2\pi}}{9\sqrt{3}} \\
&= 4.
\end{aligned}$$

□

Before proceeding with the following theorem we are going to define the following set

$$C^{2,1}(E((x,t);r)) = \{u : E((x,t);r) \to \mathbb{R} : u, \partial_i\partial_j u, \partial_t u \in C(E((x,t);r)), \forall i,j\}.$$

So the set $C^{2,1}(E((x,t);r))$ is the set of functions, that have a second continuous spacial derivative and a continuous time derivative.

Theorem 3.3.2. For $(x_0,t_0) \in \mathbb{R}^3 \times \mathbb{R}, r > 0$, let $u \in C^{2,1}(Hb((x_0,t_0);r) \cap C^1(\overline{Hb((x_0,t_0);r)})$ solve the heat equation in $Hb((x_0,t_0);r)$. Then

$$u(x,t) = \frac{1}{4r^3} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds.$$

Proof. By dilation and translation, without loss of generality, we can assume that $x_0 = 0, t_0 = 0, r = 1$. Thus consider the following function

$$\phi(r) = \frac{1}{r^3} \int_{E_r} u(y,s) \frac{|y|^2}{s^2} dyds = \int_{E_1} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds,$$

which derivative is as follows:

$$\begin{aligned}
\phi'(r) &= \int_{E_1} \left[\sum_{i=1}^{3} y_i u_{y_i}(ry, r^2 s) + 2s u_s(ry, r^2 s) \right] \frac{|y|^2}{s^2} dyds \\
&= \frac{1}{r^3} \int_{E_r} \left[\sum_{i=1}^{3} y_i u_{y_i}(y,s) + 2s u_s(y,s) \right] \frac{|y|^2}{s^2} dyds
\end{aligned}$$

and for fixed r consider the following function:

$$\psi(y,s) = \frac{-3}{2}\ln(-4\pi s) + \frac{|y|^2}{4s} + 3\ln(r),$$

where $(y,s) \in \overline{E_r}$. Notice the following:

- $\psi \equiv 0$ on ∂E_r.
- The partial derivatives of ψ are

$$\partial_t \psi(y,s) = \frac{-3}{2s} - \frac{|y|^2}{4s^2},$$
$$\partial_{y_i} \psi(y,s) = \frac{y_i}{2s}.$$

So we have

$$\begin{aligned}\sum_{i=1}^{3} y_i u_{y_i} \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s} &= -\sum_{i=1}^{3} u_{y_i} y_i \left[4\psi_s + \frac{6}{s}\right] + 4u_s \sum_{i=1}^{3} y_i \phi_{y_i} \\ &= 4(u_s(y\cdot\psi) - \psi_s(y\cdot\nabla u) - 3\nabla u\cdot\nabla\psi).\end{aligned}$$

Consider the field $\vec{F} : \overline{E_r} \to \mathbb{R}^n \times \mathbb{R}$ be

$$\vec{F} = (\psi(y_1 u_s - 3u_{y_1}), \psi(y_2 u_s - 3u_{y_2}), \psi(y_3 u_s - 3u_{y_3}), -\psi(y\cdot\nabla u)),$$

then

$$\text{div}\left(\vec{F}\right) = u_s(y\cdot\nabla\psi = +3\psi(u_s - \Delta_y u) - 3\nabla u\cdot\nabla\psi - \psi_s(y\cdot\nabla u)$$

therefore

$$4\text{div}\left(\vec{F}\right) = \sum_{i=1}^{3} y_i u_{y_i} \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s}$$

Applying the Divergence Theorem for the fiel $\vec{F}$ and domain E_r and noting that $\psi \equiv 0$ on ∂E_r we get that

$$\int_{E_r} \text{div}\left(\vec{F}\right)(y,s)dyds = 0$$

this is the same as

$$\phi'(r) = 0$$

and this last equation is valid for every $r > 0$. So

$$\begin{aligned}\phi(r) &= \lim_{r\to 0}\phi(r)\\ &= \lim_{r\to 0}\int_{E_1} u(ry, r^2 s)\frac{|y|^2}{s^2}dyds\\ &= u(0,0)\int_{E_1}\frac{|y|^2}{s^2}dyds\\ &= 4u(0,0).\end{aligned}$$

□

Theorem 3.3.3. Let $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}, r > 0$ let $u \in C^{2,1}(Hb((x_0,t_0);r) \cap C^1(\overline{Hb((x_0,t_0);r)})$ solve the heat equation in $Hb((x_0,t_0);r)$. Then

$$u(x_0,t_0) = \frac{1}{2r^3}\int_{\partial Hb((x_0,t_0);r)} u(y,s)Q(x_0-y, t_0-s)dS_{y,s},$$

where

$$Q(y,s) = \left(\frac{4s^2}{|y|^2} + \left(1 - \frac{6s}{|y|^2}\right)^2\right)^{-1/2}.$$

Proof. Without loss of generality, we can assume that $x_0 = 0, y_0 = 0, r = 1$. Let $\varepsilon \in (0, \frac{1}{4\pi e})$, the radius of the ball $E_1 \cap s = -\varepsilon$ is given by

$$r_\varepsilon = \left[4\pi\varepsilon \ln\left(\frac{1}{\sqrt{4\pi\varepsilon}}\right)\right].$$

Now denote $K_-(x,t) = \Phi(x,-t) - 1$ and the domain

$$D_\varepsilon = \left\{(x,t) : -\varepsilon > t > \frac{1}{4\pi}, \Phi(x,-t) > 1\right\} \subset Hb((0,0);1).$$

Now notice the following:

$$\begin{aligned}(uK_-)_t + \sum_{i=1}^3 (u(K_-)_{x_i} - K_- u_{x_i})_{x_i} &= u_t K_- + u(K_-)_t + \sum_{i=1}^3 u_{x_i}(K_-)_{x_i} + u(K_-)_{x_i x_i}\\ &\quad + \sum_{i=1}^3 -(K_-)_{x_i}u_{x_i} - K_- u_{x_i x_i}\\ &= K_-(u_t - \Delta u) + u((K_-)_t + \Delta K)\\ &= u((K_-)_t + \Delta K)\\ &= 0.\end{aligned}$$

the last equality is because $\Delta K_- = \Delta\Phi, (K_-)_t = -\Phi_t$ and Φ also solves the heat equation. So consider the following vector field

$$\vec{F} = (u(K_-)_{x_1} - K_- u_{x_1}, u(K_-)_{x_2} - K_- u_{x_2}, u(K_-)_{x_3} - K_- u_{x_3}, uK_-)\,,$$

So applying the Divergence Theorem to $\vec{F}$ over the domain D_ε we get:

$$\int_{\partial D_\varepsilon} (u\nabla K_- - K_-\nabla u)\cdot \nu_y dS_{y,s} + \int_{\partial D_\varepsilon} uK_-\nu_s dS_{y,s} = \int_{D_\varepsilon} \operatorname{div}\left(\vec{F}\right) dyds = 0.$$

where (ν_y, ν_s) is the outer, unit normal vector of ∂D_ε at (y,s). Note that

$$\partial D_\varepsilon = \underbrace{\left\{(x,t) : \frac{-1}{4\pi} \le t \le -\varepsilon, \Phi(x,-t) = 1\right\}}_{:=S_{1,\varepsilon}} \cup \underbrace{\{(x,-\varepsilon) : |x| < r_\varepsilon\}}_{:=S_{2,\varepsilon}},$$

So we have the following:

- When $(y,s) \in S_{1,\varepsilon}$:

$$\nu = \frac{-4s^2}{\sqrt{4|y|^2s^2 + (|y|^2 - 6s)^2}}\left(\frac{y}{2s}, \frac{|y|^2 - 6s}{4s^2}\right)$$

  and

$$K_-(y,s) = 0, \partial_{\nu_y} K_-(y,s) = -Q(y,s).$$

- When $(y,s) \in S_{2,\varepsilon}$:

$$K_-(y,-\varepsilon) = \Phi(y,\varepsilon) - 1, \nu_y = 0, \nu_s = 1$$

So we get

$$\int_{S_{1,\varepsilon}} u(y,s)Q(y,s)dS_{y,s} = \int_{|y|<r_\varepsilon} (\Phi(y,\varepsilon) - 1)u(y,\varepsilon)dy,$$

So when $\varepsilon \to 0^+$ we have:

- $S_{1,\varepsilon} \to \partial E_1$ so

$$\int_{S_{1,\varepsilon}} u(y,s)Q(y,s)dS_{y,s} \to \int_{\partial E_1} u(y,s)Q(y,s)dS_{y,s},$$

- $r_\varepsilon \to 0^+$ so

$$\int_{|y|<r_\varepsilon} u(y,\varepsilon)dy \to 0,$$

- $\frac{r_\varepsilon^2}{\varepsilon} \to +\infty$ so

$$\int_{|y|<r_\varepsilon} \Phi(y,\varepsilon)dy = \frac{1}{\pi^{3/2}} \int_{|\eta| \le \frac{r_\varepsilon}{2\sqrt{\varepsilon}}} e^{-|\eta|^2} d\eta \to 1.$$

Then using these two last equations we have that

$$\int_{|y|<r_\varepsilon} (\Phi(y,\varepsilon) - 1)u(y,\varepsilon)dy \xrightarrow[\varepsilon\to 0]{} u(0,0).$$

□

4. Parabolic integral representations and Hodge decomposition

In Section 4.1 we are going to prove the factorization of the heat operator in terms of the parabolic Dirac operators, this approach comes from the seminal work [18], but we present the proofs here for completeness. Then in section 4.2 we define a way to take the integral over the surface $\mathscr{C}$ and in Theorem 4.2.1 we prove a relation between the surface integral, and the integral over all the space-time domain $\mathscr{C}$, we also find a orthogonal decomposition of the space $L^2(\mathscr{C})$. Then in section 4.3 we define the integral operator $T^+_{x,t}$ and we prove a relation between the operator $T^+_{x,t}$ and the parabolic Dirac operator $D_{x,t}$.

4.1 Factorizations

Now following [18], we are going to introduce the following parabolic Dirac operators, these operators act on functions that have their first partial derivative:

Definition 4.1.1. Let $G : \mathscr{C} \to C\ell_{1,4}$ then

$$D^{\pm}_{x,t}G = \sum_{i=1}^{3} e_i\partial_i G + \mathfrak{f}\partial_t G \pm \mathfrak{f}^{\dagger}G \tag{4.1}$$

and

$$\mathcal{D}^{\pm}_{x,t}G = -DG + \mathfrak{f}^{\dagger}\partial_t G \pm \mathfrak{f}G, \tag{4.2}$$

We have that the operators defined in (4.1) and (4.2) factorize the heat operator $-\Delta + \partial_t$ as well as its adjoint operator $-\Delta - \partial_t$.

Theorem 4.1.1. The operators $D^{\pm}_{x,t}, \mathcal{D}^{\pm}_{x,t}$ satisfies the following:

$$(D^{\pm}_{x,t})^2 = -\Delta \pm \partial_t,$$

$$(\mathcal{D}^{\pm}_{x,t})^2 = -\Delta \pm \partial_t,$$

Proof. Using that $\Delta = -D^2$, we obtain

$$\begin{aligned}(D^+_{x,t})^2 &= (D + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger)(D + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger)\\ &= D^2 + \mathfrak{f}^2\partial_t^2 + (\mathfrak{f}^\dagger)^2 + (D\cdot f)\partial_t + (fD)\partial_t + D\cdot\mathfrak{f}^\dagger + \mathfrak{f}^\dagger D + \mathfrak{f}\mathfrak{f}^\dagger\partial_t + \mathfrak{f}^\dagger\mathfrak{f}\partial_t\\ &= D^2 + \mathfrak{f}^2\partial_t^2 + (\mathfrak{f}^\dagger)^2 - (fD)\partial_t + (fD)\partial_t - \mathfrak{f}^\dagger D + \mathfrak{f}^\dagger D + \mathfrak{f}\mathfrak{f}^\dagger\partial_t + \mathfrak{f}^\dagger\mathfrak{f}\partial_t\\ &= -\Delta + \partial_t,\end{aligned}$$

Now for $D^-_{x,t}$:

$$\begin{aligned}(D^-_{x,t})^2 &= (D + \mathfrak{f}\partial_t - \mathfrak{f}^\dagger)(D + \mathfrak{f}\partial_t - \mathfrak{f}^\dagger)\\ &= D^2 + \mathfrak{f}^2\partial_t^2 + (\mathfrak{f}^\dagger)^2 + (D\cdot f)\partial_t + (fD)\partial_t - D\cdot\mathfrak{f}^\dagger - \mathfrak{f}^\dagger D - \mathfrak{f}\mathfrak{f}^\dagger\partial_t - \mathfrak{f}^\dagger\mathfrak{f}\partial_t\\ &= D^2 + \mathfrak{f}^2\partial_t^2 + (\mathfrak{f}^\dagger)^2 - (fD)\partial_t + (fD)\partial_t - \mathfrak{f}^\dagger D - \mathfrak{f}^\dagger D - \mathfrak{f}\mathfrak{f}^\dagger\partial_t - \mathfrak{f}^\dagger\mathfrak{f}\partial_t\\ &= -\Delta - \partial_t,\end{aligned}$$

And for $\mathcal{D}^\pm_{x,t}$ we do a similar process. □

From now on, we we refer to the parabolic Dirac operator will be the operator defined in (4.1). Here we see that we naturally get the adjoint operator $-\Delta - \partial_t$ of the operator $-\Delta + \partial_t$. But we have to keep in mind that this is only valid if we are restricted to zero trace functions. More precisely, we are going to work in the Sobolev space $W_0^{1,2}$, recall Definition 1.1.6.

Proposition 4.1.2. Let $T > 0$ be fixed and let us define $\Omega_T = \Omega \times (0,T)$. The adjoint operator of the heat operator is the operator $-\Delta - \partial_t$ when we are restricted to the Sobolev space $W_0^{1,2}(\Omega_T)$.

Proof. Let $g, h \in W_0^{1,2}(\Omega_T)$, then $g|_{\partial\Omega} = h|_{\partial\Omega} = 0$ and $g(x,0) = g(x,T) = h(x,0) = h(x,T) = 0$ for every $x \in \Omega$. Moreover,

$$\begin{aligned}\int_{\Omega_T}[(-\Delta+\partial_t)g]h\,dxdt &= \int_0^T\int_\Omega(-\Delta g)h\,dxdt + \int_0^T\int_\Omega(\partial_t g)h\,dxdt\\ &= \int_0^T\int_{\partial\Omega} g\nabla h|_{\partial\Omega}\cdot\eta - h\nabla g|_{\partial\Omega}\cdot\eta dS + \int_0^T\int_\Omega g(-\Delta h)\,dxdt - \int_0^T\int_\Omega g(\partial_t h)\,dxdt\\ &= \int_0^T\int_\Omega g(-\Delta h)\,dxdt - \int_0^T\int_\Omega g(\partial_t h)\,dxdt,\\ &= \int_{\Omega_T} g\left[(-\Delta-\partial_t)h\right]\,dxdt,\end{aligned}$$

where the second step is given by using the second Green's identity. □

Besides, in the case of $T = \infty$ we need to add a further condition to preserve the above result. More precisely, $\Delta - \partial_t$ is an adjoint operator of the heat operator when we are restricted to the subspace

$$W^{1,2}_{0,\infty}(\mathscr{C}) = \left\{ g \in W^{1,2}_0(\mathscr{C}) : \lim_{t\to\infty} g(x,t) = 0, \forall x \in \Omega \right\} \tag{4.3}$$

4.2 Parabolic Hodge Decomposition

Now, we are going to show that the kernel of the parabolic Dirac operator is closed in L^2 and we are going to characterize the orthogonal complement of this kernel. From now on let $\mathscr{C} = \Omega \times (0,\infty)$, and

$$\partial\mathscr{C} = (\partial\Omega \times (0,\infty)) \cup (\Omega \times \{0\}),$$

And we define

$$\int_{\partial\mathscr{C}} g d\sigma_{x,t} h = \int_0^\infty \int_{\partial\Omega} g\eta h dS + \int_\Omega (g\mathfrak{f}h)|_{t=0}^{t=\infty} dx, \tag{4.4}$$

where η is a normal outwards pointing vector of $\partial\Omega$, so the integral over $\partial\mathscr{C}$ is the surface integral defined over the two components of $\partial\mathscr{C}$ plus the evaluation when $t \to \infty$. Then we have the following result:

Theorem 4.2.1. Let $\mathscr{C} = \Omega \times (0,\infty)$, where Ω is a bounded domain with finite connectivity with sufficiently smooth boundary $\partial\Omega$. Let $g, h : \mathscr{C} \to C\ell_{1,4}$ be integrable functions, then we have

$$\int_{\partial\mathscr{C}} g d\sigma_{x,t} h = \int_{\mathscr{C}} (g D^-_{x,t}) h + g(D^+_{x,t} h)\, dxdt, \tag{4.5}$$

where the surface integral was defined in (4.4).

Proof. By Gauss's theorem 2.2.4, we have

$$\int_{\partial\Omega} g d\sigma^* h = \int_\Omega (gD)h + g(Dh) dx,$$

where $d\sigma^* = \eta dx$. Then integration with respect to the variable t gives us:

$$\begin{aligned}
\int_0^\infty \int_{\partial\Omega} g d\sigma^* h &= \int_0^\infty \int_\Omega (gD)h + g(Dh)\, dxdt \\
&= \int_0^\infty \int_\Omega (gD)h + g(D^+_{x,t}h) - g(\partial_t(\mathfrak{f}h)) - g\mathfrak{f}^\dagger h\, dxdt \\
&= \int_0^\infty \int_\Omega (gD)h + g(D^+_{x,t}h) + (\partial_t g)\mathfrak{f}h - \partial_t(g\mathfrak{f}h) - g\mathfrak{f}^\dagger h\, dxdt \\
&= \int_0^\infty \int_\Omega (gD^-_{x,t})h + g(D^+_{x,t}h)\, dxdt - \int_\Omega (g\mathfrak{f}h)|_{t=0}^\infty\, dx
\end{aligned}$$

By (4.4), we conclude that

$$\int_{\partial\mathscr{C}} g d\sigma_{x,t} h = \int_{\mathscr{C}} (g D^-_{x,t})h + g(D^+_{x,t}h) dxdt,$$

as we desired. □

Let us denote by $L^p(\mathscr{C}) := L^p(\mathscr{C}, C\ell_{1,4})$ to the space of L^p integrable functions defined in $\mathscr{C}$ and taking values in $C\ell_{1,4}$, for $1 < p < \infty$. It is well-known that $L^2(\mathscr{C})$ is a right Hilbert module under the scalar product:

$$\langle u, v\rangle_{L^2(\mathscr{C})} = \int_{\mathscr{C}} \overline{u}v = \int_0^\infty \int_\Omega \overline{u}v \, dxdt, \qquad \forall u, v \in L^2(\mathscr{C}). \tag{4.6}$$

Some properties satisfied by this scalar product are the following:

- For all $u, v, w \in L^2(\mathscr{C})$ we have that

$$\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle ,$$

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle .$$

- For all $q \in C\ell_{1,4}$ and $u, v \in L^2(\mathscr{C})$ we have that

$$\langle uq, v\rangle = \overline{q} \langle u, v\rangle ,$$

$$\langle u, vq\rangle = \langle u, v\rangle q,$$

- If $u \in L^2(\mathscr{C})$ is such that

$$\langle u, z\rangle = 0 \qquad \forall z \in L^2(\mathscr{C})$$

then $u \equiv 0$.

Recall the generalization of the adjoint for unbounded operators given in Definition 1.2.6.

Corollary 4.2.2. $D^-_{x,t}$ is the adjoint operator of $D^+_{x,t}$ under the inner product (4.6) when we are restricted to $W^{1,2}_{0,\infty}(\mathscr{C})$.

Proof. The proof is straightforward just taking $h \in L^2(\mathscr{C})$ and $g \in W^{1,2}_{0,\infty}(\mathscr{C})$ in (4.5). □

Proposition 4.2.3. The set $\mathrm{Ker}(D^+_{x,t}) \cap L^2(\mathscr{C})$ is closed in $L^2(\mathscr{C})$.

Proof. Let $\{g_n\} \subset \mathrm{Ker}(D^+_{x,t}) \cap L^2(\mathscr{C})$ be a sequence of functions that converges to a function g in $L^2(\mathscr{C})$, then

$$\langle g_n, \phi\rangle = \int_{\mathscr{C}} \overline{g_n}\phi \xrightarrow[n\to\infty]{} \int_{\mathscr{C}} \overline{g}\phi = \langle g, \phi\rangle, \qquad \forall \phi \in L^2(\mathscr{C}). \tag{4.7}$$

In particular, (4.7) is valid if $\phi \in C^\infty_c(\mathscr{C})$ has compact support. By Theorem 4.2.1, we have

$$\int_{\mathscr{C}} \phi(D^+_{x,t}g_n) + (\phi D^-_{x,t})g_n \, dxdt = 0, \qquad \int_{\mathscr{C}} \phi(D^+_{x,t}g) + (\phi D^-_{x,t})g \, dxdt = 0, \tag{4.8}$$

for all $\phi \in C^\infty_c(\mathscr{C})$. Using that $\overline{C^\infty_c(\mathscr{C})} = W^{1,2}_0(\mathscr{C})$, we have that $D^-_{x,t}\phi \in L^2(\mathscr{C})$ for all $\phi \in C^\infty_c(\mathscr{C})$. Consequently,

$$\lim_{n\to\infty} \int_{\mathscr{C}} (\phi D^-_{x,t})g_n \, dxdt = \int_{\mathscr{C}} (\phi D^-_{x,t})g \, dxdt, \qquad \forall \phi \in C^\infty_c(\mathscr{C}). \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\begin{aligned} \int_{\mathscr{C}} \phi(D^+_{x,t}g)\, dxdt &= -\int_{\mathscr{C}} (\phi D^-_{x,t})g \, dxdt \\ &= \lim_{n\to\infty} -\int_{\mathscr{C}} (\phi D^-_{x,t})g_n \, dxdt \\ &= \lim_{n\to\infty} \int_{\mathscr{C}} \phi(D^+_{x,t}g_n)\, dxdt \\ &= 0, \end{aligned}$$

$\forall \phi \in C^\infty_c(\mathscr{C})$. Therefore, $D^+_{x,t}g = 0$. □

Theorem 4.2.4. The following space has the next decomposition

$$L^2(\mathscr{C}) = \mathrm{Ker}(D^+_{x,t}) \cap L^2(\mathscr{C}) \oplus (D^-_{x,t})W^{1,2}_{0,\infty}(\mathscr{C}).$$

Proof. Recall the definition of the subspace $W^{1,2}_{0,\infty}(\mathscr{C})$ given previously in (4.3). Let $g \in W^{1,2}_{0,\infty}(\mathscr{C})$ and $h \in \mathrm{Ker}(D^+_{x,t}) \cap L^2(\mathscr{C})$. Observe that

$$\begin{aligned} \Big\langle h, (D + \mathfrak{f}\partial_t - \mathfrak{f}^\dagger)g \Big\rangle &= \int_{\mathscr{C}} \overline{h}(D + \mathfrak{f}\partial_t - \mathfrak{f}^\dagger)g \\ &= \int_{\mathscr{C}} \overline{h}(Dg) + \overline{h}\mathfrak{f}\partial_t g - \overline{h}\mathfrak{f}^\dagger g \\ &= \int_{\mathscr{C}} -(\overline{h}D)g + \int_0^\infty \int_{\partial\Omega} \overline{h}\eta g - \int_{\mathscr{C}} (\partial_t\overline{h})\mathfrak{f}g - \int_{\mathscr{C}} \overline{h}\mathfrak{f}^\dagger g + \int_{\mathscr{C}} \partial_t(\overline{h}\mathfrak{f}g) \\ &= \int_{\mathscr{C}} ((-\overline{h}D) - (\partial_t\overline{h})\mathfrak{f} - \overline{h}\mathfrak{f}^\dagger)g + \int_0^\infty \int_{\partial\Omega} \overline{h}\eta g + \int_\Omega \overline{h}\mathfrak{f}g\big|_{t=0}^\infty \, dx, \end{aligned}$$

where the last two integrals vanish because $\mathrm{tr}\, g = 0$ and $\lim_{t\to\infty} g(x,t) = 0$, for all $x \in \Omega$. Using the fact that $h \in \mathrm{Ker}(D^+_{x,t})$ we obtain

$$0 = \overline{D^+_{x,t}h} = -\overline{h}D - \partial_t\overline{h}\mathfrak{f} - \overline{h}\mathfrak{f}^\dagger.$$

Thus $D^-_{x,t}g$ is orthogonal to h and because h was arbitrary, then $D^-_{x,t}g \in (\mathrm{Ker}(D_{x,t}))^\perp$. In other words,

$$(D^-_{x,t})W^{1,2}_{0,\infty}(\mathscr{C}) \subset (\mathrm{Ker}(D^+_{x,t}) \cap L^2(\mathscr{C}))^\perp.$$

Now take $g \in \left(D^-_{x,t}C^\infty_c(\mathscr{C})\right)^\perp$, then we have that

$$\begin{aligned} D^+_{x,t}g = 0 &\iff \left\langle D^+_{x,t}g, h\right\rangle = 0 \quad \forall h \in C^\infty_c(\mathscr{C}) \\ &\iff \int_{\mathscr{C}} \left(\overline{D^+_{x,t}g}\right) h = 0 \quad \forall h \in C^\infty_c(\mathscr{C}) \\ &\iff \int_{\mathscr{C}} \left(\overline{g}\overline{D^+_{x,t}}\right) h = 0 \quad \forall h \in C^\infty_c(\mathscr{C}) \\ &\iff \left\langle g, D^-_{x,t}h\right\rangle = 0 \quad \forall h \in C^\infty_c(\mathscr{C}). \end{aligned}$$

Therefore, $g \in \mathrm{Ker}(D^+_{x,t})$. That is, $\left(D^-_{x,t}C^\infty_c(\mathscr{C})\right)^\perp \subset \mathrm{Ker}(D^+_{x,t})$. Finally, by taking orthogonal complements and then by taking the closure we get

$$D^-_{x,t}W^{1,2}_{0,\infty}(\mathscr{C}) \subset (\mathrm{Ker}(D^+_{x,t}))^\perp.$$

□

Corollary 4.2.5. Under the same hypothesis than Theorem 4.2.4, we have

$$L^2(\mathscr{C}) = \mathrm{Ker}D^\pm_{x,t} \cap L^2(\mathscr{C}) \oplus D^\mp_{x,t}W^{1,2}_{0,\infty}(\mathscr{C}).$$

4.3 A parabolic Teodorescu transform

An immediate consequence of the factorization $-\Delta + \partial_t = (D^+_{x,t})^2$ provided by Theorem 4.1.1 is that the function

$$E(x,t) = D^+_{x,t}e(x,t) = e(x,t)\left[\frac{-\vec{x}}{2t} + \left(\frac{|x|^2}{4t^2} - \frac{3}{2t}\right)\mathfrak{f} + \mathfrak{f}^\dagger\right], \tag{4.10}$$

for all $x \in \Omega$ and $t > 0$ is a fundamental solution of the parabolic Dirac operator $D^+_{x,t}$.

Definition 4.3.1. Let us define the *parabolic Teodoresccu transform* as follows

$$T^+_{x,t}[G](x,t) = \int_0^\infty \int_\Omega E(x-y, t-s)G(y,s)dyds, \qquad \forall (x,t) \in \mathscr{C}, \tag{4.11}$$

where G is $C\ell_{1,4}$ integrable function defined in $\mathscr{C}$ and E is the fundamental solution defined previously in (4.10).

Now, we are going to prove a similar result of Lemma 3.2.2.

Proposition 4.3.1. For all $t > 0$, we have

$$\int_{\mathbb{R}^3} E(x,t)dx = \mathfrak{f}^\dagger.$$

Proof. First, we are going to analyze every component of (4.10). So we have

$$\begin{aligned}\frac{1}{2t}\int_{\mathbb{R}^3} e(x,t)(-x_i)dx_1dx_2dx_3 &= \frac{1}{2t}\int_{\mathbb{R}^3} \frac{e^{\frac{-|x|^2}{4t}}}{\sqrt{4\pi t}^3}(-x_1)\,dx_1dx_2dx_3 \\ &= \frac{-1}{2t}\int_{\mathbb{R}} \frac{e^{\frac{-|x|^2}{4t}}}{\sqrt{4\pi t}}x_i\,dx_i \\ &= 0, \quad \forall i = 1,2,3.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^3} x_i^2 e(x,t)dx_1dx_2dx_3 &= \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} x_i^2 e^{\frac{-x_1^2}{4t}}dx_1 \\ &= \frac{2}{\sqrt{4\pi t}}\int_0^\infty x_1^2 e^{\frac{-x_1^2}{4t}}dx_1 \\ &= \frac{8t}{\sqrt{\pi}}\int_0^\infty z^2 e^{-z^2}dz \\ &= \frac{4t}{\sqrt{\pi}}\int_0^\infty x^{\frac{3}{2}-1}e^{-x}dx \\ &= 2t.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^3} e(x,t)\frac{|x|}{4t^2}dx_1dx_2dx_3 = \frac{3}{2t}.$$

By Lemma 3.2.2, we have that for all $t > 0$

$$\int_{\mathbb{R}^3} e(x,t)dx = 1.$$

Therefore,

$$\int_{\mathbb{R}^3} E(x,t)dx = \mathfrak{f}^\dagger.$$

□

The following result tell us that the operator $T^+_{x,t}$ is a right inverse of the parabolic Dirac operator $D^+_{x,t}$ up to a nilpotent constant in $C\ell_{1,4}$. Some ideas to compute the integrals were taken from [20, Ch. 2, Th. 1].

Theorem 4.3.2. Let $G : \mathscr{C} \to C\ell_{1,4}$ be $G \in C^{1,1}_c(\mathscr{C}) \cap C(\overline{\mathscr{C}})$. Then

$$D^+_{x,t}T^+_{x,t}G = \mathfrak{f}\mathfrak{f}^\dagger G.$$

Proof. Let $G : \mathscr{C} \to C\ell_{1,4}$. Applying the operator $D^+_{x,t}$, we obtain

$$D^+_{x,t}T^+_{x,t}G(x,t) = D^+_{x,t}\int_0^t\int_\Omega E(x-y,t-s)G(y,s)\,dyds = D^+_{x,t}\int_0^t\int_\Omega E(x-y,s)G(y,t-s)\,dyds.$$

Let $\varepsilon > 0$ and $\varepsilon < t$. Then we start by calculating the following

$$\partial_t\int_0^t E(x-y,s)G(y,t-s)dt = E(x-y,t)G(y,0) + \int_0^t E(x-y,s)(\partial_t G(y,t-s))dt.$$

Thus,

$$\begin{aligned}\partial_t\int_0^t\int_\Omega E(x-y,s)G(y,t-s)dyds &= \int_0^t\int_\Omega E(x-y,s)(\partial_t G(y,t-s))dyds\\ &\quad + \int_\Omega E(x-y,t)G(y,0)dy\end{aligned}$$

Therefore we now have the following

$$\begin{aligned}D^+_{x,t}\int_0^t\int_\Omega E(x-y,t-s)G(y,s)dyds &= D\int_0^t\int_\Omega E(x-y,s)G(y,t-s)dyds\\ &\quad + \mathfrak{f}\int_0^t\int_\Omega E(x-y,s)(\partial_t G(y,t-s))dyds\\ &\quad + \mathfrak{f}\int_\Omega E(x-y,t)G(y,0)dy\\ &\quad + \mathfrak{f}^\dagger\int_0^t\int_\Omega E(x-y,s)G(y,t-s)dyds\\ &= I_\varepsilon + J_\varepsilon + K,\end{aligned}$$

where the last integrals $I_\varepsilon, J_\varepsilon$ and K are defined as follows

$$\begin{aligned}I_\varepsilon &= D\int_\varepsilon^t\int_\Omega E(x-y,s)G(y,t-s)dyds + \mathfrak{f}\int_\varepsilon^t\int_\Omega E(x-y,s)(\partial_t G(y,t-s))\,dyds\\ &\quad + \mathfrak{f}^\dagger\int_\varepsilon^t\int_\Omega E(x-y,s)G(y,t-s)\,dyds, \qquad (4.12)\\ J_\varepsilon &= D\int_0^\varepsilon\int_\Omega E(x-y,s)G(y,t-s)dyds + \mathfrak{f}\int_0^\varepsilon\int_\Omega E(x-y,s)(\partial_t G(y,t-s))\,dyds\\ &\quad + \mathfrak{f}^\dagger\int_0^\varepsilon\int_\Omega E(x-y,s)G(y,t-s)\,dyds, \qquad (4.13)\\ K &= \mathfrak{f}\int_\Omega E(x-y,t)G(y,0)ds. \qquad (4.14)\end{aligned}$$

Now we estimate J_ε:

$$\begin{aligned}|J_\varepsilon| &= \Big|D_x\int_0^\varepsilon\int_\Omega E(y,s)G(x-y,t-s)dyds + \mathfrak{f}\int_0^\varepsilon\int_\Omega E(y,s)(\partial_t G(x-y,t-s))dyds\\ &\quad + \mathfrak{f}^\dagger\int_0^\varepsilon\int_\Omega E(y,s)G(x-y,t-s)dyds\Big|\\ &= \Big|\int_0^\varepsilon\int_\Omega E(y,s)D_x G(x-y,t-s)dyds + \mathfrak{f}\int_0^\varepsilon\int_\Omega E(x-y,s)(\partial_t G(y,t-s))dyds\\ &\quad + \mathfrak{f}^\dagger\int_0^\varepsilon\int_\Omega E(x-y,s)G(y,t-s)dyds\Big|\\ &\leq \sqrt{2}\left(||DG||_{L^\infty} + ||\partial_t G||_{L^\infty} + ||G||_{L^\infty}\right)\int_0^\varepsilon\int_\Omega |E|\,dyds\\ &\leq \varepsilon C, \qquad (4.15)\end{aligned}$$

where the last inequality is due to Proposition 4.3.1. Therefore

$$\lim_{\varepsilon\to 0}|J_\varepsilon| = 0.$$

For I_ε, we will use the fact that $E(x,t)$ is a fundamental solution of $D^+_{x,t}$, which means that $D^+_{x,s}E(x-y,s) = 0$ for any $s \in [\varepsilon, t]$. Thus

$$\begin{aligned}
I_\varepsilon &= D\int_\varepsilon^t\int_\Omega E(x-y,s)G(y,t-s)dyds + \mathfrak{f}\int_\varepsilon^t\int_\Omega E(x-y,s)(\partial_t G(y,t-s))dyds \\
&+ \mathfrak{f}^\dagger\int_\varepsilon^t\int_\Omega E(x-y,s)G(y,t-s)dyds \\
&= \int_\varepsilon^t\int_\Omega DE(x-y,s)G(y,t-s)dyds + \mathfrak{f}\int_\varepsilon^t\int_\Omega \partial_s E(x-y,s)(G(y,t-s))dyds \\
&+ \mathfrak{f}\int_\Omega E(x-y,\varepsilon)G(y,t-\varepsilon)dy - \mathfrak{f}\int_\Omega E(x-y,t)G(y,0)dy + \mathfrak{f}^\dagger\int_\varepsilon^t\int_\Omega E(x-y,s)G(y,t-s)dyds \\
&= \int_\varepsilon^t\int_\Omega \left[D^+_{x,s}E(x-y,s)\right]G(y,t-s)dyds + \mathfrak{f}\int_\Omega E(x-y,\varepsilon)G(y,t-\varepsilon)dydt - K \\
&= \mathfrak{f}\int_\Omega E(x-y,\varepsilon)G(y,t-\varepsilon)dydt - K
\end{aligned}$$

Now we only need to compute the following

$$\lim_{\varepsilon\to 0}\mathfrak{f}\int_\Omega E(x-y,\varepsilon)G(y,t-\varepsilon)dy.$$

First, we are going to calculate

$$\int_\Omega e(x-y,\varepsilon)\left(-\frac{\vec{x}-\vec{y}}{2\varepsilon}\right)G(y,t-\varepsilon)dy.$$

Using the fact that $x \in \Omega$ there exists $r > 0$ such that $B(x,r) \subset \Omega$ and $r < 2\sqrt{(4\pi\varepsilon)^3}\varepsilon^2$, then we have

$$\begin{aligned}
\left|\int_{\Omega\setminus B(x,r)} e(x-y,\varepsilon)\left(-\frac{\vec{x}-\vec{y}}{2\varepsilon}\right)G(y,t-\varepsilon)dy\right| &\le \sum_{i=1}^3\sum_{A\in\mathscr{P}(S)}\int_{\Omega\setminus B(x,r)} e(x-y,\varepsilon)\frac{|x_i-y_i|}{2t}|G_A(y,t-\varepsilon)|dy \\
&\le e(r,\varepsilon)\sum_{A\in\mathscr{P}(S)}\int_{\Omega\setminus B(x,\varepsilon)}\sum_{i=1}^3\frac{|x_i-y_i|}{2t}\,||G_A||_\infty\,dy
\end{aligned}$$

Since Ω is bounded, we have

$$\left|\int_{\Omega\setminus B(x,r)} e(x,\varepsilon)\left(-\frac{\vec{x}-\vec{y}}{2\varepsilon}\right)G(y,t-\varepsilon)dy\right| \le e(r,\varepsilon)C \xrightarrow[\varepsilon\to 0]{} 0,$$

and on the other hand

$$\begin{aligned}
\left|\int_{B(x,r)} e(x-y,\varepsilon)\left(-\frac{\vec{x}-\vec{y}}{2\varepsilon}\right)G(y,t-\varepsilon)dy\right| &\le \sum_{i=1}^3\sum_{A\in\mathscr{P}(S)}\int_{B(x,r)} e(x-y,\varepsilon)\frac{|x_i-y_i|}{2t}|G_A(y,t-\varepsilon)|dy \\
&\le \sum_{A\in\mathscr{P}(S)}\int_{B(x,r)} e(x-y,\varepsilon)\sum_{i=1}^3\frac{|x_i-y_i|}{2t}\,||G_A||_\infty\,dy \\
&\le C\frac{3r}{\sqrt{(4\pi\varepsilon)^3}2\varepsilon} \le 3C\varepsilon \xrightarrow[\varepsilon\to 0]{} 0.
\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\Omega} e(x-y,\varepsilon)\left(-\frac{\vec{x}-\vec{y}}{2\varepsilon}\right)G(y,t-\varepsilon)dy = 0.$$

Now let us to compute

$$\lim_{\varepsilon\to 0} \text{ff}^{\dagger}\int_{\Omega} e(x-y,\varepsilon)G(y,t-\varepsilon)dy.$$

Due to G is continuous on Ω and has compact support, then for all $\eta > 0$ there exist $\delta > 0$ such that

$$|G(y,s) - G(x,t)| < \eta,$$

whenever $|(y,s)-(x,t)| < \delta$. Because $x \in \Omega$ and Ω open, there exist $r_0 > 0$ such that $B(x,r_0) \subset \Omega$. Let $r = \min\{r_0, \delta/2\}$, let $\varepsilon < \delta/2$.

$$|(y,t-\varepsilon)-(x,t)| \leq |x-y| + |\varepsilon| < \delta,$$

Notice that $y \in \Omega$, then $1_{\Omega}(y) = 1$.

$$|G(y,t-\varepsilon) - G(x,t)| < \eta.$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} e(x-y,\varepsilon)\left[1_{\Omega}(y)G(y,t-\varepsilon) - G(x,t)\right]dy = 0,$$

Thus,

$$\lim_{\varepsilon\to 0}\int_{\Omega} e(x-y,\varepsilon)G(y,t-\varepsilon)dy = G(x,t).$$

Consequently,

$$D^{+}_{x,t}\int_{\mathscr{C}} E(x-y,t-s)G(y,s)dyds = \text{ff}^{\dagger}G.$$

□

5. The Parabolic Dirac Operator

In this Chapter we are interested in the characterization of the kernel of the parabolic Dirac operator $D^{+}_{x,t}$. In Theorem 5.1.1 we are going to give an equivalent system called *generalized parabolic div-curl system* which characterizes $\mathrm{Ker}D^{+}_{x,t}$ and in Theorem 5.1.2 we construct solutions of the operator $D^{+}_{x,t}$ using only four solutions of the heat operator as initial data. Finally, in Section 5.2 we are going to show some mean value-type properties of the solutions of the parabolic Dirac operator.

5.1 Solutions of $D^{\pm}_{x,t}$

In this Section, we are going to characterize the kernel of the parabolic Dirac operator $D^{+}_{x,t}$. It is straightforward that $\mathrm{Ker}D^{\pm}_{x,t} \neq \emptyset$, due to $\mathfrak{f}^{\dagger} \in \mathrm{Ker}D^{\pm}_{x,t}$. Moreover, $\mathrm{Ker}D^{+}_{x,t}$ restricted to $C\ell_{0,3}$ valued functions is trivial. By the factorization

$$-\Delta \pm \partial_t = (D^{\pm}_{x,t})^2$$

provided in Lemma 4.1.1. The following contention is obvious

$$\mathrm{Ker}D^{\pm}_{x,t} \subset \mathrm{Ker}(-\Delta \pm \partial_t).$$

Following [19], each $C\ell_{1,4}$-valued function $G : \mathscr{C} \to C\ell_{1,4}$ will be written as a linear combination of $C\ell_{0,3}$-valued functions as follows

$$G = g_0 + \vec{g_0} + (g_1 + \vec{g_1})\, e_+ + (g_2 + \vec{g_2})\, e_- + (g_3 + \vec{g_3})\, e_+ e_-, \tag{5.1}$$

where $\vec{g_i} = g_{i,1}e_1 + g_{i,2}e_2 + g_{i,3}e_3$ is a purely vectorial function, $i = 0, 1, 2, 3$. Notice that g_i and $g_{i,j}$ are scalar functions, for each $i = 0, 1, 2, 3$ and $j = 1, 2, 3$. This way to rewrite $C\ell_{1,4}$-valued functions will the key in the next results.

Next Theorem 5.1.1 is obtained by using that $D^{+}_{x,t}$ is linear. This fact will be used component-wise, and we will employ then the multiplication rules (2.1) and (2.2).

Theorem 5.1.1. If $G\colon \mathscr{C} \to C\ell_{1,4}$, then

$$D^{+}_{x,t}G = 0,$$

if and only if G satisfies the following *generalized parabolic div-curl system*

$$\begin{cases} -\operatorname{div}(\vec{g_0}) + \frac{1}{2}(\partial_t g_1 + \partial_t g_2 + g_1 - g_2) = 0, \\ \nabla g_0 + \frac{1}{2}(-\partial_t \vec{g_1} - \partial_t \vec{g_2} - \vec{g_1} + \vec{g_2}) = 0, \\ -\operatorname{div}(\vec{g_1}) + \frac{1}{2}(\partial_t g_0 - \partial_t g_3 + g_0 + g_3) = 0, \\ \nabla g_1 + \frac{1}{2}(-\partial_t \vec{g_0} + \partial_t \vec{g_3} - \vec{g_0} - \vec{g_3}) = 0, \\ -\operatorname{div}(\vec{g_2}) + \frac{1}{2}(-\partial_t g_0 + \partial_t g_3 + g_0 + g_3) = 0, \\ \nabla g_2 + \frac{1}{2}(\partial_t \vec{g_0} - \partial_t \vec{g_3} - \vec{g_0} - \vec{g_3}) = 0, \\ -\operatorname{div}(\vec{g_3}) + \frac{1}{2}(\partial_t g_1 + \partial_t g_2 - g_1 + g_2) = 0, \\ \nabla g_3 + \frac{1}{2}(-\partial_t \vec{g_1} - \partial_t \vec{g_2} + \vec{g_1} - \vec{g_2}) = 0. \\ \operatorname{curl}(\vec{g_0}) = \operatorname{curl}(\vec{g_1}) = \operatorname{curl}(\vec{g_2}) = \operatorname{curl}(\vec{g_3}) = 0. \end{cases} \tag{5.2}$$

where $G = (g_0 + \vec{g_0}) + (g_1 + \vec{g_1})e_+ + (g_2 + \vec{g_2})e_- + (g_3 + \vec{g_3})e_+e_-$.

Let $C^2_1(\mathscr{C})$ be denote the space of functions with continuous second-order spatial partial derivatives and continuous first-order temporal partial derivative. Another important theorem is the following, which states that we only need to know 4 scalar solutions g_i of the heat equation in order to construct a $C\ell_{1,4}$ function in $\operatorname{Ker} D^{+}_{x,t}$ whose scalar parts coincides with the prescribed data g_i. More precisely,

Theorem 5.1.2. If g_0, g_1, g_2, g_3 are scalar solutions of the heat equation and $g_i \in C^2_1(\mathscr{C})$, then there exist a function $G = g^0 + g^1 e_+ + g^2 e_- + g^3 e_+ e_- \in \operatorname{Ker} D^{+}_{x,t}$, where $g^i \in C\ell_{0,3}$ is paravector-valued and satisfies

$$\operatorname{Sc}(g^i) = g_i \qquad \forall i = 0, 1, 2, 3.$$

Proof. For every (x,t) in $\mathscr{C}$ define the following functions

$$\begin{aligned} \vec{\psi}(x,t) &:= \nabla(g_1 + g_2)(x,t), \\ \vec{\phi}(x,t) &:= \nabla(g_3 - g_0)(x,t), \\ \vec{\Psi}(x,t) &:= \int_0^t \nabla(g_1 - g_2)(x,s)ds - \overrightarrow{T}_{1,\Omega}[g_1 - g_2](x,0), \\ \vec{\Phi}(x,t) &:= \int_0^t \nabla(g_0 + g_3)(x,s)ds - \overrightarrow{T}_{1,\Omega}[g_0 + g_3](x,0). \end{aligned}$$

and let

$$\vec{g_0} = \frac{1}{2}\left[\vec{\psi} + \vec{\Psi}\right], \qquad \vec{g_1} = \frac{1}{2}\left[\vec{\Phi} - \vec{\phi}\right],$$
$$\vec{g_2} = \frac{1}{2}\left[\vec{\Phi} + \vec{\phi}\right], \qquad \vec{g_3} = \frac{1}{2}\left[\vec{\psi} - \vec{\Psi}\right].$$

Finally, let us consider the function $G = (g_0+\vec{g_0})+(g_1+\vec{g_1})e_+ +(g_2+\vec{g_2})e_- +(g_3+\vec{g_3})e_+e_-$, this function G satisfies the system (5.2), which implies that $G \in \mathrm{Ker}D^+_{x,t}$ as we desired. □

Is worth mentioning that the construction of the solutions is not unique. For instance if $G \in \mathrm{Ker}(D^+_{x,t})$ is constructed as in 5.1.2 and h_1, h_2 are time-independent harmonic functions in Ω, then

$$\tilde{G} = (g^0 + \nabla h_1) + (g^1 + \nabla h_2)e_+ + (g^2 + \nabla h_2)e_- + (g^3 - \nabla h_1)e_+e_-,$$

is still a solution to the parabolic Dirac operator.

5.2 Mean Value for solutions of the Dirac Operator

Here the set $S = \{0, 1, 2, 3, +, -\}$ will be used to refer to the basic elements that are being multiplied, for example, if $A = \{1, 2, +\} \subset S$, then

$$e_A = e_1e_2e_+,$$

and for the set $A = \{+, -\}$ we are always going to refer to $e_A = e_+e_-$.

So now with theorems 3.3.2 and 3.3.3 we have the following:

Corollary 5.2.1. For $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}, r > 0$, let $G : \Omega \to C\ell_{1,3}$ such that $G \in C^{2,1}(Hb((x_0, t_0); r) \cap C^1(\overline{Hb((x_0, t_0); r)})$ and $G \in \mathrm{Ker}(D^+_{x,t})$ on $Hb((x_0, t_0); r)$. Then

$$G(x,t) = \frac{1}{4r^3}\int_{E(x,t;r)} G(y,s)\frac{|x-y|^2}{(t-s)^2}dyds.$$

Proof. So if

$$G = \sum_{A\in\mathscr{P}(S)} e_A g_A$$

is a solution to the parabolic Dirac operator $D^+_{x,t}$ every component of G is a solution to the heat equation, therefore we can apply theorem 3.3.2 to every component of G, so

$$g_A(x,t) = \frac{1}{4r^3}\int_{E(x,t;r)} g_A(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \qquad \forall A \in \mathscr{P}(S),$$

so we have

$$e_A g_A(x,t) = \frac{1}{4r^3}\int_{E(x,t;r)} e_A g_A(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \qquad \forall A \in \mathscr{P}(S),$$

now by adding over every set A, we get

$$\sum_{A\in\mathscr{P}(S)} e_A g_A(x,t) = \frac{1}{4r^3}\int_{E(x,t;r)} \sum_{A\in\mathscr{P}(S)} e_A g_A(y,s)\frac{|x-y|^2}{(t-s)^2}dyds,$$

which is

$$G(x,t) = \frac{1}{4r^3}\int_{E(x,t;r)} G(y,s)\frac{|x-y|^2}{(t-s)^2}dyds.$$

□

Corollary 5.2.2. Let $(x_0,t_0) \in \mathbb{R}^3\times\mathbb{R}, r>0$ let $G:\Omega \to C\ell_{1,3}$ such that $G \in C^{2,1}(Hb((x_0,t_0);r)\cap C^1(\overline{Hb((x_0,t_0);r)})$ and $G \in \mathrm{Ker}(D^+_{x,t})$ on $Hb((x_0,t_0);r)$. Then

$$G(x_0,t_0) = \frac{1}{2r^3}\int_{\partial Hb((x_0,t_0);r)} G(y,s)Q(x_0-y,t_0-s)dS_{y,s},$$

where

$$Q(y,s) = \left(\frac{4s^2}{|y|^2} + \left(1-\frac{6s}{|y|^2}\right)^2\right)^{-1/2}.$$

Proof. So if

$$G = \sum_{A\in\mathscr{P}(S)} e_A g_A$$

is a solution to the parabolic Dirac operator $D^+_{x,t}$ every component of G is a solution to the heat equation, therefore we can apply theorem 3.3.3 to every component of G, so

$$g_A(x_0,t_0) = \frac{1}{2r^3}\int_{\partial E(x_0,t_0;r)} g_A(y,s)Q(x_0-y,t_0-s)dS_{y,s} \qquad \forall A \in \mathscr{P}(S),$$

so we have

$$e_A g_A(x_0,t_0) = \frac{1}{2r^3}\int_{\partial E(x_0,t_0;r)} e_A g_A(y,s)Q(x_0-y,t_0-s)dS_{y,s} \qquad \forall A \in \mathscr{P}(S),$$

now by adding over $A \subset \{0,1,2,+,-\}$ we get

$$\sum_{A\in\mathscr{P}(S)} e_A g_A(x_0,t_0) = \frac{1}{2r^3}\int_{\partial E(x_0,t_0;r)} \sum_{A\in\mathscr{P}(S)} e_A g_A(y,s)Q(x_0-y,t_0-s)dS_{y,s},$$

which is

$$G(x_0,t_0)=\frac{1}{2r^3}\int_{\partial E(x_0,t_0;r)}G(y,s)Q(x_0-y,t_0-s)dS_{y,s}.$$

□

Also, we have a uniqueness for initial boundary value problems.

Proposition 5.2.3. Let $\Phi\in C(\partial\mathscr{C}), F\in C(\mathscr{C})$. Then there exists at most one solution $G\in C_1^2(\mathscr{C})\cap C(\partial\mathscr{C})$ to the problem

$$\begin{aligned} D_{x,t}^{+}G &= F \qquad \text{in } \mathscr{C}\\ G &= \Phi \qquad \text{in } \partial\mathscr{C}\end{aligned}$$

Proof. Let Φ be a $C\ell_{1,4}$-valued function, and let $G=\sum_A g_A e_A, \tilde{G}=\sum_A \tilde{g}_A e_A$ be solutions to the problem. Then for every $T>0$ we have that $w_A^{\pm}=\pm(g_A-\tilde{g}_A)$ is a solution to the problem

$$\begin{cases} (-\Delta+\partial_t)w^{\pm} &= 0 \qquad \text{in } \Omega\times(0,T),\\ w^{\pm} &= 0 \qquad \text{in } \partial\Omega\times(0,T],\\ w^{\pm} &= 0 \qquad \text{in } \Omega\times\{0\}.\end{cases}$$

Therefore by the Maximum Principle of the heat equation [20, Ch. 2, Th. 4] we have

$$g_A=\tilde{g}_A \qquad \forall(x,t)\in\Omega\times(0,T),$$

for all $T>0$. Therefore $G=\tilde{G}$. □

6. Conclusions

In this thesis, we use the properties of the Clifford algebras, mainly the Clifford algebra $C\ell_{1,4}$ to define the parabolic Dirac operator which factors the heat operator, where the parabolic Dirac operator, then we relate the solutions of this operator and relate them to a general div-rot system, which in turn find a process to construct solutions for this operator using less functions that the ones that are needed in more general spaces like in [19]. Further work could analyze more properties of the operator $T^{+}_{x,t}$ and see what properties it has, mainly if it can be an inverse of the parabolic Dirac operator, so we can follow the ideas of the theory of monogenic functions.

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