*CENTRO DE CIENCIAS BÁSICAS*

*DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA*

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TESIS

# *Design and analysis of dissipation-preserving models for Higgs' boson equation in the de Sitter space-time*

PRESENTA

*Luis Fernando Muñoz Pérez*

PARA OBTENER EL GRADO DE

*Doctor en Ciencias Aplicadas y Tecnología*

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Aguascalientes, Ags., a 14 de agosto del 2023

Jorge Martín Alférez Chávez
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Por medio del presente como tutor designado del estudiante **LUIS FERNANDO MUÑOZ PÉREZ** con ID 108852 quien realizó la tesis titulada **DESIGN AND ANALYSIS OF DISSIPATION-PRESERVING MODELS FOR HIGGS BOSON EQUATION IN THE DE SITTER SPACE-TIME**, un trabajo propio, innovador, relevante e inédito y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia doy mi consentimiento de que la versión final del documento ha sido revisada y las correcciones se han incorporado apropiadamente, por lo que me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla así como continuar con el procedimiento administrativo para la obtención del grado.

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**PROGRAMA:** Doctorado en Ciencias Aplicadas y Tecnología **LGAC (del posgrado):** Matemáticas Aplicada

**TIPO DE TRABAJO:** ( X ) Tesis ( ) Trabajo Práctico

**TITULO:** Design and analysis of dissipation-preserving models for Higgs' boson equation in the de Sitter space-time

**IMPACTO SOCIAL (señalar el impacto logrado):** Se propuso por primera vez un esquema hamiltoniano para la ecuación del bosón de Higgs mediante un modelo generalizado de la ecuación del bosón de Higgs en el espacio-tiempo de De Sitter. Se analizaron variaciones matemáticas tanto variacional como numéricamente.

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# On the solution of a generalized Higgs boson equation in the de Sitter space-time through an efficient and Hamiltonian scheme

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A B S T R A C T

The present work is the first paper in the literature to report on a Hamiltonian discretization of the (fractional) Higgs boson equation in the de Sitter space-time, and its theoretical analysis. More precisely, we design herein a numerically efficient finite-difference Hamiltonian technique for the solution of a fractional extension of the Higgs boson equation in the de Sitter space-time. The model under investigation is a multidimensional equation with generalized potential and Riesz space-fractional derivatives of orders in $(1, 2]$. An energy integral for the model is readily available, and we propose a nonlinear, implicit and consistent numerical technique based on fractional-order centered differences, with similar Hamiltonian properties in the discrete scenario. A fractional energy approach is used then to prove the properties of stability and convergence of the technique. For simulation purposes, we consider both the classical and the fractional Higgs real-valued scalar fields in the de Sitter space-time, and find results qualitatively similar to those available in the literature. For the sake of convenience, we provide the Matlab code of an alternative linear discretization of the method presented in this work. This linear implicit approach is thoroughly analyzed also.



## 1. Introduction

The design of energy-preserving methods for physical systems has been a fruitful avenue of research in the last decades. Historically, the problem of designing energy-conserving methods may date back to the decade of the 1970s [1,2] or before. However, it is worth mentioning that L. Vázquez and coauthors were probably the first researchers who pointed out the physical and mathematical significance of designing this type of schemes [3]. Various seminal papers by Vázquez and his coworkers were published in the 1990s, including various energy-conserving numerical schemes to solve partial differential equations like the Schrödinger equation [4], the sine-Gordon equation [5,6], the Klein–Gordon equations [7], and even systems consisting of ordinary differential equations [8]. In those papers, the authors established thoroughly the capability of their schemes to preserve the energy properties of the continuous problem. Moreover, they employed a discrete form of

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# A dissipation-preserving finite-difference scheme for a generalized Higgs boson equation in the de Sitter space–time

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ABSTRACT

For the first time in the literature, a dissipation-preserving computational technique to approximate the solutions of a dissipative generalization of the Higgs boson equation in the de Sitter space–time is proposed. The model is a multidimensional system which considers a generalized potential and a general time-dependent diffusion coefficient. The system has an associated energy functional which is dissipated in time. Motivated by this fact, we propose a finite-difference methodology to approximate the solutions of the mathematical model. In addition to the numerical approximation for the solutions of the system, we propose also a discrete energy functional which is dissipated with respect to the discrete time. Using a computer implementation in two spatial dimensions, we provide some simulations that confirm the presence of bubble-like solutions, in agreement with the theory available in the literature.



## 1. Introduction

For the sake of convenience, let $I_n = \{0, 1, \dots, n\}$ and $\overline{I}_n = I_n \cup \{-1\}$, for each $n \in \mathbb{N}$. Let $p \in \mathbb{N}$, and let $x \in \mathbb{R}^p$ be a vector which is represented component-wise as $x = (x_1, x_2, \dots, x_p)$. Define the spatial domain $B = \prod_{i=1}^{p}(a_i, b_i) \subseteq \mathbb{R}^p$, where $-\infty < a_i < b_i < \infty$, for each $i = 1, 2 \dots, p$. Let $T > 0$, and define $\Omega = B \times (0, T)$ as the space–time domain.

Throughout, let $\gamma \in \overline{\mathbb{R}^+}$ and suppose that $f : \overline{\mathbb{R}^+} \to \mathbb{R}$ is a differentiable function. Let $F : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that $\phi_0, \phi_1 : B \to \mathbb{R}$ are continuously differentiable functions. In the present

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# An implicit and convergent method for radially symmetric solutions of Higgs' boson equation in the de Sitter space–time

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A B S T R A C T

The present work introduces a numerical scheme that preserves the dissipation of energy of the Higgs boson equation in the de Sitter space-time. More precisely, the model considered in this work is a mathematical generalization of Higgs' model which includes a general time-dependent diffusion coefficient and a generalized potential. The mathematical system is dissipative, and we propose an implicit discrete method which approximates consistently the radially symmetric solutions of the continuous system. At the same time, a discrete energy functional is presented, and we prove that, as its continuous counterpart, the numerical technique dissipates the energy of the discrete system. The properties of consistency, stability and convergence of the numerical model are proved rigorously. To confirm the theoretical results, we approximate some radially symmetric solutions of the classical Higgs boson equation in the de Sitter space-time. In particular, the numerical results confirm the stability and the formation of bubble-like solutions.



## 1. Introduction

The Higgs boson equation is a fundamental system in the unification of various theories in physics. Indeed, Higgs boson equation in the Minkowski and in the de Sitter space-times are employed to unify the theories on weak, strong and electromagnetic interactions [61]. Since the publication of the report confirming experimentally the existence of Higgs' boson [2], this model has been studied extensively mainly from the physical point of view. Indeed, there are many physical studies focusing on the phenomenological investigation of Higgs boson [9,17], some of them making theoretical advances within the frame of the Standard Model of particle physics [7]. Some other recent experimental and theoretical studies concentrate on observing the decay of Higgs boson to bottom quarks [51], on its experimental production at hadron colliders using high-energy physics [38] or on next-to-next-to-leading order corrections with top quark mass effects [22]. Other works provide experimental evidence for the Higgs boson decay to a bottom quark–antiquark pair [49], or investigate second-order quantum chromodynamics effects in Higgs boson production through vector boson fusion [11], highly boosted Higgs bosons decaying to bottom quark-antiquark pairs [50] or the decaying of the Higgs boson to charm quarks [1]. Moreover, the investigation of the mass of this particle is also an active topic of research, as shown by reports on the measurement of the Higgs

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# Dedication

A mi amada familia, cuyo amor y aliento han sido mi fuerza constante. A mi esposa, por ser mi compañera de vida y por brindarme la inspiración y el coraje para perseguir mis sueños. A mis hijos, por llenar cada día de alegría y por recordarme la importancia de esforzarme sin cesar.

A mi madre, cuya dedicación y sacrificio han sido mi ejemplo a seguir. Tu amor y sabiduría han sido un faro en mi vida. A mis hermanos, por compartir risas, desafíos y triunfos, por ser mi red de apoyo en cada paso del camino, y porque me han demostrado que podemos con cualquier reto que nos propongamos.

A todos aquellos que han cruzado mi camino, a mis amigos y mentores, su influencia y amistad han dejado una huella imborrable en mi camino hacia esta meta.

Que este trabajo sea un tributo modesto a cada uno de ustedes, quienes han sido los cimientos de mi éxito. Su huella permanece grabada en cada página de este logro, y en mi corazón para siempre.

Con amor y gratitud.

# Contents

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# Resumen

Desde la confirmación de la existencia del bosón de Higgs, su modelo ha tenido gran impacto en la comunidad científica, especialmente en el modelo estándar de la física de partículas, la necesidad de la constante cosmológica, así como su creación experimental en colisionadores de adrones e investigaciones sobre la masa de dicha partícula. En este trabajo, presentamos una serie de esquemas en diferencias finitas con diferentes enfoques para la obtención de una solución aproximada a la ecuación del bosón de Higgs en el espacio-tiempo de de Sitter. Dichos esquemas conservarán las propiedades variacionales presentes en su forma continua, tales como la disipación de la energía en el tiempo. Cabe destacar que la mayoría de los resultados obtenidos en esta disertación son válidos para $p \in \mathbb{N}$ dimensiones espaciales, aunque las simulaciones son realizadas para $p \leqslant 3$. Además, en los diferentes esquemas considerados en esta tesis, se llevó a cabo un análisis riguroso de consistencia, unicidad y estabilidad. Todos los esquemas son convergentes con un orden cuadrático en tiempo y en espacio. El análisis se efectuó para un modelo mucho más general donde el coeficiente de difusión (que depende del tiempo en nuestro trabajo) y el potencial (que es una función no lineal de la solución) son funciones diferenciables en general. Es importante mencionar que también se consideró una extensión del modelo de Higgs que considera la presencia de difusión fraccionaria de Riesz. En las implementaciones computacionales mostradas en este trabajo, hacemos énfasis especial en demostrar la presencia de soluciones tipo “burbuja”. Dichas soluciones son típicas en el modelo de Higgs y son de gran relevancia en el mundo científico.

**Keywords**: fractional Higgs boson equation; de Sitter space-time; Riesz space-fractional equations; fractional centered differences; fractional energy method; stability and convergence analyses

# Abstract

Since the confirmation of the existence of the Higgs boson, his model has a great impact on the scientific community, especially on the standard model of particle physics. The need for the cosmological constant, its experimental creation in adron colliders, and investigations on the mass of the said particle. In this work, we present a series of finite difference schemes with different approaches to obtain an approximate solution to the Higgs boson equation in Sitter spacetime. Said schemes ging to preserve the variational properties present in their continuous form, such as the dissipation of energy over time. Most of the results obtained in this dissertation are valid for $p \in \mathbb{N}$ spatial dimensions, although the simulations are performed for $p \leqslant 3$. Furthermore, in the different schemes treated in this thesis, a rigorous analysis of consistency, uniqueness, and stability was carried out. All the schemes are convergent with a quadratic order in time and space. The analysis was carried out for a much more general model where the diffusion coefficient (which depends on time in our work) and the potential (which is a non-linear function of the solution) are generally differentiable functions. It is important to mention that it is also considered an extension of the Higgs model that considers the presence of fractional Riesz diffusion. In the computational implementations shown in this work, we place special emphasis on demonstrating the presence of "bubble" type solutions. These solutions are typically in the Higgs model and are of great relevance in the scientific world.

**Keywords**: fractional Higgs boson equation; de Sitter space-time; Riesz space-fractional equations; fractional centered differences; fractional energy method; stability and convergence analyses

# Introduction

The study of the Higgs boson has been of great interest from the perspective of phenomenology since the introduction of the theoretical concept of spontaneous gauge symmetry breaking in particle physics in 1964 [43, 27, 42]. This pioneering work was independently carried out by Peter Higgs, François Englert, Robert Brout, Gerald Guralnik, C. R. Hagen, and Tom Kibble. Their theory aimed to explain how elementary particles acquire mass, a fundamental property that governs their interactions and gives rise to the formation of structure in the Universe.

The existence of the Higgs boson was experimentally confirmed in 2012 by the ATLAS and CMS experiments at the Large Hadron Collider (LHC) at CERN [2, 18]. This historic detection was a monumental achievement for particle physics and was recognized with the Nobel Prize in Physics in 2013, awarded to François Englert and Peter Higgs for their pioneering work in predicting the existence of the Higgs boson.

The Standard Model of particle physics is the theoretical framework that describes the interactions between the fundamental particles that constitute all matter in the Universe. This model includes quarks and leptons [112], as well as the fundamental forces associated with gauge symmetries. However, in its original formulation, elementary particles were massless, which contradicted experimental observations [39]. The incorporation of the Higgs mechanism addressed this critical issue in the theory. The Higgs mechanism was proposed as a solution to this problem. It introduces the Higgs field, a scalar field that permeates all of space. The Higgs field is characterized by its unique property of spontaneous symmetry breaking, where its ground state is not invariant under certain transformations [43]. As a result of this symmetry breaking, the Higgs field acquires a non-zero vacuum expectation value, leading to the emergence of mass for certain particles. Specifically, in the Higgs mechanism, the interactions between particles and the Higgs field endow some particles, like quarks and charged leptons, with mass [44]. The massive particles acquire mass by interacting with the Higgs field, while other particles, like the photon and the gluon, remain massless because they do not interact with the Higgs field. The Higgs boson is a fundamental piece of the Standard Model as it is responsible for endowing particles with mass through its interaction with the Higgs field. Without this interaction, all particles would be completely massless, and the structure of the Universe as we know it could not form. Therefore, the presence of the Higgs boson is essential for our understanding of physics at both the subatomic and cosmic levels.

In addition to explaining the generation of mass, the Higgs boson also has implications in cosmology and the evolution of the Universe. Some studies suggest that during the first fractions of a second after the Big Bang, the Higgs field played a significant role in the process of cosmic inflation, driving an extremely rapid expansion of spacetime [10]. This inflationary period is responsible for the observed features in the distribution of the cosmic microwave background radiation, providing valuable insights into the initial conditions of the Universe.

The discovery of the Higgs boson has not only been a milestone in particle physics but has also opened new possibilities for exploration in understanding the nature of dark matter and dark energy [7], which constitute the majority of the content of the Universe and lie beyond the scope of the Standard Model. The detection and detailed study of the Higgs boson at the LHC continue to play a fundamental role in the search for new particles and physical phenomena that challenge the current description of nature.

The present work is divided into four sections, each focusing on different aspects of the Higgs boson equation and its implications.

In the first section, an approximation to the solution of the Higgs boson equation in de Sitter space is presented, utilizing finite differences in a multidimensional system with a generalized potential and a time-dependent diffusion coefficient.

The second section focuses on a particular case of 3+1 dimensions with radially symmetric solutions, proposing a discrete implicit scheme for its study.

In the third section, a discretization of a fractional extension of the Higgs boson equation in de Sitter spacetime is proposed, introducing Riesz space-fractional derivatives of orders in the interval $(1, 2]$. All these models are accompanied by associated energy functionals that evolve in time, providing insights into the dynamics of the Higgs field.

Finally, in the fourth section, the conclusions of the work are presented, summarizing the contributions and potential implications of the findings.

# 1 A two-dimensional Higgs boson equation

*FOR THE FIRST TIME IN THE LITERATURE, a dissipation-preserving computational technique to approximate the solutions of a dissipative generalization of the Higgs boson equation in the de Sitter space-time is proposed. The model is a multidimensional system which considers a generalized potential and a general time-dependent diffusion coefficient. The system has an associated energy functional which is dissipated in time. Motivated by this fact, we propose a finite-difference methodology to approximate the solutions of the mathematical model. In addition to the numerical approximation for the solutions of the system, we propose also a discrete energy functional which is dissipated with respect to the discrete time. Using a computer implementation in two spatial dimensions, we provide some simulations that confirm the presence of bubble-like solutions, in agreement with the theory available in the literature.*

## 1.1 Background

For the sake of convenience, let $I_n = \{0, 1, \ldots, n\}$ and $\overline{I}_n = I_n \cup \{-1\}$, for each $n \in \mathbb{N}$. Let $p \in \mathbb{N}$, and let $x \in \mathbb{R}^p$ be a vector which is represented component-wise as $x = (x_1, x_2, \ldots, x_p)$. Define the spatial domain $B = \prod_{i=1}^{p} (a_i, b_i) \subseteq \mathbb{R}^p$, where $-\infty < a_i < b_i < \infty$, for each $i = 1, 2 \ldots, p$. Let $T > 0$, and define $\Omega = B \times (0, T)$ as the space-time domain.

Throughout, let $\gamma \in \overline{\mathbb{R}^+}$ and suppose that $f : \overline{\mathbb{R}^+} \to \mathbb{R}$ is a differentiable function. Let $F : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that $\phi_0, \phi_1 : B \to \mathbb{R}$ are continuously differentiable functions. In the present chapter, we will investigate the numerical solution of the initial-boundary-value problem

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - f(t)\Delta\phi(x,t) + \gamma\frac{\partial\phi(x,t)}{\partial t} + F'(\phi(x,t)) = 0, \quad \forall (x,t)\in\Omega,$$
$$\text{such that } \begin{cases} \phi(x,0) = \phi_0(x), & \forall x\in B, \\ \dfrac{\partial\phi(x,0)}{\partial t} = \phi_1(x), & \forall x\in B, \\ \nabla\phi(x,t)\cdot\mathbf{n} = 0, & \forall (x,t)\in\partial B\times[0,T]. \end{cases} \tag{1.1.1}$$

Here, $\Delta$ represents the spatial Laplacian. Moreover, we use $\nabla$ to represent the gradient operator, and $\mathbf{n}$ is the vector which is normal to the boundary of $B$. Notice that we obtain the standard Higgs boson equation when $f(t) = e^{-2t}$, $\gamma = p$ and $F(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{q+1}\lambda|\phi|^{q+1}$, for each $t\in[0,T]$ and $\phi\in\mathbb{R}$ (see [57, 45]). In that context, the following is one of the most important results on the qualitative behavior of solutions of Higgs boson equation.

**Theorem 1.1.1** (Yagdjian [114]). *Let $2\le r<\infty$, and suppose that $\phi\in\mathcal{C}([0,\infty];L^r(\mathbb{R}^p))$ is a global weak solution of the standard Higgs boson equation in the de Sitter space-time. Suppose that the initial data satisfy*

$$\sigma\left[\left(\frac{p}{2}\sqrt{\frac{p^2}{4}+\mu^2}\right)\phi_0(x)+\phi_1(x)\right] > 0, \qquad \forall x\in\mathbb{R}^p, \tag{1.1.2}$$

*where $\sigma = 1$ (respectively, $\sigma = -1$), and that*

$$\sigma\int_{\mathbb{R}^p}|\phi(x,t)|^{q-1}\phi(x,t)dx \le 0, \tag{1.1.3}$$

*is satisfied for all $t$ outside of a sufficiently small neighborhood of 0. Then the solution $\phi$ cannot be an asymptotically time-weighted $L^q$-non-positive (respectively, -nonnegative) solution, where $a_0 = \sqrt{\frac{n^2}{4}+\mu^2} - \frac{n}{2}$, and either $a_\phi < a_0$ and $b_\phi\in\mathbb{R}$, or $a_\phi = a_0$ and $b_\phi < 2$.* □

**Corollary 1.1.2** (Yagdjian [114]). *Bubble-like solutions of the standard Higgs boson equation in the de Sitter space-time will form if the initial data satisfy the conditions of Theorem 1.1.1.*

It is well known that Higgs boson equation in the de Sitter space-time has an associated energy. More precisely, it is easy to verify that the total energy of the system (1.1.1) at the time $t$ is given by

$$\mathcal{E}(t) = e^{\gamma t}\left[\frac{1}{2}\left\|\frac{\partial\phi}{\partial t}\right\|_{x,2}^2 + \frac{\gamma}{2}\left\langle\frac{\partial\phi}{\partial t},\phi\right\rangle_{x,2} + \frac{f(t)}{2}\|\nabla\phi\|_{x,2}^2 + \langle F(\phi),1\rangle\right], \quad \forall t\in(0,T). \tag{1.1.4}$$

Notice that if $\phi$ is a solution of (1.1.1) then the local energy density of the system at the point $(x,t)$ is given by

$$\mathcal{H}(x,t) = e^{\gamma t}\left[\frac{1}{2}\left|\frac{\partial\phi(x,t)}{\partial t}\right|^2 + \frac{\gamma}{2}\frac{\partial\phi(x,t)}{\partial t}\phi(x,t) + \frac{f(t)}{2}|\nabla\phi(x,t)|^2 + F(\phi(x,t))\right]. \tag{1.1.5}$$

Moreover, differentiating $\mathcal{E}$ with respect to $t$, it is easy to show that the rate of change of the energy of (1.1.1) is given by

$$\mathcal{E}'(t) = e^{\gamma t} \left[ \frac{f'(t)}{2} \|\nabla \phi\|_{x,2}^2 - \frac{\gamma}{2} \langle F'(\phi), \phi \rangle_x + \gamma \langle F(\phi), 1 \rangle_x \right], \quad \forall t \in (0,T). \tag{1.1.6}$$

Higgs boson equation in the de Sitter space-time has been extensively investigated in the literature. Indeed, some analytical results are known on the solutions of this system [114, 119, 121]. Nevertheless, we must mention that there are very few papers available in the literature which propose reliable numerical methodologies to solve Higgs boson equation in the de Sitter space-time, that are capable of preserving the energy properties of this system. Among the recent progresses in that direction, we can mention some articles which propose high-performance implementations of Runge–Kutta schemes [6]. Unfortunately, those discretizations are not capable of preserving the energy properties of the continuous model. Following some recent successful derivations of energy-conserving systems [50, 76, 111, 35, 13], we will propose a discretization of the model (1.1.1) which is capable of preserving the energy features.

## 1.2 Numerical method

Divide the interval $[0,T]$ into $K \in \mathbb{N}$ subintervals of length $\tau = T/K$. For each $i = 1,2,\ldots,p$, divide the interval $[a_i,b_i]$ into $M_i \in \mathbb{N}$ subintervals of length $h_i = (b_i - a_i)/M_i$. Obviously, the nodes of these partitions are

$$t_k = k\tau \qquad \forall k \in I_K, \tag{1.2.1}$$

$$x_{i,j} = a_i + jh_i \qquad \forall i = 1,2,\ldots,p, \ \forall j \in I_{M_i}. \tag{1.2.2}$$

Define $J = \prod_{i=1}^p I_{M_i}$ and $\overline{J} = \prod_{i=1}^p \overline{I}_{M_i+1}$. Moreover, for any multi-index $j = (j_1, j_2, \ldots, j_p) \in \overline{J}$, let us agree that $x_j = (x_{1,j_1}, x_{2,j_2}, \ldots, x_{p,j_p})$, and let $\partial J = \{j \in J : x_j \in \partial B\}$. Let $h = (h_1, h_2, \ldots, h_p)$ and $h_* = h_1 h_2 \cdots h_p$. Fix the grid $\mathcal{R}_h = \{x_j : j \in J\}$, and use $\mathcal{V}_h$ to denote the set of all real functions defined on $\mathcal{R}_h$. If $\Psi \in \mathcal{V}_h$ then we let $\Psi_j = \Psi(x_j)$, for each $j \in J$. Moreover, if $(j,k) \in J \times I_K$ then we convey that $\Phi_j^k$ is a numerical approximation to $\phi(x_j, t_k)$. Obviously, $\Phi^k = (\Phi_j^k)_{j \in J}$ is a member of $\mathcal{V}_h$, for each $k \in \overline{I}_K$. For the remainder, we set $\Phi = (\Phi^k)_{k \in \overline{I}_K}$ unless we say otherwise.

**Definition 1.2.1** (Discrete temporal operators)**.** Let $(\Psi^k)_{k \in \overline{I}_K} \subseteq \mathcal{V}_h$. Define the discrete linear temporal operators

$$\mu_t \Psi_j^k = \frac{\Psi_j^{k+1} + \Psi_j^k}{2}, \qquad \forall (j,k) \in J \times I_{K-1}, \tag{1.2.3}$$

$$\mu_t^{(1)} \Psi_j^k = \frac{\Psi_j^{k+1} + \Psi_j^{k-1}}{2}, \qquad \forall (j,k) \in J \times I_{K-1}, \tag{1.2.4}$$

$$\delta_t \Psi_j^k = \frac{\Psi_j^{k+1} - \Psi_j^k}{\tau}, \qquad \forall (j,k) \in J \times I_{K-1}. \tag{1.2.5}$$

We also define $\delta_t^{(1)} \Psi_j^k = (\mu_t \circ \delta_t) \Psi_j^{k-1}$ and $\delta_t^{(2)} \Phi_j^k = (\delta_t \circ \delta_t) \Psi_j^{k-1}$, for each $(j,k) \in J \times I_{K-1}$. On the other hand, if $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $(j,k) \in J \times I_{K-1}$, then we define the discrete nonlinear temporal operator

$$\delta_{\Psi,t} F(\Psi_j^k) = \begin{cases} \dfrac{F(\Psi_j^{k+1}) - F(\Psi_j^k)}{\Psi_j^{k+1} - \Psi_j^k}, & \text{if } \Psi_j^{k+1} \neq \Psi_j^k, \\ F'(\Psi_j^k), & \text{otherwise.} \end{cases} \tag{1.2.6}$$

**Definition 1.2.2** (Discrete spatial operators)**.** Let $(\Psi^k)_{k \in \overline{I}_K}$ be a finite sequence in $\mathcal{V}_h$. For each $i \in I_p$ and $(j,k) \in J \times I_K$, we define the discrete linear spatial operators

$$\delta_{x_i} \Psi_j^k = \frac{\Psi^k_{j_1,\ldots,j_{i-1},j_i+1,j_{i+1},\ldots,j_p} - \Psi_j^k}{h_i}, \quad \forall (j,k) \in J \times I_{K-1}, \tag{1.2.7}$$

$$\delta_{x_i}^{(1)} \Psi_j^k = \frac{\Psi^k_{j_1,\ldots,j_{i-1},j_i+1,j_{i+1},\ldots,j_p} - \Psi_{j_1,\ldots,j_{i-1},j_i-1,j_{i+1},\ldots,j_p}}{2h_i}, \quad \forall (j,k) \in J \times I_{K-1}, \tag{1.2.8}$$

$$\delta_{x_i}^{(2)} \Psi_j^k = \frac{\Psi_{j_1,\ldots,j_{i-1},j_i+1,j_{i+1},\ldots,j_p} - 2\Psi_j^k + \Psi_{j_1,\ldots,j_{i-1},j_i-1,j_{i+1},\ldots,j_p}}{h_i^2}, \quad \forall (j,k) \in J \times I_{K-1}. \tag{1.2.9}$$

Moreover, the spatial Laplacian will be approximated with a quadratic order of consistency using the discrete operator

$$\delta_x^{(2)} \Psi_j^k = \sum_{i=1}^p \delta_{x_i}^{(2)} \Psi_j^k, \quad \forall (j,k) \in J \times I_{K-1}. \tag{1.2.10}$$

Meanwhile, define $\Lambda_x \Psi_j^k = (\delta_{x_1} \Psi_j^k, \delta_{x_2} \Psi_j^k, \ldots, \delta_{x_p} \Psi_j^k)$ and $\Lambda_x^{(1)} \Psi_j^k = (\delta_{x_1}^{(1)} \Psi_j^k, \delta_{x_2}^{(1)} \Psi_j^k, \ldots, \delta_{x_p}^{(1)} \Psi_j^k)$, for each $(j,k) \in \overline{J} \times \overline{I}_K$.

Using these conventions, the numerical scheme method to solve (1.1.1) is given by the discrete system

$$\begin{gathered} (\mu_t \circ \delta_t^{(2)}) \Phi_j^k - \mu_t \left( f(t_k) \delta_x^{(2)} \Phi_j^k \right) + \gamma \delta_t \Phi_j^k + \delta_{\Phi,t} F(\Phi_j^k) = 0, \quad \forall (j,k) \in J \times I_{K-2}, \\ \text{such that } \begin{cases} \Phi_j^0 = \mu_t^{(1)} \Phi_j^0 = \phi_0(x_j), & \forall j \in J, \\ \delta_t^{(1)} \Phi_j^0 = \phi_1(x_j), & \forall j \in J, \\ (\Lambda_x^{(1)} \Phi_j^k) \cdot \mathbf{n} = 0, & \forall (j,k) \in \partial J \times \overline{I}_K. \end{cases} \end{gathered} \tag{1.2.11}$$

Associated to this finite-difference scheme, we define the discrete energy density at the point $x_j$ and time $t_k$ for each $(j,k) \in J \times I_{K-1}$ by

$$H_j^k = e^{\gamma t_k} \left[ \frac{1}{2} (\delta_t \Phi_j^k)(\delta_t \Phi_j^{k-1}) + \frac{\gamma}{2} (\delta_t \Phi_j^k)(\mu_t \Phi_j^{k-1}) - \frac{f(t_k)}{2} \Phi_j^k (\delta_x^{(2)} \Phi_j^k) + F(\Phi_j^k) \right]. \tag{1.2.12}$$

The following result is a discrete form of the product rule. Its proof is straightforward.

**Lemma 1.2.3.** *If $(\Phi^k)_{k\in\overline{I}_K}$ and $(\Psi^k)_{k\in\overline{I}_K}$ are sequences in $\mathcal{V}_h$ then $\delta_t(\Phi_j^k\Psi_j^k)=\Phi_j^k\delta_t\Psi_j^k+\Psi_j^{k+1}\delta_t\Phi_j^k$, for each $k\in I_{K-1}$.* □

**Lemma 1.2.4.** *Suppose that $f:[0,T]\to\mathbb{R}$ is nonnegative, and let $(\Phi^k)_{k\in\overline{I}_K}$ and $(\Psi^k)_{k\in\overline{I}_K}$ be sequences in $\mathcal{V}_h$. Then*

(a) $\langle\mu_t\delta_t^{(2)}\Phi^k,\delta_t\Phi^k\rangle=\frac{1}{2}\delta_t\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle$, *for each $k\in I_{K-2}$.*

(b) $\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle=\mu_t\|\delta_t\Phi^{k-1}\|_2^2-\frac{1}{2}\tau^2\|\delta_t^{(2)}\Phi^k\|_2^2$, *for each $k\in I_{K-1}$.*

(c) $\langle\delta_{\Phi,t}F(\Phi^k),\delta_t\Phi^k\rangle=\delta_t\langle F(\Phi^k),1\rangle$, *for each $k\in I_{K-1}$.*

(d) $\delta_t\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle=\langle\delta_t\Phi^{k+1},\delta_t^{(1)}\Phi^k\rangle+\langle\delta_t^{(2)}\Phi^k,\mu_t\Phi^{k-1}\rangle$, *for each $k\in I_{K-2}$.*

(e) *For each $k\in I_{K-1}$ and $i=1,2,\ldots,p$,*

$$2\left\langle-\mu_t\left(f(t_k)\delta_{x_i}^{(2)}\Phi^k\right),\delta_t\Phi^k\right\rangle=\delta_t\left(f(t_k)\|\delta_{x_i}\Phi^k\|_2^2\right)-(\delta_tf(t_k))\langle\delta_{x_i}\Phi^{k+1},\delta_{x_i}\Phi^k\rangle. \tag{1.2.13}$$

*Proof.* The identities (a)–(c) are straightforward, so we will only establish (d) and (e). To prove (d), notice that

$$\begin{aligned}\delta_t\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle=\frac{1}{\tau}\Big[&\langle\delta_t\Phi^{k+1},\mu_t\Phi^k\rangle-\langle\delta_t\Phi^{k+1},\mu_t\Phi^{k-1}\rangle\\&+\langle\delta_t\Phi^{k+1},\mu_t\Phi^k\rangle-\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle\Big]\end{aligned} \tag{1.2.14}$$

holds for each $k\in I_{K-2}$. Identity (d) readily follows after an algebraic simplification. To establish (e) now, fix the value of $i\in\{1,2,\ldots,p\}$. Using the definitions of the discrete operators, the distributivity property of the inner product and the square-root properties of fractional-ordered centered differences, we obtain

$$\begin{aligned}\langle-\mu_t(f(t_k)\delta_{x_i}^{(2)}\Phi^k),\delta_t\Phi^k\rangle&=\frac{1}{2\tau}\Big[f(t_{k+1})\langle-\delta_{x_i}^{(2)}\Phi^{k+1},\Phi^{k+1}\rangle\\&\quad-f(t_{k+1})\langle-\delta_{x_i}^{(2)}\Phi^{k+1},\Phi^k\rangle+f(t_k)\langle-\delta_{x_i}^{(2)}\Phi^k,\Phi^{k+1}\rangle-f(t_k)\langle-\delta_{x_i}^{(2)}\Phi^k,\Phi^k\rangle\Big]\\&=\frac{1}{2\tau}\Big[f(t_{k+1})\|\delta_{x_i}\Phi^{k+1}\|_2^2-f(t_k)\|\delta_{x_i}\Phi^k\|_2^2+(f(t_k)-f(t_{k+1}))\langle\delta_{x_i}\Phi^{k+1},\delta_{x_i}\Phi^k\rangle\Big],\end{aligned} \tag{1.2.15}$$

for each $k\in I_{K-1}$, which leads to (e). The identities of this result have been readily established now. □

**Lemma 1.2.5.** *Let $(\Phi^k)_{k\in\overline{I}_K}$ be any sequence in $\mathcal{V}_h$. For each $k\in I_{K-2}$, the following identities are satisfied:*

$$\frac{1}{2}\delta_t\left[e^{\gamma t_k}\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle\right]=\frac{1}{2}\delta_te^{\gamma t_k}\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle+e^{\gamma t_{k+1}}\langle\mu_t\delta_t^{(2)}\Phi^k,\delta_t\Phi^k\rangle, \tag{1.2.16}$$

$$\delta_t\left[e^{\gamma t_k}\langle F(\Phi^k),1\rangle\right]=\delta_te^{\gamma t_k}\langle F(\Phi^k),1\rangle+e^{\gamma t_{k+1}}\langle\delta_{\Phi,t}F(\Phi^k),\delta_t\Phi^k\rangle, \tag{1.2.17}$$

| Parameter | $B$ | $\gamma$ | $\lambda$ | $\mu$ | $p$ | $q$ | $\tau$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| Value | $(-1,1)\times(-1,1)$ | 2 | 0.1 | 0.1 | 2 | 3 | 0.001 | 0.01 |
| Function | $f(t)=e^{-2t}$ | | | | $F(\phi)=-\frac{1}{2}\mu^2\phi^2+\frac{1}{q+1}\lambda\vert\phi\vert^{q+1}$ | | | |

**Table 1.1** Table of the values of the parameters and the expressions of the functions employed to obtain the simulations of Section 1.3.

$$\frac{\gamma}{2}\delta_t\left[e^{\gamma t_k}\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle\right]=\frac{\gamma}{2}\delta_t e^{\gamma t_k}\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle+\frac{\gamma}{2}e^{\gamma t_{k+1}}\langle\delta_t^{(2)}\Phi^{k+1},\mu_t\Phi^k\rangle \\ +\frac{\gamma}{2}e^{\gamma t_{k+1}}\langle\delta_t\Phi^k,\delta_t^{(1)}\Phi^k\rangle, \tag{1.2.18}$$

*and*

$$\frac{1}{2}\delta_t\left[e^{\gamma t_k}f(t_k)\|\Lambda_x\Phi^k\|_2^2\right]=\frac{1}{2}\delta_t e^{\gamma t_k}f(t_k)\|\Lambda_x\Phi^k\|_2^2+\frac{1}{2}e^{\gamma t_{k+1}}\delta_t f(t_k)\langle\Lambda_x\Phi^{k+1},\Lambda_x\Phi^k\rangle \\ -e^{\gamma t_{k+1}}\langle\mu_t[f(t_k)\delta_x^{(2)}\Phi^k],\delta_t\Phi^k\rangle. \tag{1.2.19}$$

*Proof.* The proofs of all these identities are established using Lemmas 1.2.4 and 1.2.3. □

**Theorem 1.2.6.** *Let* $\Phi$ *be a solution of* (1.2.11)*, and define the discrete energy*

$$E^k=h_*\sum_{j\in J}H_j^k=e^{\gamma t_k}\left[\frac{1}{2}\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle+\frac{\gamma}{2}\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle+\frac{f(t_k)}{2}\|\Lambda_x\Phi^k\|_2^2+\langle F(\Phi^k),1\rangle\right], \tag{1.2.20}$$

*for each* $k\in I_{K-1}$*. Then the following identity holds, for each* $k\in I_{K-2}$*:*

$$\delta_t E^k=-\gamma e^{\gamma t_{k+1}}\|\delta_t\Phi^k\|_2^2 \\ +\frac{e^{\gamma t_{k+1}}}{2}\left[\gamma\langle\delta_t^{(2)}\Phi^{k+1},\mu_t\Phi^k\rangle+\gamma\langle\delta_t\Phi^k,\delta_t^{(1)}\Phi^k\rangle+(\delta_t f(t_k))\langle\Lambda_x\Phi^{k+1},\Lambda_x\Phi^k\rangle\right] \\ +\frac{\delta_t e^{\gamma t_k}}{2}\left[\langle\delta_t\Phi^k,\delta_t\Phi^{k-1}\rangle+\gamma\langle\delta_t\Phi^k,\mu_t\Phi^{k-1}\rangle+f(t_k)\|\Lambda_x\Phi^k\|_2^2+2\langle F(\Phi^k),1\rangle\right]. \tag{1.2.21}$$

*Proof.* The proof readily follows from the identities in the lemma and straightforward algebraic calculations. □

## 1.3 Results

We restrict our attention to the two-dimensional scenario. Throughout, we will use the parameters in Table 1.1. For each $x_0\in\mathbb{R}^2$ and $R>0$, define $\mathrm{B}(\cdot;x_0,R):\mathbb{R}^2\to\mathbb{R}$ by $\mathrm{B}(x;x_0,R)=0$ if $\|x-x_0\|_2\geq R$, and by

$$\mathrm{B}(x;x_0,R)=\exp\left(\frac{1}{R^2}-\frac{1}{R^2-\|x-x_0\|_2^2}\right),\quad \text{if } \|x-x_0\|_2<R. \tag{1.3.1}$$

Define $x_1 = (0.5, 0)$, $x_2 = (-0.5, 0)$, $x_3 = (0, -0.5)$ and $x_4 = (0, 0.5)$ of $\mathbb{R}^2$. We consider the problem (1.1.1) with

$$\begin{cases} \phi_0(x) = \mathrm{B}(x; x_1, 0.3) + \mathrm{B}(x; x_2, 0.3) + \mathrm{B}(x; x_3, 0.3) + \mathrm{B}(x; x_4, 0.3), & \forall x \in B, \\ \phi_1(x) = -5\phi_0(x), & \forall x \in B. \end{cases} \tag{1.3.2}$$

Figure 1.1 shows some snapshots of the solutions of (1.1.1). The results show how the four localized initial bubbles interact and generate a complex behavior. Eventually, new bubbles begin to form, in agreement with the qualitative behavior witnessed using some standard Runge–Kutta methods [6]. For convenience, Figure 1.2 shows the dynamics of the total energy of the system for two, four and eight initial bubbles, using $T = 1$. The graph shows that the total energy of the Higgs boson equation is decreasing, in agreement with the theory on this continuous model.

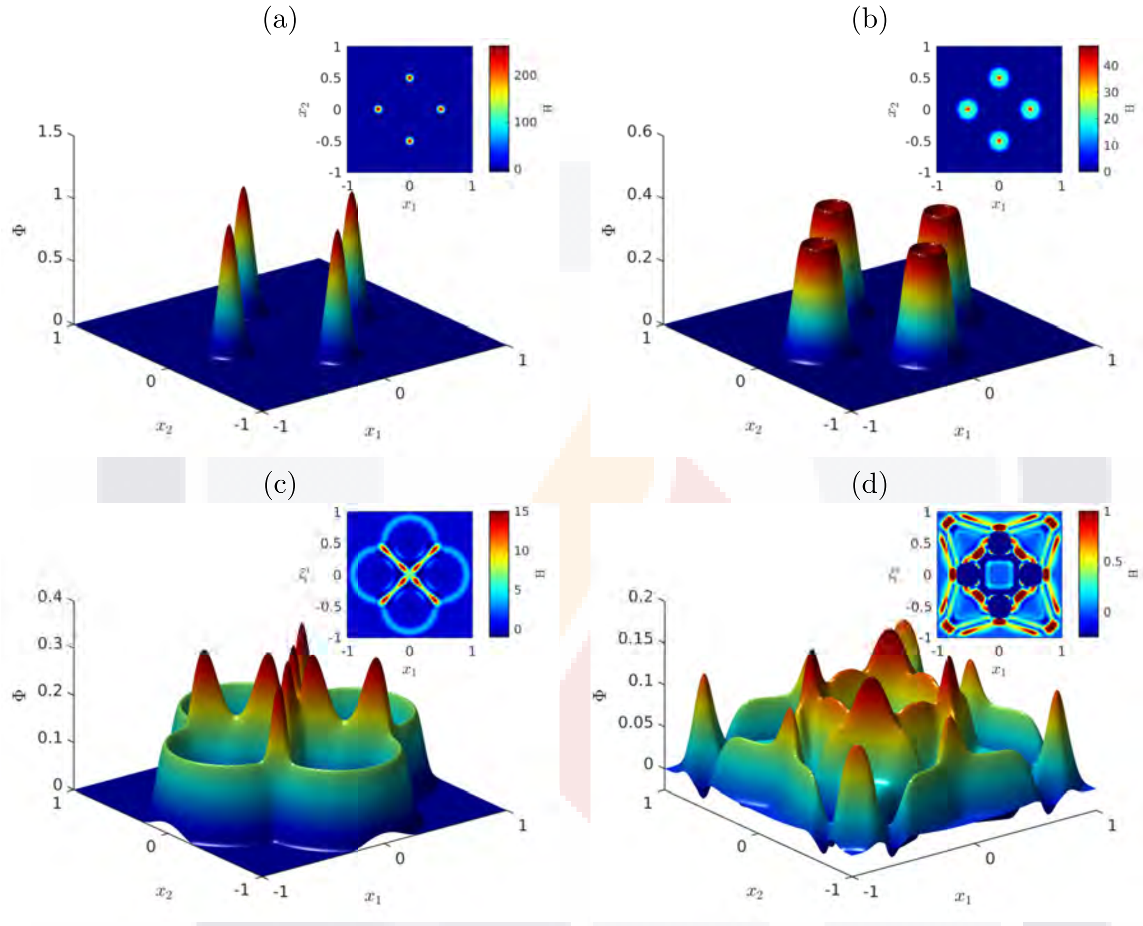


**Figure 1.1** Graphs of the approximate solution of (1.1.1) versus $(x_1, x_2) \in [-1,1] \times [-1,1]$, obtained using the finite-difference scheme (1.2.11). Various times were employed, namely, (a) 0, (b) 0.1, (c) 0.5 and (d) 2. Throughout, we used the parameters in Table 1.1, and the initial conditions (1.3.2). Also, we let $f(t) = e^{-2t}$, $\gamma = p$ and $F(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{q+1}\lambda|\phi|^{q+1}$, for each $t \in [0,T]$ and $\phi \in \mathbb{R}$. The insets provide the corresponding interpolated checkerboard plots of the local energy densities.

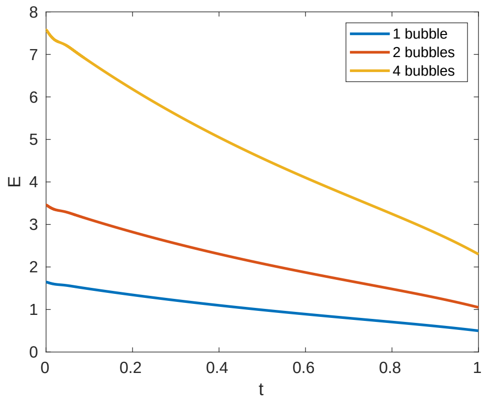


**Figure 1.2** Graphs of the total energy of the system (1.1.1) versus $t$, obtained using the finite-difference scheme (1.2.11). Throughout, we used the parameters in Table 1.1, and the initial conditions (1.3.2). Also, we let $f(t) = e^{-2t}$, $\gamma = p$ and $F(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{q+1}\lambda|\phi|^{q+1}$, for each $t \in [0,T]$ and $\phi \in \mathbb{R}$. The graphs correspond to initial data with one bubble-type initial condition (blue), two bubbles (red) and four bubbles (yellow).

# 2 A radially symmetric Higgs boson equation

*THE PRESENT CHAPTER* introduces a numerical scheme that preserves the dissipation of energy of the Higgs boson equation in the de Sitter space-time. More precisely, the model considered in this chapter is a mathematical generalization of Higgs' model which includes a general time-dependent diffusion coefficient and a generalized potential. The mathematical system is dissipative, and we propose an implicit discrete method which approximates consistently the radially symmetry solutions of the continuous system. At the same time, a discrete energy functional is presented, and we prove that, as its continuous counterpart, the numerical technique dissipates the energy of the discrete system. The properties of consistency, stability and convergence of the numerical model are proved rigorously. To confirm the theoretical results, we approximate some radially symmetric solutions of the classical Higgs boson equation in the de Sitter space-time. In particular, the numerical results confirm the stability and the formation of bubble-like solutions.

## 2.1 Background

The Higgs boson equation is a fundamental system in the unification of various theories in physics. Indeed, Higgs boson equation in the Minkowski and in the de Sitter space-times are employed to unify the theories on weak, strong and electromagnetic interactions [113]. Since the publication of the report confirming experimentally the existence of Higgs' boson [2], this model has been studied extensively mainly from the physical point of view. Indeed, there are many physical studies focusing on the phenomenological investigation of Higgs boson [14, 26], some of them making theoretical advances within the frame of the Standard Model of particle physics [10]. Some other recent experimental and theoretical studies concentrate on observing the decay of Higgs boson to bottom quarks [98], on its experimental production at hadron colliders using high-energy physics [78] or on next-to-next-to-leading order corrections with top quark mass effects [41]. Other works provide experimental evidence for the Higgs boson decay to a bottom quark–antiquark pair [96], or investigate second-order quantum chromodynamics effects in Higgs boson production through vector boson fusion [19], highly boosted Higgs bosons decaying to bottom quark-antiquark pairs [97] or the decaying of the Higgs boson to charm quarks [1]. Moreover, the investigation of the mass of this particle is also an active topic of research, as shown by reports on the measurement of the Higgs boson mass in the diphoton decay channel [99], the reconciliation of the Effective Field Theory and hybrid calculations of the light Minimal Supersymmetric Standard Model Higgs-boson mass [5, 28], among various other articles [74].

From the mathematical point of view, the investigation of Higgs boson equation in the Minkowski and the de Sitter space-times has proved to be a hard task. There are some recent works which investigate the behavior of the global solutions of these models from a rigorous point of view. As an example, there are reports which provide sufficient conditions for the existence of the zeros of global solutions [114]. As a consequence, some conditions for the creation of the so-called "bubbles" have been established therein. Here, it is important to point out that those "bubbles" are of considerable theoretical interest in particle physics and inflationary cosmology [60, 110]. Additionally, some conditions for the global solutions to be oscillatory in time have been thoroughly proven. However, to the best of our knowledge the proof of the existence of global solutions for this model is still an open problem of research to-day. Nevertheless, some other efforts have been reported in the mathematical literature, like some analytical results on semilinear hyperbolic partial differential equations in curved space-times [117], a maximum principle and sign changing solutions of hyperbolic equations with the Higgs potential [119], the Huygens' principle for the Klein-Gordon equation in the de Sitter space-time [116], the global existence of the self-interacting scalar field in the de Sitter universe [118] and global solutions of semilinear system of Klein-Gordon equations in the de Sitter space-time [115], among other important reports by K. Yagdjian and coworkers.

In view of the lack of a mathematically rigorous apparatus to elucidate the existence of global-in-time solutions of the Higgs boson equation in the de Sitter space-time, some reports have turned the attention to the numerical investigation of this model [6, 107]. However, it is worth pointing out here that most of those reports employ discretizations to solve first-order systems that approximate partial differential equations, like commercial Runge–Kutta methods which are already built in Matlab® or Mathematica®. Motivated by these limitations, the authors of the present work have devoted some efforts to propose and analyze a discretization of the Higgs boson equation in the de Sitter space-time which are capable of resembling the Hamiltonian structure of the problem. Historically, the problem of designing energy-conserving methods may date back to the decade of the 1970s [4, 95]. However, it is worth mentioning that L. Vázquez and coauthors were probably the first researchers who pointed out the physical and mathematical significance of designing this type of schemes [85]. Various seminal papers by Vázquez and his coworkers were published in the 1990s, including various energy-conserving numerical schemes to solve partial differential equations like the Schrödinger equation [102], the sine-Gordon equation [9, 30], Klein–Gordon equations [100], and even systems consisting of ordinary differential equations [29]. In those papers, the authors established thoroughly the capability of their schemes to preserve the energy properties of the continuous problem. They employed a discrete form of the energy method to establish rigorously the stability and the convergence properties of the schemes. After the publication of those works, the investigation on energy-conserving schemes became a highly transited route of research, and many interesting articles were proposed in the specialized literature [46, 56, 91]. It is worth mentioning here that Vázquez and coworkers still continue to propose contributions in various areas of the physical sciences [93, 94, 108], including some recent progresses in research related to the future missions to Mars [90, 51, 109].

Some reports are already available in the literature to solve the Higgs boson equation in the de Sitter space-time using Hamiltonian discretizations [79, 80]. Those articles show various limitations, including that the methods proposed are practically impossible to be extended to the three-dimensional scenario. In the present chapter, we will propose and theoretically analyze a Hamiltonian finite-difference scheme to solve our mathematical model in the three-dimensional case. We will limit our attention to the approximation of radially symmetric solutions, whence a convenient simplification of the mathematical model will be readily at hand. A sufficiently general form of Higgs boson equation in the de Sitter space-time will be considered, and an energy functional is available for that system. An obvious advantage of studying radial solutions will be that a reduced $(1+1)$-dimensional system will result [100], though the solutions are obviously valid for the three-dimensional case. Moreover, the new $(1+1)$-dimensional system will present a singularity at the origin, whence the stability and the convergence analysis of the discrete model proposed will be a mathematical challenge that will be solved in this paper. We will prove thoroughly the dissipation properties of the finite-difference scheme, and the quadratic order of convergence of the discrete model will be rigorously established. Some numerical simulations will be provided to show the capability of the scheme to approximate the radially symmetric solutions of the Higgs boson equation in the de Sitter space-time, and the presence of bubble-like solutions will be shown in that context. The advantage of our approach is obviously the simplicity of the discretization. In addition, the numerical scheme proposed in this chapter is uniquely solvable, preserves the dissipation of the energy, and is numerically efficient (that is, it is consistent, stable and convergent). We will provide a computational code in detail at the end of this chapter for the sake of convenience.

## 2.2 Physical model

For the remainder of this chapter and unless we mention otherwise, we consider an open and bounded spatial domain $D \subseteq \mathbb{R}^3$, and a temporal interval $[0,T]$, for some $T > 0$. We define the space-time domain $\Omega = D \times (0,T)$, and use $\overline{D}$ and $\overline{\Omega}$ to denote, respectively, the closures of $D$ and $\Omega$ in the standard topology of $\mathbb{R}^4$. Let $f:[0,T]\to\mathbb{R}$ and $F:\mathbb{R}\to\mathbb{R}$ be differentiable functions, and assume that $\phi_0,\phi_1:D\to\mathbb{R}$ are continuously differentiable. The point of departure of our investigation is the following initial-boundary-value problem, which is governed by a general form of the Higgs boson equation in the de Sitter space-time:

$$\begin{aligned}&\frac{\partial^2\phi(x,t)}{\partial t^2}-f(t)\Delta\phi(x,t)+3\frac{\partial\phi(x,t)}{\partial t}+F'(\phi(x,t))=0, \quad \forall(x,t)\in\Omega,\\ &\text{such that }\begin{cases}\phi(x,0)=\phi_0(x), & \forall x\in D,\\ \dfrac{\partial\phi(x,0)}{\partial t}=\phi_1(x), & \forall x\in D,\\ \phi(x,t)=0, & \forall(x,t)\in\partial D\times[0,T].\end{cases}\end{aligned} \tag{2.2.1}$$

In the initial-boundary-value problem (2.2.1), the symbol $\Delta$ represents the standard Laplacian in the three spatial variables. Meanwhile, $\nabla$ will represent the gradient operator in the spatial coordinates, and $\partial D$ denotes the boundary of $D$. From a variational point of view, the total energy associated to the system (2.2.1) is given by (see [79])

$$\mathcal{E}(t)=e^{3t}\int_{\overline{D}}\left[\frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2+\frac{3}{2}\frac{\partial\phi}{\partial t}\phi+\frac{f(t)}{2}|\nabla\phi|^2+F(\phi)\right]dx, \quad \forall t\in(0,T). \tag{2.2.2}$$

In particular, the energy density is given by the Hamiltonian function

$$\mathcal{H}(x,t)=e^{3t}\left[\frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2+\frac{3}{2}\frac{\partial\phi}{\partial t}\phi+\frac{f(t)}{2}|\nabla\phi|^2+F(\phi)\right], \quad \forall(x,t)\in\Omega. \tag{2.2.3}$$

Moreover, differentiating (2.2.2) with respect to $t$, it is easy to show that the rate of change of the energy of the initial-boundary-value problem (2.2.1) is given by

$$\mathcal{E}'(t)=e^{3t}\int_{\overline{D}}\left[\frac{f'(t)}{2}|\nabla\phi|^2-\frac{3}{2}F'(\phi)\phi+3F(\phi)\right]dx, \quad \forall t\in(0,T). \tag{2.2.4}$$

It is worth pointing out here that if $f(t)=e^{-2t}$ and $F(\phi)=-\frac{1}{2}\mu^2\phi^2+\frac{1}{4}\lambda\phi^4$, then we readily obtain the classical expression of the Higgs boson equation in the de Sitter space-time [45, 57]. In that context, the following is one of the most important results on the qualitative behavior of the solutions of Higgs' equation.

**Theorem 2.2.1** (Yagdjian [114]). *Assume that $f(t)=e^{-2t}$ and $F(\phi)=-\frac{1}{2}\mu^2\phi^2+\frac{1}{4}\lambda\phi^4$. Let $2\le c<\infty$, and suppose that $\phi\in\mathcal{C}([0,\infty];L^c(\mathbb{R}^3))$ is a global weak solution of the standard Higgs boson equation in the de Sitter space-time. Suppose that the initial data satisfy*

$$\sigma\left[\left(\frac{3}{2}+\sqrt{\frac{9}{4}+\mu^2}\right)\phi_0(x)+\phi_1(x)\right]>0, \qquad \forall x\in\mathbb{R}^3, \tag{2.2.5}$$

*where $\sigma = 1$ (respectively, $\sigma = -1$), and that*

$$\sigma \int_{\mathbb{R}^p} |\phi(x,t)|^{q-1}\phi(x,t)\,dx \leq 0, \tag{2.2.6}$$

*is satisfied for all $t$ outside of a sufficiently small neighborhood of 0. Then the solution $\phi$ cannot be an asymptotically time-weighted $L^q$-non-positive (respectively, -nonnegative) solution with the weighted function $\nu_\phi(t) = e^{a_\phi t} t^{b_\phi}$, where either $a_\phi < 2\sqrt{\frac{9}{4} + \mu^2} - 3$ and $b_\phi \in \mathbb{R}$, or $a_\phi = 2\sqrt{\frac{9}{4} + \mu^2} - 3$ and $b_\phi < 2$.* □

**Corollary 2.2.2** (Yagdjian [114])**.** *Bubble-like solutions of the standard Higgs boson equation in the de Sitter space-time will form if the initial data satisfy the conditions of Theorem 2.2.1.* □

Bubble-like solutions of the Higgs boson equation are radially symmetric functions. Physically, this fact is justified partly due to the cosmological principle, which assumes that the Universe is homogeneous and isotropic on large scales. Based on these facts, we will let $r = \|x\|$, where $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^3$ of $x$, and we will suppose that $\phi = \phi(r,t)$ is a radially symmetric solution of the system (2.2.1). Moreover, we will assume that $D$ is the open sphere in $\mathbb{R}^3$ with center at the origin and radius equal to $L > 0$. Under these circumstances, the nonlinear partial differential equation of (2.2.1) is transformed into the following model, assuming that the solutions possess radial symmetry:

$$\frac{\partial^2\phi}{\partial t^2} - f(t)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right) + 3\frac{\partial\phi}{\partial t} + F'(\phi) = 0, \quad \forall (r,t) \in (0,L)\times(0,T). \tag{2.2.7}$$

Let us consider the transformation $\psi(r,t) = r\phi(r,t)$. It is obvious that $\psi(0,t) = 0$, for each $t \in [0,t]$. Also, the following identities trivially hold, for each $0 < r < L$:

$$\frac{1}{r}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\phi}{\partial t^2}, \tag{2.2.8}$$

$$\frac{1}{r}\frac{\partial^2\psi}{\partial r^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}. \tag{2.2.9}$$

As a consequence, the initial-boundary-value problem (2.2.1) is equivalent to

$$\begin{aligned} &\frac{\partial^2\psi}{\partial t^2} - f(t)\frac{\partial^2\psi}{\partial r^2} + 3\frac{\partial\psi}{\partial t} + rF'(\psi/r) = 0, \quad \forall (r,t) \in (0,L)\times[0,T], \\ &\text{such that } \begin{cases} \psi(r,0) = r\phi_0(r), & \forall r \in (0,L), \\ \dfrac{\partial\psi}{\partial t}(r,0) = r\phi_1(r), & \forall r \in (0,L), \\ \psi(0,t) = \psi(L,t) = 0, & \forall t \in [0,T]. \end{cases} \end{aligned} \tag{2.2.10}$$

In turn, the expression of the total energy for the system (2.2.10) in terms of the new scalar function $\psi$ is provided by the following result.

**Theorem 2.2.3.** *If $\phi$ be a radially symmetric solution of* (2.2.1) *and* $\psi(r,t)=r\phi(r,t)$, *for each* $(r,t)\in\overline{\Omega}$, *then the total energy of the system at the time* $t\in[0,T]$ *is given by*

$$\mathcal{E}(t)=4\pi e^{3t}\int_0^L\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2+\frac{3}{2}\frac{\partial\psi}{\partial t}\phi+\frac{f(t)}{2}\left(\frac{\partial\psi}{\partial r}\right)^2+r^2F(\psi/r)\right]dr. \tag{2.2.11}$$

*Proof.* Recall firstly that the expression for the gradient in radially symmetric coordinates is given by $\nabla\phi(r,t)=\partial\phi/\partial r$, and $\psi=r\phi$. Thus, the energy equation (2.2.2) is transformed into

$$\mathcal{E}(t)=e^{3t}\iiint_D\left\{\frac{1}{2}\left[\frac{\partial\psi}{\partial t}\right]^2+\frac{3}{2}\frac{\partial\psi}{\partial t}\psi+\frac{f(t)}{2}\left[\frac{\partial\psi}{\partial r}\right]^2+r^2F(\psi/r)\right\}\sin\Xi dr d\Xi d\Theta, \tag{2.2.12}$$

for each $t\in[0,T]$. The conclusion of this result readily follows now. □ □

In light of this theorem, the energy density of the model (2.2.10) at the point $(r,t)$ is

$$\mathcal{H}(r,t)=4\pi e^{3t}\left\{\frac{1}{2}\left[\frac{\partial\psi}{\partial t}\right]^2+\frac{3}{2}\frac{\partial\psi}{\partial t}\psi+\frac{f(t)}{2}\left[\frac{\partial\psi}{\partial r}\right]^2+r^2F(\psi/r)\right\}. \tag{2.2.13}$$

In similar fashion, we apply a change of variables in (2.2.4) to obtain the expression for the rate of change of energy of the system (2.2.10) at time $t$. One can readily establish that

$$\mathcal{E}'(t)=4\pi e^{3t}\int_0^L\left\{\frac{f'(t)}{2}\left[\frac{\partial\psi}{\partial r}\right]^2-\frac{3}{2}rF'(\psi/r)\psi+3r^2F(\psi/r)\right\}dr. \tag{2.2.14}$$

## 2.3 Numerical method

To approximate numerically the solution of the model (2.2.10) over the space-time domain $\overline{\Omega}$, we will follow a finite-difference approach and let $M,N\in\mathbb{N}$. To that end, let us fix a regular partition of the interval $[0,L]$, of the form $0=r_0<r_1<\cdots<r_M=L$. Also, take a regular partition of $[0,T]$, of the form $0=t_0<t_1<\cdots<t_N=T$. Define the spatial and temporal partition norms, respectively, by $h=L/M$ and $\tau=T/N$.

For the sake of convenience, let $I_n=\{0,1,\ldots,n\}$ and $\overline{I}_n=I_n\cup\{-1\}$, for each $n\in\mathbb{N}$. In this chapter, we use the symbol $\mathcal{V}_h$ to denote the real vector space of all real functions defined on the grid $\{r_j:j\in\overline{I}_M\}$. For the remainder of this chapter and unless we say otherwise, the numerical approximation of the value $\psi_j^n=\psi(r_j,t_n)$ will be denoted by $\Psi_j^n$, for each $(j,n)\in I_M\times I_N$. Moreover, let $\Psi^n=(\Psi_j^n)_{j\in\overline{I}_M}\in\mathcal{V}_h$, for each $n\in\overline{I}_N$. It is obvious that the finite sequence $r=(r_j)_{j\in\overline{I}_M}$ can be considered also a member of $\mathcal{V}_h$.

**Definition 2.3.1.** If $\Phi$ and $\Psi$ are any functions in $\mathcal{V}_h$, then we define component-wise the sum, the difference, the multiplication and the division (whenever they are defined) of $\Phi$

and $\Psi$. More concretely, if $*$ represents any of those operations, we let $(\Phi * \Psi)_j = \Phi_j * \psi_j$, for each $j \in \overline{I}_M$. Moreover, if $G : \mathbb{R} \to \mathbb{R}$ is any function and $\Psi \in \mathcal{V}_h$, then we also define the composition $G \circ \Psi$ component-wise, which means that $(G \circ \Psi)_j = G(\Psi_j)$, for each $j \in \overline{I}_M$.

**Definition 2.3.2** (Discrete temporal operators)**.** Let $(\Psi^n)_{n \in \overline{I}_N}$ be any finite sequence in $\mathcal{V}_h$. Define the discrete operators

$$\delta_t \Psi_j^n = \frac{\Psi_j^{n+1} - \Psi_j^n}{\tau}, \quad \forall (j,n) \in \overline{I}_M \times \overline{I}_{N-1}, \tag{2.3.1}$$

$$\delta_t^{(1)} \Psi_j^n = \frac{\Psi_j^{n+1} - \Psi_j^{n-1}}{2\tau}, \quad \forall (j,n) \in \overline{I}_M \times I_{N-1}, \tag{2.3.2}$$

$$\delta_t^{(2)} \Psi_j^n = \frac{\Psi_j^{n+1} - 2\Psi_j^n + \Psi_j^{n-1}}{\tau^2}, \quad \forall (j,n) \in \overline{I}_M \times I_{N-1}. \tag{2.3.3}$$

In addition, if $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $(j,n) \in I_M \times I_{N-1}$, then we define

$$\delta_{\Psi,t}^{(1)} F(\Psi_j^n) = \begin{cases} \dfrac{F(\Psi_j^{n+1}) - F(\Psi_j^{n-1})}{\Psi_j^{n+1} - \Psi_j^{n-1}}, & \text{if } \Psi_j^{n+1} \neq \Psi_j^{n-1}, \\ F'(\Psi_j^n), & \text{otherwise.} \end{cases} \tag{2.3.4}$$

**Definition 2.3.3** (Discrete spatial operators)**.** Let $(\Psi^n)_{n \in \overline{I}_N}$ be any finite sequence in $\mathcal{V}_h$. Define the discrete operators

$$\delta_r \Psi_j^n = \frac{\Psi_{j+1}^n - \Psi_j^n}{h}, \quad \forall (j,n) \in \overline{I}_{M-1} \times \overline{I}_N \tag{2.3.5}$$

$$\delta_r^{(1)} \Psi_j^n = \frac{\Psi_{j+1}^n - \Psi_{j-1}^n}{2h}, \quad \forall (j,n) \in I_{M-1} \times \overline{I}_N, \tag{2.3.6}$$

$$\delta_r^{(2)} \Psi_j^n = \frac{\Psi_{j+1}^n - 2\Psi_j^n + \Psi_{j-1}^n}{h^2}, \quad \forall (j,n) \in I_{M-1} \times \overline{I}_N. \tag{2.3.7}$$

Moreover, we agree that $\delta_{rt} = \delta_r \circ \delta_t$ and $\delta_{tr} = \delta_t \circ \delta_r$. Obviously, $\delta_{rt} = \delta_{tr}$.

**Definition 2.3.4.** Let $p \in [1, \infty)$. We define the inner product $\langle \cdot, \cdot \rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ and the norm $\| \cdot \|_p : \mathcal{V}_h \to \mathbb{R}$, respectively, by

$$\langle \Phi, \Psi \rangle = h \sum_{j \in I_{M-1}} \Phi_j \Psi_j, \quad \forall \Phi, \Psi \in \mathcal{V}_h, \tag{2.3.8}$$

$$\|\Phi\|_p = \left[ h \sum_{j \in I_{M-1}} |\Phi_j|^p \right]^{1/p}, \quad \forall \Phi \in \mathcal{V}_h. \tag{2.3.9}$$

The Euclidean norm induced by $\langle \cdot, \cdot \rangle$ is obviously $\| \cdot \|_2$. Meanwhile, $\| \cdot \|_\infty : \mathcal{V}_h \to \mathbb{R}$ will be the infinity norm in $\mathcal{V}_h$, which is defined as $\|\Phi\|_\infty = \max\{|\Phi_j| : j \in I_{M-1}\}$, for each $\Phi \in \mathcal{V}_h$.

In the sequel, we will restrict our attention to functions in $\mathcal{V}_h$ which vanish at $r_0$ and $r_M$.

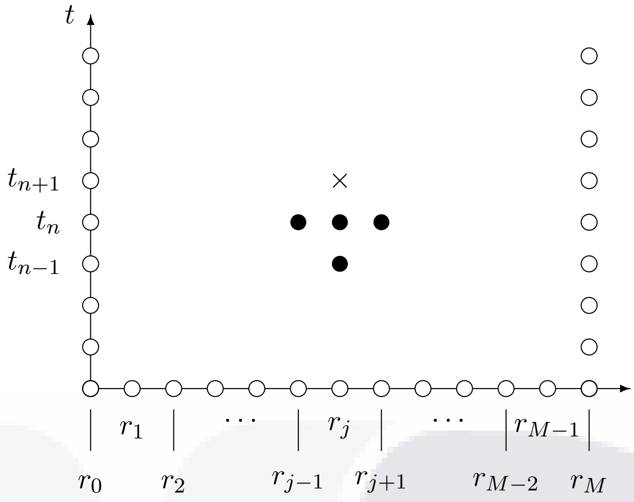


**Figure 2.1** Forward-difference stencil for the approximation to the exact solution of the mathematical model (2.2.10) at the time $t_n$, using the finite-difference scheme (2.3.11). The black circles represent the known approximations at the times $t_{n-1}$ and $t_n$, while the cross denotes the unknown approximation at the time $t_{n+1}$.

**Lemma 2.3.5.** *If $\Phi, \Psi \in \mathcal{V}_h$, then $\langle -\delta_r^2 \Phi, \Psi \rangle = \langle \Phi, -\delta_r^2 \Psi \rangle = \langle \delta_r \Phi, \delta_r \Psi \rangle$. Moreover,*

$$\|\delta_r \Phi\|_2^2 \leq 4h^{-1} \|\Phi\|_2^2, \quad \forall \Phi \in \mathcal{V}_h. \tag{2.3.10}$$

*Proof.* The proof is a particular case of Lemma 4.3 and property (C) in [72] □

Let $\phi_0$ and $\phi_1$ be smooth initial conditions, and suppose that $f$ and $F$ are differentiable functions. The scheme to approximate the solutions of the differential model (2.2.10) is given by the system of discrete equations

$$\begin{gathered}\delta_t^{(2)} \Psi_j^n - f(t_n) \delta_r^{(2)} \Psi_j^n + 3\delta_t^{(1)} \Psi_j^n + r_j \delta_{\Psi,t}^{(1)} F(\Psi_j^n / r_j) = 0, \quad \forall (j,n) \in I_{M-1} \times \overline{I}_{N-1}, \\ \text{such that} \begin{cases} \Psi_j^0 = r_j \phi_0(x_j), & \forall j \in I_{M-1}, \\ \delta_t^{(1)} \Psi_j^0 = r_j \phi_1(x_j), & \forall j \in I_{M-1}, \\ \Psi_0^n = \Psi_M^n = 0, & \forall n \in I_{N-1}. \end{cases}\end{gathered} \tag{2.3.11}$$

It is clear that this scheme is a three-step implicit discrete model. As a consequence, to solve the discrete model (2.3.11), we will require to use the Newton–Raphson method for nonlinear systems of algebraic equations. On the other hand, letting $n = 0$ in the main equation of (2.3.11), and using the condition $\delta_t^{(1)} \Psi_j^0 = r_j \phi_1(x_j)$, we readily obtain that

$$\begin{aligned}\Psi_j^1 = r_j \phi_0(x_j) + \tau r_j \phi_1(x_j) + \frac{\tau^2}{2} f(t_0) \delta_r^{(2)} (r_j \phi_0(x_j)) - \frac{3}{2} \tau^2 r_j \phi_1(x_j) \\ - \tau r_j \frac{F(\Psi_j^1 / r_j) - F(\Psi_j^1 / r_j - 2\tau \phi_1(x_j))}{4\phi_1(x_j)}, \quad \forall j \in I_{M-1}.\end{aligned} \tag{2.3.12}$$

These relations are employed to obtain the first numerical approximations of the finite-difference scheme. In such way, the fictitious approximations at the time $t_{-1}$ are not employed in the computational implementation of the numerical method. For the sake of convenience, Figure 2.1 shows the stencil of the implicit method (2.3.11) (see [69]).

The following proposition will be used to prove the existence of solutions of (2.3.11).

**Lemma 2.3.6** (Brouwer's fixed-point theorem)**.** *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathcal{V}$. Assume that $r:\mathcal{V}\to\mathcal{V}$ is continuous, and that there exists $\lambda>0$ such that $\langle r(\Phi),\Phi\rangle\geq 0$, for all $\Phi\in\mathcal{V}$ with $\|\Phi\|=\lambda$. Then there exists $\Phi\in\mathcal{V}$ with $\|\Phi\|\leq\lambda$, such that $r(\Phi)=0$.* □

Our first step will be to prove the existence of solutions of the finite-difference scheme (2.3.11). To that end, let $\Phi\in\mathcal{V}_h$, and suppose that $(\Psi^n)_{n\in\overline{I}_N}$ is a finite sequence in $\mathcal{V}_h$. In the next result, we will require to use the following nonlinear operator:

$$\delta^{(1)}_{\Phi,\Psi,t}F^n_j=\begin{cases}\dfrac{F(\Phi_j)-F(\Psi^{n-1}_j)}{\Phi_j-\Psi^{n-1}_j}, & \text{if } \Phi_j\neq\Psi^{n-1}_j,\\ F'(\Psi^n_j), & \text{otherwise.}\end{cases}\tag{2.3.13}$$

**Theorem 2.3.7** (Existence of solutions)**.** *Suppose that there exists $K_1\geq 0$ such that $|f(t)|\leq K_1$, for all $t\in[0,T]$. If $F'\in L^\infty(\mathbb{R})$, then the finite-difference scheme* (2.3.11) *is solvable for any set of initial conditions.*

*Proof.* Notice that $\Psi^0$ and $\Psi^1$ are defined through the initial data. So let $n\in I_{N-1}$, and assume that we have been already calculated the solutions $\Psi^{n-1}$ and $\Psi^n$ at the times $t_{n-1}$ and $t_n$, respectively. The assumption on the regularity of $F$ assures then that there exists a constant $K_2\geq 0$, such that $\|\delta^{(1)}_{\Phi,\Psi,t}F^n\|_2\leq K_2$. Let $s:\mathcal{V}_h\to\mathcal{V}_h$ be the continuous function whose $j$th component $s_j:\mathcal{V}_h\to\mathbb{R}$ is defined by

$$s_j(\Phi)=\frac{\Phi_j-2\Psi^n_j+\Psi^{n-1}_j}{\tau^2}-f(t_n)\delta^{(2)}_r\Psi^n_j+3\frac{\Phi_j-\Psi^{n-1}_j}{2\tau}+r_j\delta^{(1)}_{\Psi,\Phi,t}F^n_j,\tag{2.3.14}$$

for each $\Phi\in\mathcal{V}_h$. Applying the Cauchy–Schwarz inequality, using the inequality in Lemma 2.3.5 along with the square-root property stated in that result, we obtain that

$$\begin{aligned}\langle s(\Phi),\Phi\rangle &\geq \frac{1}{\tau^2}\left(\|\Phi\|_2^2-2\|\Psi^n\|_2\|\Phi\|_2-\|\Psi^{n-1}\|_2\|\Phi\|_2\right)-f(t_n)\langle\delta^{(1)}_r\Psi^n,\delta^{(1)}_r\Phi\rangle\\ &\quad+\frac{3}{2\tau}\left(\|\Phi\|_2^2-\|\Psi^{n-1}\|_2\|\Phi\|_2\right)-\|r\delta^{(1)}_{\Phi,\Psi,t}F^n\|_2\|\Phi\|_2\\ &\geq\frac{\|\Phi\|_2}{2\tau^2}\left[(2+3\tau)\|\Phi\|_2-4\|\Psi^n\|_2-(2+3\tau)\|\Psi^{n-1}\|_2-2LK_2\tau^2\right]\\ &\quad-K_1\|\delta^{(1)}_r\Psi^n\|_2\|\delta^{(1)}_r\Phi\|_2\\ &\geq\frac{2+3\tau}{2\tau^2}\|\Phi\|_2\left[\|\Phi\|_2-\lambda\right],\end{aligned}\tag{2.3.15}$$

for each $\Phi \in \mathcal{V}_h$. Here, we used the constant

$$\lambda = \frac{(4h+8\tau^2K_1)\,h^{-1}\|\Psi^n\|_2 + (2+3\tau)\|\Psi^{n-1}\|_2 + 2LK_2\tau^2}{2+3\tau} > 0. \tag{2.3.16}$$

Notice that $\langle s(\Phi),\Phi\rangle \geq 0$ is satisfied, for each $\Phi \in \mathcal{V}_h$ with $\|\Phi\|_2 = \lambda$. By Lemma 2.3.6, there exists $\Psi^{n+1} \in \mathcal{V}_h$ with $\|\Psi^{n+1}\|_2 \leq \lambda$, such that $s(\Psi^{n+1}) = 0$. This means that $\Psi^{n+1}$ is a solution of the $n$th recursive equation in (2.3.11). The theorem follows now by induction. □

**Definition 2.3.8.** Define the approximation of the Hamiltonian (2.2.13) at the point $(x_j,t_n)$ for each $(j,n) \in I_{M-1} \times \overline{I}_{N-1}$ by

$$H_j^n = 4\pi e^{3t_n}\left[\frac{1}{2}(\delta_t\Psi_j^n)^2 + \frac{3}{2}\delta_t\Psi_j^n\mu_t\Psi_j^n + \frac{1}{2}\mu_t f(t_n)\delta_r\Psi_j^{n+1}\delta_r\Psi_j^n + r_j^2\mu_t F(\Psi_j^n/r_j)\right]. \tag{2.3.17}$$

Additionally, the discrete total energy at time $t_n$ is defined as

$$\begin{aligned} E^n = 4\pi e^{3t_n}\Bigg[&\frac{1}{2}\|\delta_t\Psi^n\|_2^2 + \frac{3}{2}\langle\delta_t\Psi^n,\mu_t\Psi^n\rangle \\ &+\frac{1}{2}\mu_t f(t_n)\langle\delta_r\Psi^{n+1},\delta_r\Psi^n\rangle + \langle\mu_t F(\Psi^n/r),r^2\rangle\Bigg], \quad \forall n \in \overline{I}_{N-1} \end{aligned} \tag{2.3.18}$$

We will require the following technical result to calculate the discrete rate of change of the total energy of the finite-difference system (2.3.11).

**Lemma 2.3.9.** *Let $F:\mathbb{R}\to\mathbb{R}$ be nonnegative, and let $(\Psi^n)_{n\in\overline{I}_N}$ be a sequence in $\mathcal{V}_h$. Then*

(a) $\langle\delta_t^{(2)}\Psi^n,\delta_t^{(1)}\Psi^n\rangle = \frac{1}{2}\delta_t\|\delta_t\Psi^{n-1}\|^2$, *for each* $n \in \overline{I}_{N-1}$.

(b) $\delta_t\langle\delta_t\Psi^{n-1},\mu_t\Psi^{n-1}\rangle = \langle\delta_t\Psi^{n-1},\delta_t^{(1)}\Psi^n\rangle + \langle\delta_t^{(2)}\Psi^n,\mu_t\Psi^n\rangle$, *for each* $n \in \overline{I}_{N-1}$.

(c) $\langle r\delta_{\Psi,t}^{(1)}F(\Psi^n/r),\delta_t^{(1)}\Psi^n\rangle = \delta_t\langle r^2,\mu_t F(\Psi^{n-1}/r)\rangle$, *for each* $n \in \overline{I}_{N-1}$.

(d) $\langle-\delta_r^{(2)}\Psi^n,\delta_t^{(1)}\Psi^n\rangle = \frac{1}{2}\delta_t\langle\delta_r\Psi^n,\delta_r\Psi^{n-1}\rangle$, *for each* $n \in \overline{I}_{N-1}$.

(e) $\langle\delta_r\Psi^{n+1},\delta_r\Psi^n\rangle = \mu_t\|\delta_r\Psi^n\|_2^2 - \frac{1}{2}\tau^2\|\delta_{rt}\Psi^n\|_2^2$, *for each* $n \in \overline{I}_{N-1}$.

*Proof.* The identity (a) is the result of using the definitions of the discrete temporal operators and some algebra. To establish (b), notice that

$$\delta_t\langle\delta_t\Psi^{n-1},\mu_t\Psi^{n-1}\rangle = \frac{1}{\tau}\left[\langle\delta_t\Psi^{n-1},\mu_t\Psi^n - \mu_t\Psi^{n-1}\rangle + \langle\delta_t\Psi^n - \delta_t\Psi^{n-1},\mu_t\Psi^n\rangle\right], \tag{2.3.19}$$

holds for each $n \in \overline{I}_{N-2}$, whence the result follows. In the case of (c), it is enough to use the formula (2.3.4) and the definitions of the discrete operators. To show (d), it suffices to notice that the following hold, for each $n \in I_{N-1}$:

$$\begin{aligned}\langle \delta_r^{(2)} \Psi^n, \delta_t^{(1)} \Psi^n \rangle &= \frac{1}{2\tau} \left[ \langle \delta_r^{(2)} \Psi^n, \Psi^{n+1} \rangle - \langle \delta_r^{(2)} \Psi^n, \Psi^{n-1} \rangle \right] \\ &= -\frac{1}{2\tau} \left[ \langle \delta_r \Psi^n, \delta_r \Psi^{n+1} \rangle - \langle \delta_r \Psi^{n-1}, \delta_r \Psi^n \rangle \right].\end{aligned} \tag{2.3.20}$$

Finally, the formula of (e) follows after noting that

$$\begin{aligned}\langle \delta_r \Psi^{n+1}, \delta_r \Psi^n \rangle &= \mu_t \|\delta_r \Psi^n\|_2^2 \\ &+ \frac{1}{2} \langle \delta_r \Psi^n, \delta_r (\Psi^{n+1} - \Psi^n) \rangle - \frac{1}{2} \langle \delta_r \Psi^{n+1}, \delta_r (\Psi^{n+1} - \Psi^n) \rangle.\end{aligned} \tag{2.3.21}$$

The identities of this result have been established now. □

**Lemma 2.3.10** (Product rule)**.** *Assume that $(\Phi^n)_{n \in \overline{I}_N}$ and $(\Psi^n)_{n \in \overline{I}_N}$ are any finite sequences in $\mathcal{V}_h$. If $(j,n) \in I_{M-1} \times \overline{I}_{N-1}$, then $\delta_t(\Psi_j^n \Phi_j^n) = \Psi_j^n \delta_t \Phi_j^n + \Phi_j^{n+1} \delta_t \Psi_j^n$.* □

**Lemma 2.3.11.** *If $(\Psi^n)_{n \in \overline{I}_N}$ is any finite sequence in $\mathcal{V}_h$, then the following identities are satisfied for each $n \in I_{N-2}$:*

$$\frac{1}{2} \delta_t \left[ e^{3t_n} \|\delta_t \Psi^n\|_2^2 \right] = \frac{1}{2} \delta_t e^{3t_n} \|\delta_t \Psi^{n+1}\|_2^2 + e^{3t_n} \langle \delta_t^{(2)} \Psi^{n+1}, \delta_t^{(1)} \Psi^{n+1} \rangle, \tag{2.3.22}$$

$$\delta_t \left[ e^{3t_n} \langle \mu_t F(\Psi^n/r), r^2 \rangle \right] = \delta_t e^{3t_n} \langle \mu_t F(\Psi^{n+1}/r), r^2 \rangle + e^{3t_n} \langle r \delta_{\Psi,t} F(\Psi^{n+1}/r), \delta_t^{(1)} \Psi^{n+1} \rangle, \tag{2.3.23}$$

$$\begin{aligned}\frac{3}{2} \delta_t \left[ e^{3t_n} \langle \delta_t \Psi^n, \mu_t \Psi^n \rangle \right] &= \frac{3}{2} \delta_t e^{3t_n} \langle \delta_t \Psi^{n+1}, \mu_t \Psi^{n+1} \rangle + \frac{3}{2} e^{3t_n} \langle \delta_t^{(2)} \Psi^{n+1}, \mu_t \Psi^{n+1} \rangle \\ &+ \frac{3}{2} e^{3t_n} \langle \delta_t \Psi^n, \delta_t^{(1)} \Psi^{n+1} \rangle\end{aligned} \tag{2.3.24}$$

*and*

$$\begin{aligned}\frac{1}{2} \delta_t \left[ e^{3t_n} (\mu_t f(t_n)) \langle \delta_r \Psi^{n+1}, \delta_r \Psi^n \rangle \right] &= \frac{1}{2} (\delta_t e^{3t_n}) (\mu_t f(t_{n+1})) \langle \delta_r \Psi^{n+1}, \delta_r \Psi^n \rangle \\ + \frac{1}{2} e^{3t_n} (\delta_t^{(1)} f(t_{n+1})) \langle \delta_r \Psi^{n+1}, \delta_r \Psi^n \rangle &- e^{3t_{n+1}} (\mu_t f(t_{n+1})) \langle \delta_r^{(2)} \Psi^{n+1}, \delta_t^{(1)} \Psi^{n+1} \rangle.\end{aligned} \tag{2.3.25}$$

*Proof.* These identities are straightforward results of Lemmas 2.3.9 and 2.3.10. □

For the remainder of this chapter, we will let $\Psi = (\Psi^n)_{n \in \overline{I}_N}$ represent any solution of the equation (2.3.11).

**Theorem 2.3.12** (Energy dissipation)**.** *If* $\Psi$ *is a solution of* (2.3.11) *and* $n \in \overline{I}_{N-2}$*, then*

$$\begin{aligned}\delta_t E^n = 4\pi \Bigg[ & \frac{1}{2}(\delta_t e^{3t_n})\|\delta_t \Psi^{n+1}\|_2^2 + e^{3t_n}\langle \delta_t^{(2)}\Psi^{n+1}, \delta_t^{(1)}\Psi^{n+1}\rangle \\ & + \frac{3}{2}(\delta_t e^{3t_n})\langle \delta_t \Psi^{n+1}, \mu_t \Psi^{n+1}\rangle + \frac{3}{2}e^{3t_n}\langle \delta_t^{(2)}\Psi^{n+1}, \mu_t \Psi^{n+1}\rangle \\ & + \frac{3}{2}e^{3t_n}\langle \delta_t \Psi^n, \delta_t^{(1)}\Psi^{n+1}\rangle + \frac{1}{2}(\delta_t e^{3t_n})(\mu_t f(t_{n+1}))\langle \delta_r \Psi^{n+1}, \delta_r \Psi^n\rangle \\ & + \frac{1}{2}e^{3t_n}(\delta_t^{(1)} f(t_{n+1}))\langle \delta_r \Psi^{n+1}, \delta_r \Psi^n\rangle - e^{3t_{n+1}}(\mu_t f(t_{n+1}))\langle \delta_r^{(2)}\Psi^{n+1}, \delta_t^{(1)}\Psi^{n+1}\rangle \\ & + (\delta_t e^{3t_n})\langle \mu_t F(\Psi^{n+1}/r), r^2\rangle + e^{3t_n}\langle r\delta_{\Psi,t}F(\Psi^{n+1}/r), \delta_t^{(1)}\Psi^{n+1}\rangle \Bigg].\end{aligned} \tag{2.3.26}$$

*Proof.* The result readily follows from Lemma 2.3.11 and the discrete total energy (2.3.18). □

## 2.4 Numerical properties

The purpose of this section is to establish the main numerical properties of the finite-difference scheme (2.3.11). In particular, we will show that the numerical model is quadratically consistent, stable and quadratically convergent. To establish the consistency of the numerical model (2.3.11), we need to define the differential operator

$$\mathcal{L}(\psi(r,t)) = \frac{\partial^2 \psi}{\partial t^2} - f(t)\frac{\partial^2 \psi}{\partial r^2} + 3\frac{\partial \psi}{\partial t} + rF'(\psi/r), \quad \forall (r,t) \in (0,L)\times(0,T), \tag{2.4.1}$$

and the difference operator

$$L(\psi_j^n) = \delta_t^{(2)}\psi_j^n - f(t_n)\delta_r^{(2)}\psi_j^n + 3\delta_t^{(1)}\psi_j^n + r_j \delta_{\psi,t}^{(1)} F(\psi_j^n/r_j), \quad \forall (j,n) \in I_{M-1}\times I_{N-1}. \tag{2.4.2}$$

**Theorem 2.4.1** (Consistency)**.** *Let* $f : [0,T] \to \mathbb{R}$ *be bounded, and suppose that* $F \in \mathcal{C}^2(\mathbb{R})$ *is such that* $F'' \in L^\infty(\mathbb{R})$*. If* $\psi \in \mathcal{C}_{x,t}^{4,3}(\overline{\Omega})$ *then there exist nonnegative constants* $C$ *and* $C'$ *which are independent of* $\tau$ *and* $h$*, with the property that*

$$\left| \mathcal{L}(\psi(r_j,t_n)) - L(\psi_j^n) \right| \le C(\tau^2 + h^2), \quad \forall (j,n) \in I_{M-1}\times I_{N-1}, \tag{2.4.3}$$

$$\left| \mathcal{H}(r_j, t_{n+\frac{1}{2}}) - H_j^n \right| \le C'(\tau^2 + h^2), \quad \forall (j,n) \in I_{M-1}\times I_{N-1}. \tag{2.4.4}$$

*Proof.* Since $f$ is bounded, then there exists a nonnegative number $K$ such that $|f(t)| \le K$, for each $t \in [0,T]$. Using the usual argument based on Taylor's theorem along with the regularity of $F$ and $\psi$, there exist nonnegative constants $C_1$, $C_2$, $C_3$ and $C_4$ which are independent of $\tau$ and $h$, such that the following inequalities hold:

$$\left| \frac{\partial^2 \psi(r_j,t_n)}{\partial t^2} - \delta_t^{(2)}\psi_j^n \right| \le C_1 \tau^2, \quad \forall (j,n) \in I_{M-1}\times I_{N-1}, \tag{2.4.5}$$

$$\left| f(t_n)\frac{\partial^2\psi(r_j,t_n)}{\partial r^2} - f(t_n)\delta_r^{(2)}\psi_j^n \right| \leq KC_2h^2, \quad \forall (j,n) \in I_{M-1}\times I_{N-1}, \tag{2.4.6}$$

$$\left| \frac{\partial\psi(r_j,t_n)}{\partial t} - \delta_t^{(1)}\psi_j^n \right| \leq C_3\tau^2, \quad \forall (j,n) \in I_{M-1}\times I_{N-1}, \tag{2.4.7}$$

$$\left| r_jF'(\psi(r_j,t_n)/r_j) - r_j\delta_{\psi,t}^{(1)}F(\psi_j^n/r_j) \right| \leq LC_4\tau^2, \quad \forall (j,n) \in I_{M-1}\times I_{N-1}. \tag{2.4.8}$$

The first inequality of this theorem readily follows with $C = C_1 + KC_2 + 3C_3 + LC_4$, which is a nonnegative constant that is independent of $\tau$ and $h$, as desired. The existence of the constant $C'$ is established in similar fashion. □

The aim of the following discussion is to establish the stability and convergence properties of the finite-difference method (2.3.11). To that end, various assumptions and technical results will be required. In particular, we will assume that $f$ is a positive and continuously differentiable function on $[0,T]$. In particular, these hypothesis guarantee the existence of positive constants $C_0$, $C_1$ and $C_2$, such that

- $|\delta_t^{(1)}f(t_n)| \leq C_0$, for each $n \in \overline{I}_N$, and
- $C_1 \leq f(t_n) \leq C_2$, for each $n \in \overline{I}_N$.

**Lemma 2.4.2** (Macías-Díaz [72]). *Let $F \in \mathcal{C}^2(\mathbb{R})$ and $F'' \in L^\infty(\mathbb{R})$, and suppose that $(\Psi^n)_{n\in\overline{I}_N}$, $(\widetilde{\Psi}^n)_{n\in\overline{I}_N}$ and $(R^n)_{n\in\overline{I}_N}$ are sequences in $\mathcal{V}_h$. For each $n \in \overline{I}_N$, let $\epsilon^n = \Psi^n - \widetilde{\Psi}^n$ and*

$$\widetilde{F}^n = \delta_{\Psi,t}^{(1)}F(\Psi^n/r) - \delta_{\widetilde{\Psi},t}^{(1)}F(\widetilde{\Psi}^n/r), \quad \forall n \in I_{N-1}. \tag{2.4.9}$$

*Then there exists a constant $C_3 \in \mathbb{R}^+$ which depends only on $F$, such that for each $m \in I_{N-1}$,*

$$2\tau\left|\sum_{n=1}^{m}\langle R^n - r\widetilde{F}^n, \delta_t^{(1)}\epsilon^n\rangle\right| \leq 2\tau\sum_{n=0}^{m}\|R^n\|_2^2 + C_3\|\epsilon^0\|_2^2 + C_3\tau\sum_{n=0}^{m}\|\delta_t\epsilon^n\|_2^2. \tag{2.4.10}$$

The following lemma is a discrete form of Gronwall's inequality.

**Lemma 2.4.3** (Pen-Yu [84]). *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative real numbers, and suppose that there exists $C \geq 0$ such that*

$$\omega^m \leq \rho^m + C\tau\sum_{n=0}^{m-1}\omega^n, \qquad \forall m \in \overline{I}_N. \tag{2.4.11}$$

*Then $\omega^n \leq \rho^n e^{Cn\tau}$, for each $n \in \overline{I}_N$.* □

In the following, we will assume that $\Psi$ is the solution of the discrete initial-boundary-value problem (2.3.11) associated to the initial data $(\phi_0, \phi_1)$. We will consider also a second set of initial data $(\widetilde{\phi}_0, \widetilde{\phi}_1)$, and we will assume that $\widetilde{\Psi}$ is the corresponding solution of (2.3.11). In other words, the following discrete problem is satisfied:

$$\delta_t^{(2)}\widetilde{\Psi}_j^n - f(t_n)\delta_r^{(2)}\widetilde{\Psi}_j^n + 3\delta_t^{(1)}\widetilde{\Psi}_j^n + r_j\delta_{\widetilde{\Psi},t}^{(1)}F(\widetilde{\Psi}_j^n/r_j) = 0, \quad \forall (j,n) \in I_{M-1}\times \overline{I}_{N-1},$$
$$\text{such that} \begin{cases} \widetilde{\Psi}_j^0 = r_j\widetilde{\phi}_0(x_j), & \forall j \in I_{M-1}, \\ \delta_t^{(1)}\widetilde{\Psi}_j^0 = r_j\widetilde{\phi}_1(x_j), & \forall j \in I_{M-1}, \\ \widetilde{\Psi}_0^n = \widetilde{\Psi}_M^n = 0, & \forall n \in I_{N-1}. \end{cases} \tag{2.4.12}$$

**Theorem 2.4.4** (Stability)**.** *Suppose that $f$ is positive and continuously differentiable in $[0,T]$, and let $F \in \mathcal{C}^2(\mathbb{R})$ satisfy $F'' \in L^\infty(\mathbb{R})$. Let $(\phi_0,\phi_1)$ and $(\widetilde{\phi}_0,\widetilde{\phi}_1)$ be two sets of initial conditions, and let $\Psi$ and $\widetilde{\Psi}$ be the respective numerical solutions obtained using* (2.3.11)*. Define $\epsilon_j^n = \Psi_j^n - \widetilde{\Psi}_j^n$, for each $(j,n) \in \overline{I}_M \times \overline{I}_N$, and suppose that*

$$\tau\left(C_3 + \frac{2\tau}{h}C_2\right) < \frac{1}{2}. \tag{2.4.13}$$

*Then there exists a constant $C_5$ which is independent of $\tau$ and $h$, such that for each $n \in \overline{I}_{N-1}$,*

$$\frac{1}{2}\|\delta_t\epsilon^n\|_2^2 + C_1\mu_t\|\delta_r\epsilon^n\|_2^2 \le C_5\left(\|\delta_t\epsilon^0\|_2^2 + C_2\mu_t\|\delta_r\epsilon^0\|_2^2 + C_3\|\epsilon^0\|_2^2\right). \tag{2.4.14}$$

*Proof.* Throughout, the constants $C_0$, $C_1$, $C_2$ and $C_3$ will be those provided in the previous paragraphs. Observe that $\epsilon$ satisfies the following discrete initial-boundary-value problem:

$$\delta_t^{(2)}\epsilon_j^n - f(t_n)\delta_r^{(2)}\epsilon_j^n + 3\delta_t^{(1)}\epsilon_j^n + r_j\widetilde{F}_j^n = 0, \quad \forall (j,n) \in I_{M-1}\times \overline{I}_{N-1},$$
$$\text{such that} \begin{cases} \epsilon_j^0 = r_j[\phi_0(x_j) - \widetilde{\phi}_0(x_j)], & \forall j \in I_{M-1}, \\ \delta_t^{(1)}\epsilon_j^0 = r_j[\phi_1(x_j) - \widetilde{\phi}_1(x_j)], & \forall j \in I_{M-1}, \\ \epsilon_0^n = \epsilon_M^n = 0, & \forall n \in I_{N-1}. \end{cases} \tag{2.4.15}$$

Here, $\widetilde{F}_j^n$ is as in Lemma 2.4.2, for each $(j,n) \in I_{M-1}\times \overline{I}_{N-1}$. Let $m$ be a arbitrary (though fixed) element of $I_{N-1}$. Using the identity of Lemma 2.3.9(a) along with the formula for telescoping sums, it is easy to check that

$$\sum_{n=1}^{m}\langle\delta_t^{(2)}\epsilon^n, \delta_t^{(1)}\epsilon\rangle = \frac{1}{2}\sum_{n=1}^{m}\delta_t\|\delta_t\epsilon^{n-1}\|_2^2 = \frac{1}{2\tau}\left[\|\delta_t\epsilon^m\|_2^2 - \|\delta_t\epsilon^0\|_2^2\right]. \tag{2.4.16}$$

Using now the identities (d) and (e) of Lemma 2.3.9, a discrete form of the product rule and the formula for telescoping sums, we readily notice that

$$\begin{aligned}\sum_{n=1}^{m}\langle -f(t_n)\delta_r^{(2)}\epsilon^n,\delta_t^{(1)}\epsilon^n\rangle &= \frac{1}{2}\sum_{n=1}^{m} f(t_n)\delta_t\langle\delta_r\epsilon^n,\delta_r\epsilon^{n-1}\rangle \\ &= \frac{1}{2}\sum_{n=1}^{m}\delta_t\left[f(t_{n-1})\langle\delta_r\epsilon^n,\delta_r\epsilon^{n-1}\rangle\right]-\frac{1}{2}\sum_{n=1}^{m}(\delta_t f(t_{n-1}))\langle\delta_r\epsilon^n,\delta_r^{n-1}\rangle \\ &= \frac{f(t_m)}{2\tau}\left[\mu_t\|\delta_r\epsilon^m\|_2^2-\frac{\tau^2}{2}\|\delta_{rt}\epsilon^m\|_2^2\right]-\frac{f(t_0)}{2\tau}\left[\mu_t\|\delta_r\epsilon^0\|_2^2-\frac{\tau^2}{2}\|\delta_{rt}\epsilon^0\|_2^2\right] \\ &\quad -\frac{1}{2}\sum_{n=1}^{m}(\delta_t f(t_{n-1}))\langle\delta_r\epsilon^n,\delta_r\epsilon^{n-1}\rangle.\end{aligned} \tag{2.4.17}$$

Calculate the inner product of $\delta_t^{(1)}\epsilon^n$ with the recursive equation of (2.4.15) at the time $t_n$, and take the sum over all indexed $n\in I_m$. Substitute then the identities (2.4.16) and (2.4.17), rearrange terms algebraically and employ the bound provided by Lemma 2.4.2 with $R_j^n=0$, for each $(j,n)\in\overline{I}_M\times\overline{I}_N$. For the sake of simplicity, we let

$$\omega_0^m = \|\delta_t\epsilon^m\|_2^2+C_1\mu_t\|\delta_r\epsilon^m\|_2^2, \tag{2.4.18}$$

$$\rho = \|\delta_t\epsilon^0\|_2^2+C_2\mu_t\|\delta_r\epsilon^0\|_2^2+C_3\|\epsilon^0\|_2^2. \tag{2.4.19}$$

Use now Young's inequality and the bound provided by Lemma 2.3.5. In such way, one may check then that the following inequalities are satisfied:

$$\begin{aligned}\omega_0^m &\le \|\delta_t\epsilon^m\|_2^2+f(t_m)\mu_t\|\delta_r\epsilon^m\|_2^2 \\ &\le \|\delta_t\epsilon^0\|_2^2+f(t_0)\mu_t\|\delta_r\epsilon^0\|_2^2+2\tau\sum_{n=1}^{m}(\delta_t f(t_{n-1}))\langle\delta_r\epsilon^n,\delta_r^{n-1}\rangle \\ &\qquad -2\tau\sum_{n=1}^{m}\langle r\widetilde{F}^n,\delta_r^{(1)}\epsilon^n\rangle+\frac{1}{2}\tau^2 f(t_m)\|\delta_{rt}\epsilon^m\|_2^2 \\ &\le \rho+C_4\tau\sum_{n=0}^{m-1}\left[\frac{1}{2}\|\delta_t\epsilon^n\|_2^2+C_1\mu_t\|\delta_r\epsilon^{n-1}\|_2^2\right]+\tau\left(C_3+\frac{2\tau}{h}C_2\right)\|\delta_t\epsilon^m\|_2^2 \\ &\le \rho+C_4\tau\sum_{n=0}^{m-1}\left[\frac{1}{2}\|\delta_t\epsilon^n\|_2^2+C_1\mu_t\|\delta_r\epsilon^{n-1}\|_2^2\right]+\frac{1}{2}\|\delta_t\epsilon^m\|_2^2.\end{aligned} \tag{2.4.20}$$

Here, we let

$$C_4 = \frac{2C_0}{C_1}+2C_3. \tag{2.4.21}$$

Obviously, we employed the condition (2.4.13) in the last step. Subtract $\frac{1}{2}\|\delta_t\epsilon^m\|_2^2$ on both ends of the inequalities (2.4.13). Applying Lemma 2.4.3 with $\omega^n=\omega_0^n-\frac{1}{2}\|\delta_t\epsilon^n\|_2^2$ for each

$n \in \overline{I}_{N-1}$, it follows that $\omega^n \leq C_5\rho$ is satisfied for each $n \in \overline{I}_N$. Here, we define $C_5 = \exp(C_4 T)$, which is independent of $\tau$ and $h$. The conclusion readily follows now. □

As a consequence of Theorem 2.4.4, we establish the uniqueness of solutions of the finite-difference scheme (2.3.11). This property is stated in the following result.

**Corollary 2.4.5** (Uniqueness of solutions)**.** *Let $f$ be positive and continuously differentiable in $[0,T]$, and let $F \in \mathcal{C}^2(\mathbb{R})$ satisfy $F'' \in L^\infty(\mathbb{R})$. Let $\Psi$ and $\widetilde{\Psi}$ be two solutions of* (2.3.11) *corresponding to the initial data $(\phi_0, \phi_1)$, and suppose that* (2.4.13) *holds. Then $\Psi = \widetilde{\Psi}$.*

*Proof.* Let $\epsilon_j^n = \Psi_j^n - \widetilde{\Psi}_j^n$, for each $(j,n) \in \overline{I}_M \times \overline{I}_N$. Theorem 2.4.4 guarantees then that (2.4.14) is satisfied for each $n \in \overline{I}_{N-1}$. This means that all the terms on the right-hand side of that inequality are equal to zero. It follows that

$$0 \leq \frac{1}{\tau}\left| \|\epsilon^{n+1}\|_2 - \|\epsilon^n\|_2 \right| \leq \|\delta_t \epsilon^n\|_2 = 0, \quad \forall n \in \overline{I}_{N-1}, \tag{2.4.22}$$

which implies that $\|\epsilon^{n+1}\|_2 = \|\epsilon^n\|_2$, for each $n \in \overline{I}_{N-1}$. Using induction, we may readily check that $\|\epsilon^n\|_2 = \|\epsilon^0\|_2 = 0$, for each $n \in \overline{I}_N$. In turn, this implies that $\epsilon^n = 0$, for each $n \in \overline{I}_N$, whence the result readily follows. □

Finally, we tackle the problem of the convergence of the scheme. Recall that $\psi$ represents the exact solution of the problem (2.2.10) for a set of initial conditions $(\phi_0, \phi_1)$, while $\Psi$ denotes a solution to the discrete model (2.3.11) for the same initial data. Assuming that (2.4.13) is satisfied, the solution $\Psi$ is unique. Moreover, the following discrete initial-boundary-value problem is satisfied by the function $\psi$:

$$\begin{aligned} &\delta_t^{(2)}\psi_j^n - f(t_n)\delta_r^{(2)}\psi_j^n + 3\delta_t^{(1)}\psi_j^n + r_j\delta_{\psi,t}^{(1)}F(\psi_j^n/r_j) = R_j^n, \quad \forall (j,n) \in I_{M-1} \times \overline{I}_{N-1}, \\ &\text{such that } \begin{cases} \psi_j^0 = r_j\phi_0(x_j), & \forall j \in I_{M-1}, \\ \delta_t^{(1)}\psi_j^0 = r_j\phi_1(x_j), & \forall j \in I_{M-1}, \\ \psi_0^n = \psi_M^n = 0, & \forall n \in I_{N-1}. \end{cases} \end{aligned} \tag{2.4.23}$$

Here, $R_j^n$ represents the local truncation error which, under the hypotheses of Theorem 2.4.1, satisfies $|R_j^n| \leq C(\tau^2 + h^2)$, for some constant $C \geq 0$ which is independent of $\tau$ and $h$. It follows that there exists a constant $C_* \geq 0$, such that $\|R^n\|_2^2 \leq C_*^2(\tau^2 + h^2)^2$, for each $n \in \overline{I}_N$. These facts will be employed in the proof of the following result.

**Theorem 2.4.6** (Convergence)**.** *Assume that $f$ is positive and continuously differentiable in $[0,T]$, and let $F \in \mathcal{C}^2(\mathbb{R})$ be such that $F'' \in L^\infty(\mathbb{R})$. If the inequality* (2.4.13) *is satisfied and $\psi \in \mathcal{C}_{x,t}(\overline{\Omega})$, then the solution of the numerical model* (2.3.11) *converges to the solution of the problem* (2.2.10)*, with order of convergence $\mathcal{O}(\tau^2 + h^2)$ in the norm $\|\cdot\|_2$.*

*Proof.* Consider a fixed set of initial data $(\phi_0, \phi_1)$, and notice that the numerical approximation $\Psi$ and the exact solution $\psi$ satisfy, respectively, the discrete initial-boundary-value problems

(2.3.11) and (2.4.23). Subtracting those problems and letting $\epsilon_j^n = \Psi_j^n - \psi_j^n$, for each $(j,n) \in \overline{I}_M \times \overline{I}_N$, one readily checks that the following problem is satisfied:

$$\delta_t^{(2)} \epsilon_j^n - f(t_n)\delta_r^{(2)}\epsilon_j^n + 3\delta_t^{(1)}\epsilon_j^n + r_j \widetilde{F}_j^n + R_j^n = 0, \quad \forall (j,n) \in I_{M-1} \times \overline{I}_{N-1},$$
$$\text{such that } \begin{cases} \epsilon_j^0 = 0, & \forall j \in I_{M-1}, \\ \delta_t^{(1)}\epsilon_j^0 = 0, & \forall j \in I_{M-1}, \\ \epsilon_0^n = \epsilon_M^n = 0, & \forall n \in I_{N-1}. \end{cases} \tag{2.4.24}$$

In this case, $\widetilde{F}_j^n = \delta_{\Psi,t}^{(1)} F(\Psi_j^n/r_j) - \delta_{\psi,t}^{(1)} F(\psi_j^n/r_j)$, for each $(j,n) \in I_{M-1} \times \overline{I}_{N-1}$. Assume that $m$ is an arbitrary (though fixed) element of $I_{N-1}$. Notice that the identities (2.4.16) and (2.4.17) hold in this case also. Following the idea in the proof of Theorem 2.4.4, we let $\omega_0^m$ be as therein and, for each $m \in I_{N-1}$, we define the constants

$$\rho^m = \|\delta_t \epsilon^0\|_2^2 + C_2\mu\|\delta_r\epsilon^0\|_2^2 + C_3\|\epsilon^0\|_2^2 + 2\tau\sum_{n=1}^m \|R^n\|_2^2 = 2\tau\sum_{n=1}^m \|R^n\|_2^2. \tag{2.4.25}$$

Obviously, the simplification at the right-hand side of this last identity is obtained using the initial conditions of (2.4.24). Next, take the inner product of the main equation of (2.4.24) at the time $t_n$ with the vector $\delta_t^{(1)}\epsilon^n$, and take the sum over all indexes $n \in I_m$. Use then the identities (2.4.16) and (2.4.17), and rearrange algebraically all the terms involved. Employ then Lemma 2.4.2, Young's inequality together with the constants $C_0$, $C_1$ and $C_2$ defined in the previous paragraphs. In such way, it is possible to check that the following inequality holds:

$$\omega_0^m \leq \rho^m + C_4\tau\sum_{n=0}^{m-1}\omega^n + \frac{1}{2}\|\delta_t\epsilon^m\|_2^2. \tag{2.4.26}$$

Here, $C_4$ is as in the proof of Theorem 2.4.4. Subtract $\frac{1}{2}\|\delta_t\epsilon^m\|_2^2$ on both sides of this last inequality, and let $C_5$ be as in the proof of Theorem 2.4.4. The hypotheses of Lemma 2.4.3 are satisfied, whence it follows that, for each $m \in \overline{I}_{M-1}$,

$$\frac{1}{2}\|\delta_t\epsilon^m\|_2^2 \leq \omega^m \leq C_5\rho^m = 2C_5\tau\sum_{n=1}^m \|R^n\|_2^2 \leq 2C_5 T C_*^2(\tau^2 + h^2)^2. \tag{2.4.27}$$

This implies that $\|\delta_t\epsilon^m\|_2 \leq C_6(\tau^2 + h^2)$, for each $m \in \overline{I}_{N-1}$. Here, we let $C_6 = 2C_*\sqrt{C_5 T}$. Use then the triangle inequality, multiply by $\tau$ and sum over all $m \in \overline{I}_{n-1}$, for any $n \in I_{N-1}$.

If $C_7 = C_6 T$, then it is readily checked that

$$\begin{aligned} \|\epsilon^n\|_2 = \|\epsilon^n\|_2 - \|\epsilon^0\|_2 &= \sum_{m=0}^{n-1} (\|\epsilon^{m+1}\|_2 - \|\epsilon^m\|_2) \\ &\leq \tau \sum_{m=0}^{n-1} \|\delta_t \epsilon^m\|_2 \\ &\leq C_6 \tau n(\tau^2 + h^2) \leq C_7(\tau^2 + h^2), \quad \forall n \in I_{N-1}. \end{aligned} \tag{2.4.28}$$

We conclude that the finite-difference scheme (2.3.11) converges to the solution of (2.2.10) with quadratic order of convergence in the $\|\cdot\|_2$ norm, as desired. □

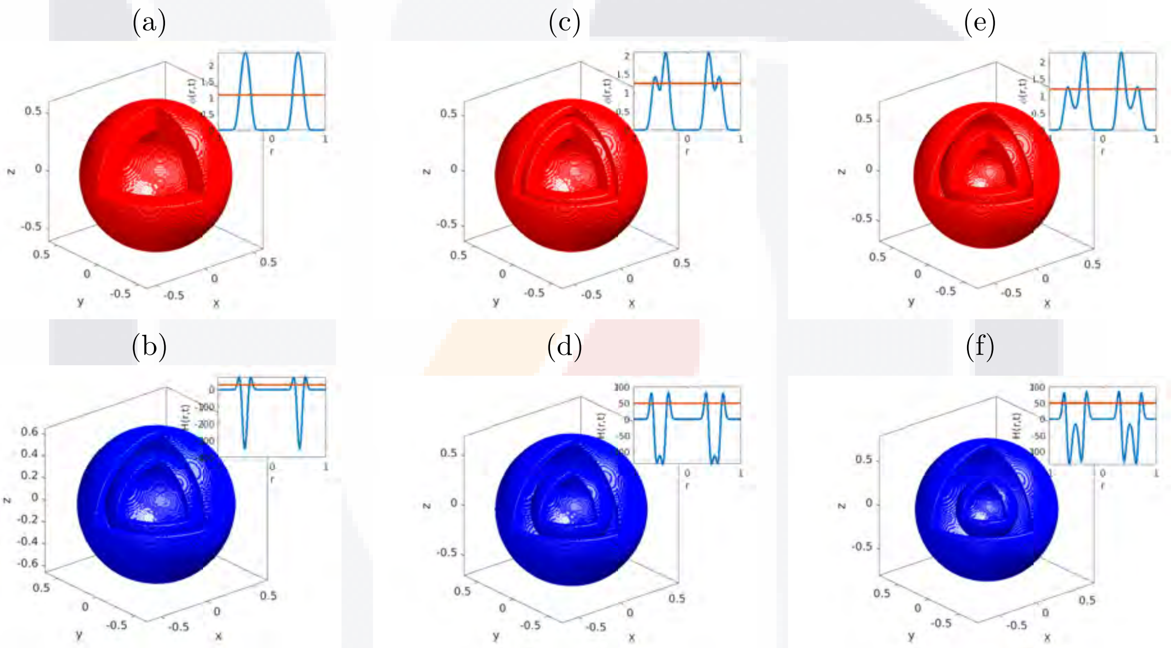


**Figure 2.2** Solutions of the problem (2.5.1) at the times $t = 0.060$ (left column), $t = 0.120$ (middle column) and $t = 0.180$ (right column), obtained using the scheme (2.3.11). The graphs on the top row show the approximate solution $\psi(x,t)$, for each $x \in D$. The insets are the corresponding graph of $\psi(r,t)$ in the radial coordinate $r$. The graphs on the bottom row correspond to those of the Hamiltonians. We used the parameters $\tau = h = 0.002$, $L = 1$, $\lambda = 2$ and $\mu = 3$. The initial data were $\phi_0(r) = \mathrm{B}(r; 0.5, 0.3)$ and $\phi_1(r) = -5\phi_0(r)$, for each $r \in [0,1]$.

## 2.5 Results

The purpose of this section is to provide some illustrative simulations using a computational implementation of the scheme (2.3.11). A Matlab® implementation of this scheme is provided in the Appendix A.1 for the sake of convenience. It is important to mention that the program was employed to produce the simulations of the present section, and that it is provided specifically to approximate radially symmetric solutions of the three-dimensional Higgs boson equation in the de Sitter space-time.

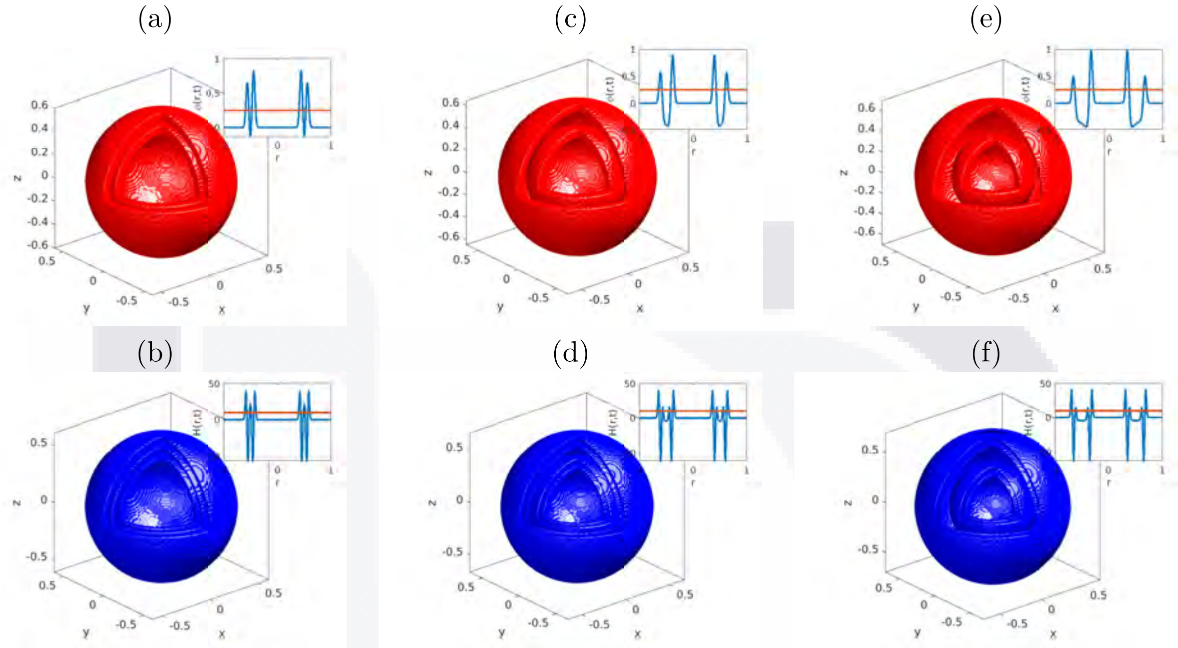


**Figure 2.3** Solutions of the problem (2.5.1) at the times $t = 0.060$ (left column), $t = 0.120$ (middle column) and $t = 0.180$ (right column), obtained using the scheme (2.3.11). The graphs on the top row show the approximate solution $\psi(x,t)$, for each $x \in D$. The insets are the corresponding graph of $\psi(r,t)$ in the radial coordinate $r$. The graphs on the bottom row correspond to those of the Hamiltonians. We used the parameters $\tau = h = 0.002$, $L = 1$, $\lambda = 2$ and $\mu = 3$. The initial data were $\phi_0(r) = 2\mathrm{B}(r; 0.5, 0.2)$ and $\phi_1(r) = 5\phi_0(r)$, for each $r \in [0,1]$.

For the remainder of this section, we will let $D \subseteq \mathbb{R}^3$ be the open ball with center at the origin and radius $L > 0$, and we will consider the following initial-boundary-value problem, governed by the three-dimensional Higgs boson equation in the de Sitter space-time:

$$
\begin{aligned}
&\frac{\partial^2 \phi(x,t)}{\partial t^2} - e^{-2t}\Delta\phi(x,t) + 3\frac{\partial \phi(x,t)}{\partial t} - \mu^2\phi(x,t) + \lambda|\phi(x,t)|^{q-1}\phi(x,t) = 0, \\
&\qquad\text{such that } \begin{cases} \phi(x,0) = \phi_0(x), & \forall x \in B, \\ \dfrac{\partial \phi(x,0)}{\partial t} = \phi_1(x), & \forall x \in B, \\ \phi(x,t) = 0, & \forall (x,t) \in \partial B \times [0,T], \end{cases}
\end{aligned}
\tag{2.5.1}
$$

for each $(x,t) \in B \times \overline{\mathbb{R}^+}$. It is easy to check that this model is a particular form of (2.2.1). Moreover, we will focus our attention on the radially symmetric solutions of (2.5.1). To that end, we will let $\phi(x,t) = \phi(r,t)$, and consider the transformation $\psi(r,t) = r\phi(r,t)$, for each $(r,t) \in [0,L] \times [0,T]$. As we saw previously, $\phi$ satisfies (2.5.1) if and only if $\psi$ satisfies (2.2.10). To approximate the solutions of $\phi$, we employ the finite-difference method (2.3.11). We need only mention that the energy associated to (2.5.1) is given by

$$\mathcal{E}(t) = e^{3t}\left[\frac{1}{2}\left\|\frac{\partial\phi}{\partial t} + \frac{3}{2}\phi\right\|_{x,2}^2 + \frac{e^{-2t}}{2}\|\nabla\phi\|_{x,2}^2 - \left(\frac{9}{8} + \frac{\mu^2}{2}\right)\|\phi\|_{x,2}^2 + \frac{\lambda\|\phi\|_{x,q+1}^{q+1}}{q+1}\right], \tag{2.5.2}$$

for each $t \in (0,T)$. This system is dissipative in view that

$$\frac{d\mathcal{E}(t)}{dt} = -e^{3t}\left(e^{-2t}\|\nabla\phi\|_{x,2}^2 + \frac{3\lambda(q-1)}{2(q+1)}\|\phi\|_{x,q+1}^{q+1}\right), \quad \forall t \in (0,T). \tag{2.5.3}$$

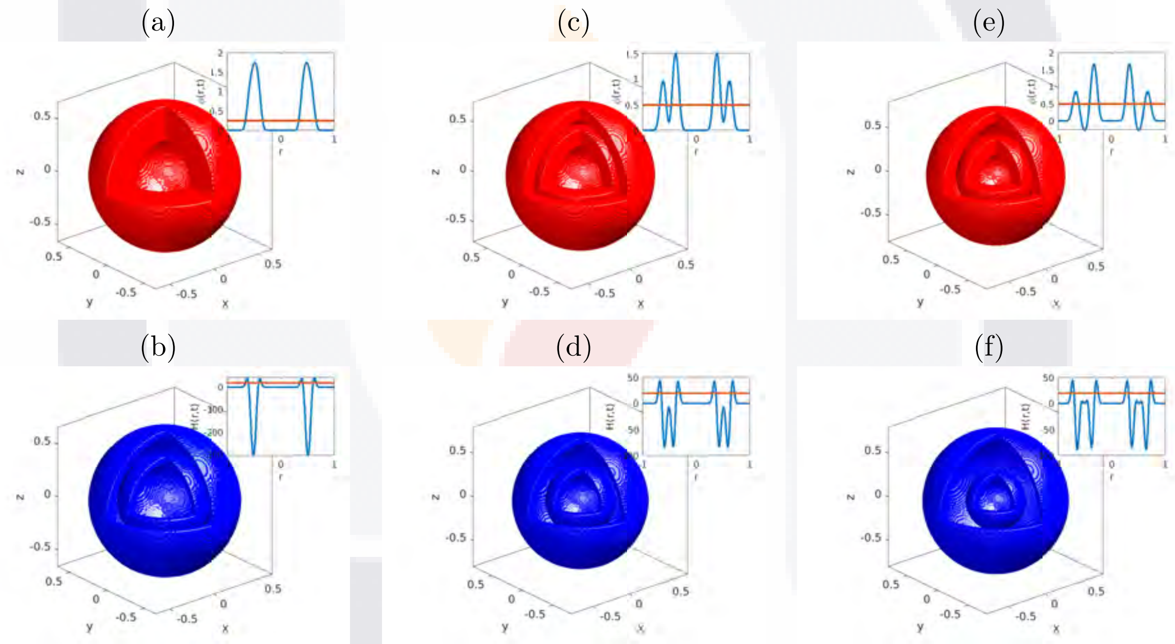


**Figure 2.4** Solutions of the problem (2.5.1) at the times $t = 0.060$ (left column), $t = 0.120$ (middle column) and $t = 0.180$ (right column), obtained using the scheme (2.3.11). The graphs on the top row show the approximate solution $\psi(x,t)$, for each $x \in D$. The insets are the corresponding graph of $\psi(r,t)$ in the radial coordinate $r$. The graphs on the bottom row correspond to those of the Hamiltonians. We used the parameters $\tau = h = 0.002$, $L = 1$, $\lambda = 2$ and $\mu = 3$. The initial data were $\phi_0(r) = 3\mathrm{B}(r;0.5,0.3)$ and $\phi_1(r) = 0$, for each $r \in [0,1]$.

**Definition 2.5.1.** For each $R > 0$ and $r_0 \in \mathbb{R}$, we define the function $\mathrm{B}(\cdot; r_0, R) : \mathbb{R}^3 \to \mathbb{R}$ by

$$\mathrm{B}(r; r_0, R) = \begin{cases} \exp\left(\dfrac{1}{R^2} - \dfrac{1}{R^2 - |r - r_0|^2}\right), & \text{if } |r - r_0| < R, \\ 0, & \text{if } |r - r_0| \geq R. \end{cases} \tag{2.5.4}$$

In all our examples, we will let $\tau = h = 0.002$, set $L = 1$ and let $D$ be the unit ball in $\mathbb{R}^3$. Moreover, we will employ the model parameters $\lambda = 2$ and $\mu = 3$. Different initial conditions will be considered for the initial-boundary-value problem (2.5.1).

In a first example, we let $\phi_0(r) = \mathrm{B}(r; 0.5, 0.3)$ and $\phi_1(r) = -5\phi_0(r)$, for each $r \in [0, 1]$. Under these circumstances, Figure 2.2 shows the solutions of the problem (2.5.1) at the times $t = 0.060$ (top row), $t = 0.120$ (middle row) and $t = 0.180$ (bottom row). The graphs on the top row of Figure 2.2 show the approximate solution $\phi(x, t)$, for each $x \in D$. Meanwhile, the insets provide the corresponding graph of the function $\phi(r, t)$ in the radial coordinate. On the other hand, the graphs on the bottom row correspond to those of the Hamiltonians. Obviously, the insets are the same Hamiltonians as functions of the radial variable $r$. The three-dimensional graphs were obtained using of the Matlab® function `isosurface`. In each graph, the corresponding `isovalue` was given by the red line of the associated inset.

In addition to the experiment described in the previous paragraph, we have used two additional sets of initial conditions, namely, $\phi_0(r) = 2\mathrm{B}(r; 0.5, 0.2)$ and $\phi_1(r) = 5\phi_0(r)$, for each $r \in [0, 1]$, and $\phi_0(r) = 3\mathrm{B}(r; 0.5, 0.3)$ and $\phi_1(r) = 0$, for each $r \in [0, 1]$. The results are shown in Figures 2.3 and 2.4, respectively. From all these simulations, some observations need to be highlighted. Firstly, the numerical simulations obtained in this section illustrate the fact that the numerical model proposed in this chapter is a stable technique when the computational parameters are sufficiently small. Secondly, the simulations above were performed to compare them against those obtained in [6]. It is worth pointing out that our results are in good qualitative agreement with those available in the literature. This provides strong evidence on the accuracy of our numerical model and its computational implementation. Finally, notice that the graphs show the formation of bubble-like solutions, in agreement with Theorem 2.2.1.

# 3 A fractional Higgs boson equation

*THE PRESENT CHAPTER is the first paper in the literature to report on a Hamiltonian discretization of the (fractional) Higgs boson equation in the de Sitter space-time, and its theoretical analysis. More precisely, we design herein a numerically efficient finite-difference Hamiltonian technique for the solution of a fractional extension of the Higgs boson equation in the de Sitter space-time. The model under investigation is a multidimensional equation with generalized potential and Riesz space-fractional derivatives of orders in* $(1,2]$*. An energy integral for the model is readily available, and we propose a nonlinear, implicit and consistent numerical technique based on fractional-order centered differences, with similar Hamiltonian properties in the discrete scenario. A fractional energy approach is used then to prove the properties of stability and convergence of the technique. For simulation purposes, we consider both the classical and the fractional Higgs real-valued scalar fields in the de Sitter space-time, and find results qualitatively similar to those available in the literature. For the sake of convenience, we provide the Matlab code of an alternative linear discretization of the method presented in this chapter. This linear implicit approach is thoroughly analyzed also.*

## 3.1 Background

The design of energy-preserving methods for physical systems has been a fruitful avenue of research in the last decades. Historically, the problem of designing energy-conserving methods may date back to the decade of the 1970s [4, 95] or before. However, it is worth mentioning that L. Vázquez and coauthors were probably the first researchers who pointed out the physical and mathematical significance of designing this type of schemes [85]. Various seminal papers by Vázquez and his coworkers were published in the 1990s, including various energy-conserving numerical schemes to solve partial differential equations like the Schrödinger equation [102], the sine-Gordon equation [9, 30], the Klein–Gordon equations [100], and even systems consisting of ordinary differential equations [29]. In those papers, the authors established thoroughly the capability of their schemes to preserve the energy properties of the continuous problem. Moreover, they employed a discrete form of the energy method to establish rigorously the stability and the convergence properties of the schemes. After the publication of those works, the investigation on energy-conserving schemes became a highly transited route of investigation, and many interesting articles were proposed in the specialized literature. As examples, some energy-preserving methods have been proposed to simulate the nonlinear dynamics of three-dimensional beams undergoing finite rotations [46], to approximate the kinematics of geometrically exact rods [91] and frictionless dynamic contact problems [56], among other interesting reports.

After those seminal works by Vázquez and coauthors, the investigation on energy-preserving schemes became a vast area of research. However, those papers by D. Furihata and collaborators published at the beginning of the millennium became a landmark in the area [33, 35]. In particular, they contributed to the state of the art by reviewing various existing methods for hyperbolic partial differential equations which conserved or dissipated the energy of the systems [34, 76]. Those works would eventually pave the road to the birth of the *discrete variational derivative method*, which is a helpful tool to construct finite-difference schemes resembling the variational properties of continuous models [36]. Various works have been published in that area, including studies for the simulation of nonlinear partial differential equations with variable coefficients [48], the solution of nonlinear systems based on the use of discrete differential forms [120], the investigation of numerical schemes using average-difference approaches [37], the two-dimensional vorticity equation [101], the solution of Hamilton's equation using variational principles [49] and the investigation of coupled partial differential equations through an alternating form of the discrete variational derivative method [54], among other interesting works. Needless to mention that this approach has been extended to consider different discretization methods, including finite elements [47] and other techniques [75].

It is worth pointing out that most of the problems investigated using the discrete variational derivative method are hyperbolic systems. However, there are some systems which have not been able to be solved using this approach, one of them being the Higgs boson equation in the de Sitter space-time. Some analytical results are known on the solutions of this system [114, 119], including some analytical approximations to its solutions [121]. About the physical relevance of this model, there are already various works available in the literature which justify its physical use [45, 57]. Nevertheless, we must mention that there are very few papers available in the literature which propose numerical methodologies to solve Higgs boson equation in the de Sitter space-time, that are capable of preserving the variational properties of this system. Among the most recent progress in this field, we can mention some articles which propose high-performance implementations of Runge–Kutta finite-difference schemes for this model [6]. Unfortunately, those discretizations are not capable of preserving the energy properties of the continuous Higgs boson equation in the de Sitter space-time. In general, the literature lacks numerical models to solve this physical equation, in such way that the energy structure of the continuous system is reflected on the discrete case. Such numerical integrators would be extremely useful, especially in the investigation of special types of solutions of this equation, like the so-called "bubbles" and other physically relevant structures [114].

On the other hand, fractional derivatives have been introduced to mathematical models in order to provide more realistic descriptions of the physical phenomena. For instance, many fractional systems have been obtained as the continuous limit of discrete systems of particles with long-range interactions [103, 104], and fractional derivatives have been successfully used in the theory of viscoelasticity [53], the theory of thermoelasticity [88], financial problems under a continuous-time frame [92], self-similar protein dynamics [40] and quantum mechanics [81]. As expected, the complexity of fractional problems is considerably higher than that of integer-order models, whence the need to design reliable numerical techniques to approximate the solutions is pragmatically justified [71]. In this direction, the literature reports on various methods to approximate the solutions of fractional systems. For example, some numerical methods have been proposed to solve fractional partial differential equations using fractional centered differences [82, 62, 63], the time-fractional diffusion equation [3], the fractional Schrödinger equation in multiple spatial dimensions [11], the nonlinear fractional Korteweg–de Vries–Burgers equation [25] and the fractional FitzHugh–Nagumo monodomain models [61], among other examples [52, 59, 32, 77, 89].

Motivated by these facts, we propose here a fractional extension of the Higgs boson equation in the de Sitter space-time that considers spatial fractional derivatives. The generalization introduced in this chapter not only extends the equation of motion of that model, but also the Hamiltonian associated to the system. We show that the total energy of the fractional system is dissipated with respect to time, and it extends the well-known formula for the integer-order case. To that end, we will use some functional properties of the fractional differential operators. A finite-difference discretization of the physical model is proposed in this chapter and we provide discrete forms of the Hamiltonian functional. We show that, like the continuous regime, the system is capable of preserving the dissipative nature of the total energy. The method is an implicit technique, and we show here that the numerical model is consistent of the second order in both space and time. Also, we employ a fractional discrete form of the energy method to prove the stability and the convergence of the finite-difference method. It is worth pointing out that the uniqueness of the discrete solutions will be a consequence of the stability of the scheme. Moreover, for the sake of convenience, we will introduce and analyze a linear form of our numerical model which is easy to implement.

The present chapter is sectioned as follows. In Section 3.2, we present the fractional partial differential equation of interest. We propose an energy density function associated to our model. We prove therein that the total energy of the system is dissipated, and it extends to the fractional scenario the well-known formulas of the Higgs boson equation in the de Sitter space-time. The boundedness of the solutions readily follows under suitable analytical assumptions. Section 3.3 introduces the discrete nomenclature employed throughout this chapter, and presents the finite-difference scheme to approximate the solutions of the mathematical model. Discrete forms of the local and total energy are presented therein, and we establish a discrete analogue of the theorem on the conservation/dissipation of energy of the continuous model. As expected, the boundedness of the numerical solutions is established under the same conditions of the continuous-case scenario. In turn, the purpose of Section 3.5 is to establish the main numerical features of our discrete model. Some illustrative simulations are shown in Section 3.6, including some numerical results that exhibit the presence of bubble-type solutions. Finally, we close this chapter with a section of concluding remarks, followed by two appendices: one which provides alternative discretizations of the terms in the discrete energy scheme (let see appendix A.2), and another in which we present the Matlab code to produce the simulations of this chapter (let see appendix A.3).

## 3.2 Physical model

In this section, we will present the partial differential equation of interest, and recall various useful properties of its solutions. In particular, we will recall the energy properties associated to the Higgs boson equation, and record some theorems on the existence of special solutions. Moreover, the concept of Riesz fractional derivative will be recalled from the literature along with some properties. For the sake of convenience, we will let $I_n = \{1, \ldots, n\}$ and $\overline{I}_n = I_n \cup \{0\}$, for each $n \in \mathbb{N}$. Also, integration in this chapter will be understood in the sense of Lebesgue. Moreover, any vector $x \in \mathbb{R}^p$ will be represented component-wise as $x = (x_1, x_2, \ldots, x_p)$, for each $p \in \mathbb{N}$.

The present chapter is motivated by the lack in the literature of Hamiltonian and numerically efficient finite-difference schemes to solve the well-known Higgs boson equation in the de Sitter space-time. Concretely, let $\phi : \mathbb{R}^p \times \overline{\mathbb{R}^+}$ be a sufficiently smooth function with $p \in \mathbb{N}$, and suppose that $\phi_0, \phi_1 : \mathbb{R}^p \to \mathbb{R}$ are two smooth functions. The model under investigation is described by the semi-linear hyperbolic partial differential equation with initial data

$$\begin{aligned} & \frac{\partial^2 \phi(x,t)}{\partial t^2} - e^{-2t} \Delta \phi(x,t) + p \frac{\partial \phi(x,t)}{\partial t} + F'(\phi(x,t)) = 0, \qquad \forall (x,t) \in \mathbb{R}^p \times \mathbb{R}^+, \\ & \text{such that } \begin{cases} \phi(x,0) = \phi_0(x), & \forall x \in \mathbb{R}^p, \\ \dfrac{\partial \phi(x,0)}{\partial t} = \phi_1(x), & \forall x \in \mathbb{R}^p, \end{cases} \end{aligned} \tag{3.2.1}$$

with $F'(\phi(x,t)) = -\mu^2 \phi(x,t) + \lambda |\phi(x,t)|^{q-1} \phi(x,t)$. In this expression, $\Delta$ denotes the $p$-dimensional Laplacian operator, $\mu$ and $\lambda$ are positive constants, and $p \geq 1$. The determination of necessary and sufficient conditions for the existence and uniqueness of solutions for this model is still an open problem of research. There are recent reports on some sufficient conditions for the global temporal existence of solutions [118], while other works report on the qualitative behavior of the global solutions of (3.2.1) and the existence of zeros [114]. However, the analytical investigation of (3.2.1) is still an ongoing topic of study.

It is worth pointing out that Higgs boson equation is an important scalar field in the standard model of particle physics [27]. Moreover, a standard transformation, variational arguments and straightforward substitutions show that the total energy of the system (3.2.1) is given for each $t \in (0,T)$ by the expression

$$\mathcal{E}(t) = e^{pt}\left(\frac{1}{2}\left\|\frac{\partial\phi}{\partial t}+\frac{p}{2}\phi\right\|_{x,2}^2+\frac{e^{-2t}}{2}\|\nabla\phi\|_{x,2}^2-\frac{1}{2}\left(\frac{p^2}{4}+\mu^2\right)\|\phi\|_{x,2}^2+\frac{\lambda}{q+1}\|\phi\|_{x,q+1}^{q+1}\right). \quad (3.2.2)$$

In this formula, the symbol $\nabla$ represents the usual gradient in the spatial variables. Using elementary analysis, one may readily check that the rate of change of energy is a non-increasing function of time. In fact, it is easy to see that

$$\frac{d\mathcal{E}(t)}{dt} = -e^{pt}\left(e^{-2t}\|\nabla\phi\|_{x,2}^2+\frac{\lambda p(q-1)}{2(q+1)}\|\phi\|_{x,q+1}^{q+1}\right), \qquad \forall t \in (0,T). \quad (3.2.3)$$

Obviously, the constants $\phi = \pm\mu/\sqrt{\lambda}$ are nontrivial solutions of (3.2.1). In either case, the total energy at the time $t$ is given by $\mathcal{E}(t) = e^{pt}\mu^2(2p^2-1)/(8\lambda)$. The following is one of the most important results on the qualitative behavior of solutions of (3.2.1).

**Theorem 3.2.1** (Yagdjian [114])**.** *Let $2 \le r < \infty$, and let $\phi \in \mathcal{C}([0,\infty];L^r(\mathbb{R}^p))$ be a global weak solution of* (3.2.1)*. Suppose that the initial data satisfy*

$$\sigma\left[\left(\frac{p}{2}\sqrt{\frac{p^2}{4}+\mu^2}\right)\phi_0(x)+\phi_1(x)\right] > 0, \qquad \forall x \in \mathbb{R}^p, \quad (3.2.4)$$

*where $\sigma = 1$ (respectively, $\sigma = -1$), and that*

$$\sigma\int_{\mathbb{R}^p}|\phi(x,t)|^{q-1}\phi(x,t)dx \le 0 \quad (3.2.5)$$

*is satisfied for all $t$ outside of a sufficiently small neighborhood of $0$. Then the solution $\phi$ cannot be an asymptotically time-weighted $L^q$-non-positive (respectively, -nonnegative) solution of the problem* (3.2.1) *with the weight $\mu_\phi(t) = e^{a_\phi t}t^{b_\phi}$, where $a_0 = \sqrt{\frac{n^2}{4}+\mu^2}-\frac{n}{2}$, and either*

(a) *$a_\phi < a_0$ and $b_\phi \in \mathbb{R}$, or*

(b) *$a_\phi = a_0$ and $b_\phi < 2$.*

**Definition 3.2.2** (Podlubny [87])**.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function, and let $n\in\mathbb{N}\cup\{0\}$ and $\alpha\in\mathbb{R}$ satisfy $n-1<\alpha<n$. The *Riesz fractional derivative* of $f$ of order $\alpha$ at $x\in\mathbb{R}$ is defined (when it exists) as

$$\frac{d^\alpha f(x)}{d|x|^\alpha}=\frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{3.2.6}$$

**Definition 3.2.3** (Podlubny [87])**.** Assume that $p\in\mathbb{N}$ and $i\in I_p$. Let $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$ satisfy $n-1<\alpha<n$, and suppose that $\phi:\mathbb{R}^p\times\mathbb{R}\to\mathbb{R}$ is a function. When it exists, the *Riesz fractional partial derivative* of $\phi$ of order $\alpha$ with respect to $x_i$ at the point $(x,t)\in\mathbb{R}^p\times\mathbb{R}$ is defined by

$$\frac{\partial^\alpha\phi(x,t)}{\partial|x_i|^\alpha}=-\frac{1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)}\frac{\partial^n}{\partial x_i^n}\int_{-\infty}^{\infty}\frac{\phi(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_p,t)}{|x_i-\xi|^{\alpha+1-n}}d\xi, \tag{3.2.7}$$

whenever it exists. For the remainder of this chapter, we will consider differentiation orders satisfying $1<\alpha<2$. Moreover, for convenience, we will agree that the Riesz partial fractional derivatives of $\phi$ of order 1 and 2 with respect to $x_i$ coincide with the usual first- and second-order partial derivatives of $\phi$ with respect to $x_i$, respectively. Finally, the *Riesz fractional Laplacian* of $\phi$ of order $\alpha$ at $(x,t)$ will be defined as

$$\Delta^\alpha\phi(x,t)=\sum_{i=1}^{p}\frac{\partial^\alpha\phi(x,t)}{\partial|x_i|^\alpha}. \tag{3.2.8}$$

Also, the corresponding fractional gradient of $\phi$ at $(x,t)$ will be given by the $p$-dimensional vector

$$\nabla^{\alpha/2}\phi(x,t)=\left(\frac{\partial^{\alpha/2}\phi(x,t)}{\partial|x_1|^{\alpha/2}},\frac{\partial^{\alpha/2}\phi(x,t)}{\partial|x_2|^{\alpha/2}},\ldots,\frac{\partial^{\alpha/2}\phi(x,t)}{\partial|x_p|^{\alpha/2}}\right). \tag{3.2.9}$$

It is interesting to point out that various different representations of the Riesz derivative are available in the literature of space-fractional quantum mechanics [8].

For the remainder of this chapter, let $p\in\mathbb{N}$ represent the number of spatial dimensions, and let $B=\prod_{i=1}^{p}(a_i,b_i)\subseteq\mathbb{R}^p$ be a spatial domain, where $-\infty<a_i<b_i<\infty$, for each $i\in I_p$. Let $T$ be a positive number, and define $\Omega=B\times(0,T)$ as the space-time domain of reference. Also, we will consider real functions defined on $\overline{\Omega}$, and we will extend them to all of $\mathbb{R}^p\times[0,T]$ by setting them equal to zero outside of $\overline{B}$.

**Definition 3.2.4.** Let $L_{x,2}(\overline{\Omega})$ denote the set of all functions $\phi:\overline{\Omega}\to\mathbb{R}$ such that $\phi(\cdot,t)\in L^2(\overline{B})$, for each $t\in[0,T]$. Moreover, for each pair $\phi,\psi\in L_{x,2}(\overline{\Omega})$, the inner product of $f$ and $g$ is the function of $t$ defined by

$$\langle\phi,\psi\rangle_x=\int_B\phi(x,t)\psi(x,t)dx,\qquad\forall t\in[0,T]. \tag{3.2.10}$$

In turn, the Euclidean norm of $\phi \in L_{x,2}(\overline{\Omega})$ is the function of $t$ defined by $\|\phi\|_{x,2} = \sqrt{\langle \phi, \phi \rangle_x}$. In general, if $1 \leqslant q < \infty$ then $L_{x,q}(\overline{\Omega})$ represents the set of all functions $\phi : \overline{\Omega} \to \mathbb{R}$ such that $\phi(\cdot, t) \in L^q(\overline{B})$, for each $t \in [0,T]$. For each such function $\phi$, we define its norm as the function of $t$ given by

$$\|\phi\|_{x,q} = \left( \int_{\overline{B}} |\phi(x,t)|^q dx \right)^{1/q}, \qquad \forall t \in [0,T]. \tag{3.2.11}$$

It is important to recall that the Riesz fractional derivative of order $\alpha \in (1,2]$ is a self-adjoint and negative operator [55]. Moreover, it is a well-known fact that positive self-adjoint operators possess positive square-roots, and that they are unique when they exist [31]. This and [55] imply that the additive inverse of the Riesz fractional derivative has a unique square-root operator. It turns out that such square root is the Riesz fractional derivative of order $\alpha/2$, and it satisfies the following property, for all functions $\phi, \psi : \overline{\Omega} \to \mathbb{R}$ and $i \in I_p$:

$$\left\langle -\frac{\partial^\alpha \phi}{\partial |x_i|^\alpha}, \psi \right\rangle_x = \left\langle \frac{\partial^{\alpha/2} \phi}{\partial |x_i|^{\alpha/2}}, \frac{\partial^{\alpha/2} \psi}{\partial |x_i|^{\alpha/2}} \right\rangle_x = \left\langle \phi, -\frac{\partial^\alpha \psi}{\partial |x_i|^\alpha} \right\rangle_x. \tag{3.2.12}$$

In this chapter, we will study an extended form of (3.2.1) which considers a general time-dependent diffusion coefficient, fractional diffusion and a generalized potential. More precisely, let $\alpha \in (0,1) \cup (1,2]$, let $\gamma \in \overline{\mathbb{R}^+}$, and suppose that $f : \overline{\mathbb{R}^+} \to \mathbb{R}$ is a differentiable function. Let $F : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that $\phi_0, \phi_1 : B \to \mathbb{R}$ are continuous functions. In the present chapter, we will investigate the numerical solution of the initial-value problem

$$\begin{gathered}
\frac{\partial^2 \phi(x,t)}{\partial t^2} - f(t) \Delta^\alpha \phi(x,t) + \gamma \frac{\partial \phi(x,t)}{\partial t} + F'(\phi(x,t)) = 0, \qquad \forall (x,t) \in \Omega, \\
\text{such that } \begin{cases} \phi(x,0) = \phi_0(x), & \forall x \in B, \\ \dfrac{\partial \phi(x,0)}{\partial t} = \phi_1(x), & \forall x \in B. \end{cases}
\end{gathered} \tag{3.2.13}$$

Obviously, the Higgs boson equation in the de Sitter space-time is obtained from (3.2.13) in the case when $f(t) = e^{-2t}$, $\gamma = p$ and $F(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{q+1}\lambda|\phi|^{q+1}$, for each $t \in \overline{\mathbb{R}^+}$ and $\phi \in \mathbb{R}$.

**Theorem 3.2.5.** *The total energy of the system* (3.2.13) *at the time $t$ is given by*

$$\mathcal{E}(t) = e^{\gamma t} \left[ \frac{1}{2} \left\| \frac{\partial \phi}{\partial t} \right\|_{x,2}^2 + \frac{\gamma}{2} \left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle_x + \frac{f(t)}{2} \left\| \nabla^{\alpha/2} \phi \right\|_{x,2}^2 + \langle F(\phi), 1 \rangle_x \right], \qquad \forall t \in (0,T]. \tag{3.2.14}$$

*Proof.* Beforehand, notice that the partial differential equation of (3.2.13) becomes an undamped system using the standard transformation $\phi(x,t) = \exp(-\frac{\gamma}{2}t)\psi(x,t)$, for each $(x,t) \in \Omega$. After a straightforward substitution and algebraic simplifications, the resulting equation of motion reads

$$\frac{\partial^2\psi(x,t)}{\partial t^2} - f(t)\Delta^\alpha\psi(x,t) - \frac{\gamma^2}{4}\psi(x,t) + e^{\frac{\gamma}{2}t}F'(\phi(x,t)) = 0, \qquad \forall (x,t)\in\Omega. \tag{3.2.15}$$

Notice that the Lagrangian of the new system (3.2.15) is given by

$$\mathcal{L}(\psi,\nabla^{\alpha/2}\psi) = -\frac{\gamma^2}{8}\psi^2(x,t) + \frac{f(t)}{2}\left\|\nabla^{\alpha/2}\psi(x,t)\right\|_2^2 + e^{\gamma t}F(\phi(x,t)), \qquad \forall (x,t)\in\Omega. \tag{3.2.16}$$

Using the variational derivative of $\mathcal{L}$ with respect to $(\psi,\nabla^{\alpha/2}\phi)$, it is easy to obtain the following expression for the total energy of (3.2.15) at the time $t$:

$$\mathcal{E}(t) = \frac{1}{2}\left\|\frac{\partial\psi}{\partial t}\right\|_{x,2}^2 + \frac{f(t)}{2}\left\|\nabla^{\alpha/2}\psi\right\|_{x,2}^2 - \frac{\gamma^2}{8}\|\psi\|_{x,2}^2 + e^{\gamma t}\langle F(\phi),1\rangle_x, \qquad \forall t\in(0,T). \tag{3.2.17}$$

The conclusion of this theorem follows now recalling that $\psi(x,t) = \exp(\frac{\gamma}{2}t)\phi(x,t)$ holds for each $(x,t)\in\Omega$, substituting this expression into (3.2.17) and simplifying algebraically. □

The following result is straightforward. We provide here an abridged version of the proof in order to be able to carry it over to the discrete-case scenario.

**Corollary 3.2.6** (Energy rate of change). *The rate of change of the energy of the system* (3.2.13) *at time $t$ is given by*

$$\mathcal{E}'(t) = e^{\gamma t}\left[\frac{f'(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 - \frac{\gamma}{2}\langle F'(\phi),\phi\rangle_x + \gamma\langle F(\phi),1\rangle_x\right], \qquad \forall t\in(0,T). \tag{3.2.18}$$

*Proof.* Beforehand, note that the following identities can be easily established using the product rule, the chain rule and functional properties of the fractional differential operators. It is important to point out that they are satisfied for all $t\in(0,T)$:

$$\frac{d}{dt}\left[\frac{e^{\gamma t}}{2}\left\|\frac{\partial\phi}{\partial t}\right\|_{x,2}^2\right] = \frac{\gamma e^{\gamma t}}{2}\left\|\frac{\partial\phi}{\partial t}\right\|_{x,2}^2 + e^{\gamma t}\left\langle\frac{\partial^2\phi}{\partial t^2},\frac{\partial\phi}{\partial t}\right\rangle_x, \tag{3.2.19}$$

$$\frac{d}{dt}\left[\frac{\gamma e^{\gamma t}}{2}\left\langle\frac{\partial\phi}{\partial t},\phi\right\rangle_{x,2}\right] = \frac{\gamma^2 e^{\gamma t}}{2}\left\langle\frac{\partial\phi}{\partial t},\phi\right\rangle_x + \frac{\gamma e^{\gamma t}}{2}\left\langle\frac{\partial^2\phi}{\partial t^2},\phi\right\rangle_x + \frac{\gamma e^{\gamma t}}{2}\left\|\frac{\partial\phi}{\partial t}\right\|_{x,2}, \tag{3.2.20}$$

$$\begin{aligned}\frac{d}{dt}\left[\frac{f(t)e^{\gamma t}}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2\right] &= \frac{\gamma e^{\gamma t}f(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 + \frac{e^{\gamma t}f'(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 \\ &\quad - e^{\gamma t}f(t)\left\langle\Delta^\alpha\phi,\frac{\partial\phi}{\partial t}\right\rangle_x,\end{aligned} \tag{3.2.21}$$

$$\frac{d}{dt}\left[e^{\gamma t}\langle F(\phi),1\rangle\right] = \gamma e^{\gamma t}\langle F(\phi),1\rangle_x + e^{\gamma t}\left\langle F'(\phi),\frac{\partial\phi}{\partial t}\right\rangle_x. \tag{3.2.22}$$

Take now the derivative of (3.2.14) with respect to $t$, substitute the identities above, collect then the last terms corresponding to the right-hand sides of the identities (3.2.19), (3.2.21) and (3.2.22), and substitute the equation of motion in (3.2.13). At the same time, collect those inner products with $\phi$. As a consequence, we readily obtain the identities

$$\begin{aligned}\mathcal{E}'(t) &= e^{\gamma t}\left[\frac{\gamma}{2}\left\langle \frac{\partial^2\phi}{\partial t^2}+\gamma\frac{\partial \phi}{\partial t},\phi\right\rangle_x + \frac{\gamma f(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 \right.\\ &\qquad \left. + \frac{f'(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 + \gamma\langle F(\phi),1\rangle_x\right] \\ &= e^{\gamma t}\left[\frac{\gamma f(t)}{2}\langle \Delta^\alpha \phi,\phi\rangle_x - \frac{\gamma}{2}\langle F'(\phi),\phi\rangle_x + \frac{\gamma f(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2\right. \\ &\qquad \left. + \frac{f'(t)}{2}\left\|\nabla^{\alpha/2}\phi\right\|_{x,2}^2 + \gamma\langle F(\phi),1\rangle_x\right].\end{aligned} \tag{3.2.23}$$

Finally, use the square-root properties of the fractional derivatives to cancel out the first and the third terms on the right-hand side of this equation. The conclusion of this result readily follows. □

**Definition 3.2.7.** Let $\phi$ be a solution of (3.2.13). We define the *energy density* of the system at the point $(x,t)$ by

$$\mathcal{H}(x,t) = e^{\gamma t}\left[\frac{1}{2}\left(\frac{\partial \phi(x,t)}{\partial t}\right)^2 + \frac{\gamma}{2}\frac{\partial\phi(x,t)}{\partial t}\phi(x,t) + \frac{f(t)}{2}\left|\nabla^{\alpha/2}\phi(x,t)\right|^2 + F(\phi(x,t))\right]. \tag{3.2.24}$$

## 3.3 Numerical method

In this section, we will introduce the discrete nomenclature and the numerical model to solve the initial-value problem (3.2.13). To that end, we will follow a finite-difference methodology. Throughout this chapter, we will let $K$ be a natural number, and consider a uniform partition of the temporal interval $[0,T]$ consisting of $K$ subintervals. Obviously, the norm of this partition is given by the number $\tau = T/K$. On the other hand, let $M_i \in \mathbb{N}$ for each $i \in I_p$, and fix a uniform partition of $[a_i,b_i]$ with norm $h_i = (b_i - a_i)/M_i$. Introduce the respective partition nodes

$$t_k = k\tau, \qquad \forall k \in \overline{I}_K, \tag{3.3.1}$$

$$x_{i,j} = a_i + jh_i, \qquad \forall i \in I_p, \forall j \in \overline{I}_{M_i}. \tag{3.3.2}$$

For convenience, we agree that $t_{k+\frac{1}{2}} = (k+\frac{1}{2})\tau$, for each $k \in \overline{I}_{K-1}$. Moreover, define the sets $J = \prod_{i=1}^p I_{M_i-2}$ and $\overline{J} = \prod_{i=1}^p \overline{I}_{M_i}$. Finally, for any multi-index $j = (j_1, j_2, \dots, j_p) \in \overline{J}$, we define $x_j = (x_{1,j_1}, x_{2,j_2}, \dots, x_{p,j_p})$.

Throughout this paper, we will convey that $h = (h_1, h_2, \dots, h_p)$ and $h_* = h_1 h_2 \cdots h_p$. In this chapter, we fix the grid set $\mathcal{R}_h = \{x_j : j \in \overline{J}\}$, and use the symbol $\mathcal{V}_h$ to denote the real vector space of all real functions defined on $\mathcal{R}_h$ which vanish on $\partial J$. If $u \in \mathcal{V}_h$ is any function then we let $u_j = u(x_j)$, for each $j \in \overline{J}$. Moreover, if $(j,k) \in \overline{J} \times \overline{I}_K$ then we convey that $\phi_j^k = \phi(x_j, t_k)$, and we let $\Phi_j^k$ represent a numerical approximation to the exact value of $\phi_j^k$. It is easy to see that $\phi^k = (\phi_j^k)_{j\in\overline{J}}$ and $\Phi^k = (\Phi_j^k)_{j\in\overline{J}}$ are actually members of the set $\mathcal{V}_h$, for each $k \in \overline{I}_K$.

For the remainder of this chapter and unless we mention otherwise, we agree that $\Phi = (\Phi^k)_{k\in\overline{I}_K}$.

**Definition 3.3.1.** Define the *inner product* $\langle\cdot,\cdot\rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ and the *norm* $\|\cdot\|_p : \mathcal{V}_h \to \mathbb{R}$ by

$$\langle \Phi, \Psi \rangle = h_* \sum_{j\in\overline{J}} \Phi_j \Psi_j, \qquad \forall \Phi, \Psi \in \mathcal{V}_h, \tag{3.3.3}$$

$$\|\Phi\|_p = \left[ h_* \sum_{j\in\overline{J}} |\Phi_j|^p \right]^{1/p}, \qquad \forall \Phi \in \mathcal{V}_h. \tag{3.3.4}$$

The Euclidean norm induced by $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|_2$, and $\|\cdot\|_\infty : \mathcal{V}_h \to \mathbb{R}$ will be the usual infinity norm in $\mathcal{V}_h$, which is defined as $\|\Phi\|_\infty = \max\{|\Phi_j| : j \in \overline{J}\}$, for each $\Phi \in \mathcal{V}_h$.

**Definition 3.3.2.** We introduce the following linear operators on $\mathcal{V}_h$, for each $(\Psi^k)_{k\in\overline{I}_K} \subseteq \mathcal{V}_h$:

$$\mu_t \Psi_j^k = \frac{\Psi_j^{k+1} + \Psi_j^k}{2}, \qquad \forall (j,k) \in J \times \overline{I}_{K-1}, \tag{3.3.5}$$

$$\mu_t^{(1)} \Psi_j^k = \frac{\Psi_j^{k+1} + \Psi_j^{k-1}}{2}, \qquad \forall (j,k) \in J \times I_{K-1}, \tag{3.3.6}$$

$$\delta_t \Psi_j^k = \frac{\Psi_j^{k+1} - \Psi_j^k}{\tau}, \qquad \forall (j,k) \in J \times \overline{I}_{K-1}, \tag{3.3.7}$$

$$\delta_t^{(1)} \Psi_j^k = \frac{\Psi_j^{k+1} - \Psi_j^{k-1}}{2\tau}, \qquad \forall (j,k) \in J \times I_{K-1}, \tag{3.3.8}$$

$$\delta_t^{(2)} \Psi_j^k = \frac{\Psi_j^{k+1} - 2\Psi_j^k + \Psi_j^{k-1}}{\tau^2}, \qquad \forall (j,k) \in J \times I_{K-1}. \tag{3.3.9}$$

If $F : \mathbb{R} \to \mathbb{R}$ is differentiable then we introduce the following nonlinear operator, for all $(j,k) \in J \times I_{K-1}$:

$$\delta_{\Psi,t}^{(1)} F(\Psi_j^k) = \begin{cases} \dfrac{F(\Psi_j^{k+1}) - F(\Psi_j^{k-1})}{\Psi_j^{k+1} - \Psi_j^{k-1}}, & \text{if } \Psi_j^{k+1} \neq \Psi_j^{k-1}, \\ F'(\Psi_j^k), & \text{otherwise.} \end{cases} \tag{3.3.10}$$

It is important to mention that the finite averages and differences in Definition 3.3.2 are quadratically consistent approximations of some suitable differential operators. More concretely, under suitable regularity conditions on the function $\Psi$, the average operator

(3.3.5) is a consistent approximation of $\Psi(x_j,t_{k+\frac{1}{2}})$, while (3.3.6) approximates consistently the value $\Psi(x_j,t_k)$.

On the other hand, the difference operators (3.3.7) and (3.3.8) approximate consistently the partial derivative of $\Psi$ with respect to $t$ at $(x_j,t_{k+\frac{1}{2}})$ and $(x_j,t_k)$, respectively. In turn, the operator (3.3.9) estimates the second-order partial derivative of $\Psi$ with respect to $t$ at $(x_j,t_k)$, and (3.3.10) estimates $F'(\Psi(x_j,t_k))$.

In this chapter, we will approximate fractional derivatives using fractional-order centered differences. We recall now the definition from the literature along with some useful properties.

**Definition 3.3.3** (Ortigueira [82]). For any function $f:\mathbb{R}\to\mathbb{R}$, any $h>0$ and $\alpha>-1$, the *fractional centered difference* of order $\alpha$ of $f$ at the point $x$ is defined as

$$\Delta_h^{(\alpha)}f(x)=\sum_{k=-\infty}^{\infty}g_k^{(\alpha)}f(x-kh),\qquad \forall x\in\mathbb{R}, \tag{3.3.11}$$

where

$$g_k^{(\alpha)}=\frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)},\qquad \forall k\in\mathbb{Z}. \tag{3.3.12}$$

**Lemma 3.3.4** (Çelik and Duman [16]). *If $1<\alpha\leqslant 2$ then the coefficients $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ satisfy various properties.*

(a) *The following recursive identities hold:*

$$g_0^{(\alpha)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2+1)^2},\quad \text{and}\quad g_{k+1}^{(\alpha)}=\left(1-\frac{\alpha+1}{\alpha/2+k+1}\right)g_k,\quad \forall k\in\mathbb{N}\cup\{0\}. \tag{3.3.13}$$

(b) $g_0^{(\alpha)}>0$.

(c) $g_k^{(\alpha)}=g_{-k}^{(\alpha)}\leqslant 0$ *for all* $k\neq 0$.

(d) $\sum_{k=-\infty}^{\infty}g_k^{(\alpha)}=0$. *As a consequence, it follows that* $g_0^{(\alpha)}=-\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}g_k^{(\alpha)}$.

**Lemma 3.3.5** (Çelik and Duman [16]). *Let $f\in\mathcal{C}^5(\mathbb{R})$ and assume that all its derivatives up to order five are integrable. If $1<\alpha\leqslant 2$ then, for almost all $x$,*

$$-\frac{\Delta_h^{\alpha}f(x)}{h^{\alpha}}=\frac{d^{\alpha}f(x)}{d|x|^{\alpha}}+\mathcal{O}(h^2). \tag{3.3.14}$$

**Definition 3.3.6.** Let $\alpha\in(1,2]$ and suppose that $(\Phi^k)_{k\in\overline{I}_K}\subseteq\mathcal{V}_h$. For each $i\in I_p$ and $(j,k)\in\overline{J}\times\overline{I}_K$, we let

$$\delta_{x_i}^{(\alpha)}\Phi_j^k=-\frac{1}{h_i^{\alpha}}\sum_{l=0}^{M_i}g_{j_i-l}^{(\alpha)}\Phi_{j_1,\dots,j_{i-1},l,j_{i+1},\dots,j_p}^k. \tag{3.3.15}$$

In light of Lemma 3.3.5, the operator $\delta_{x_i}^{(\alpha)}$ yields a quadratically consistent approximation to the Riesz fractional partial derivative of $\phi$ of order $\alpha$ with respect to $x_i$ at the point $(x_j, t_k)$. Moreover, the fractional Laplacian will be approximated with a quadratic order of consistency using the discrete operator

$$\delta_x^{(\alpha)} \Phi_j^k = \sum_{i=1}^p \delta_{x_i}^{(\alpha)} \Phi_j^k. \tag{3.3.16}$$

Meanwhile, the discrete gradient operator of order $\alpha$ is defined as

$$\delta_x^{(\alpha/2)} \Phi_j^k = \left( \delta_{x_1}^{(\alpha/2)} \Phi_j^k, \delta_{x_2}^{(\alpha/2)} \Phi_j^k, \ldots, \delta_{x_p}^{(\alpha/2)} \Phi_j^k \right). \tag{3.3.17}$$

**Lemma 3.3.7** (Macías-Díaz [72]). *If $1 < \alpha \leqslant 2$ then there exists a unique positive self-adjoint (square-root) operator $\delta_{x_i}^{(\alpha/2)} : \mathcal{V}_h \to \mathcal{V}_h$, such that $\langle -\delta_x^{(\alpha)} \Phi^k, \Psi^k \rangle_x = \langle \delta_x^{(\alpha/2)} \Phi^k, \delta_x^{(\alpha/2)} \Psi^k \rangle_x$, for each $\Phi, \Psi \in \mathcal{V}_h$.*

The following result summarizes some properties of the fractional centered differences and their square-roots.

**Lemma 3.3.8** (Macías-Díaz [62]). *Let $\alpha \in (1,2]$ and define $g_h^{(\alpha)} = 2h_* g_0^{(\alpha)} \sum_{i=1}^p h_i^{-\alpha}$. If $\Phi \in \mathcal{V}_h$ and $i \in I_p$ then*

(a) $\|\delta_{x_i}^{(\alpha/2)} \Phi\|_2^2 \leqslant 2 g_0^{(\alpha)} h_* h_i^{-\alpha} \|\Phi\|_2^2$,

(b) $\|\delta_{x_i}^{(\alpha)} \Phi\|_2^2 \leqslant \|\delta_{x_i}^{(\alpha/2)} \delta_{x_i}^{(\alpha/2)} \Phi\|_2^2$,

(c) $\|\delta_{x_i}^{(\alpha)} \Phi\|_2^2 \leqslant 2 g_0^{(\alpha)} h_* h_i^{-\alpha} \|\delta_{x_i}^{(\alpha/2)} \Phi\|_2^2 \leqslant 4 \left( g_0^{(\alpha)} h_* h_i^{-\alpha} \right)^2 \|\Phi\|_2^2$,

(d) $\sum_{i=1}^p \|\delta_{x_i}^{(\alpha)} \Phi\|_2^2 \leqslant 2 h_* g_0^{(\alpha)} \sum_{i=1}^p h_i^{-\alpha} \|\delta_{x_i}^{(\alpha/2)} \Phi\|_2^2 \leqslant 4 h_*^2 \|\Phi\|_2^2 \sum_{i=1}^p \left( g_0^{(\alpha)} h_i^{-\alpha} \right)^2$, *and*

(e) $\sum_{i=1}^p \|\delta_{x_i}^{(\alpha)} \Phi\|_2^2 \leqslant g_h^{(\alpha)} \sum_{i=1}^p \|\delta_{x_i}^{(\alpha/2)} \Phi\|_2^2 \leqslant \left( g_h^{(\alpha)} \|\Phi\|_2 \right)^2$.

Let $\phi_0, \phi_1 : B \to \mathbb{R}$ be sufficiently smooth initial conditions for the problem (3.2.13). The finite-difference method to approximate the solutions of the continuous problem (3.2.13) is given by the system of discrete equations

$$\begin{gathered} \delta_t^{(2)} \Phi_j^k - \mu_t^{(1)} \left( f(t_k) \delta_x^{(\alpha)} \Phi_j^k \right) + \gamma \delta_t^{(1)} \Phi_j^k + \delta_{\Phi,t}^{(1)} F(\Phi_j^k) = 0, \qquad \forall (j,k) \in J \times \overline{I}_{K-1}, \\ \text{such that} \begin{cases} \Phi_j^0 = \phi_0(x_j), & \forall j \in J, \\ \delta_t^{(1)} \Phi_j^0 = \phi_1(x_j), & \forall j \in J. \end{cases} \end{gathered} \tag{3.3.18}$$

It is easy to check that the discrete model (3.3.18) is a three-step implicit finite-difference model. For the sake of convenience, Figure 3.1 shows the forward-difference stencil of this scheme in the one-dimensional case. In general, the computer implementation of (3.2.13) would require an algorithm to solve coupled systems of nonlinear algebraic equations, like and implementation of the Newton–Raphson method to approximate real roots of nonlinear systems of algebraic equations.

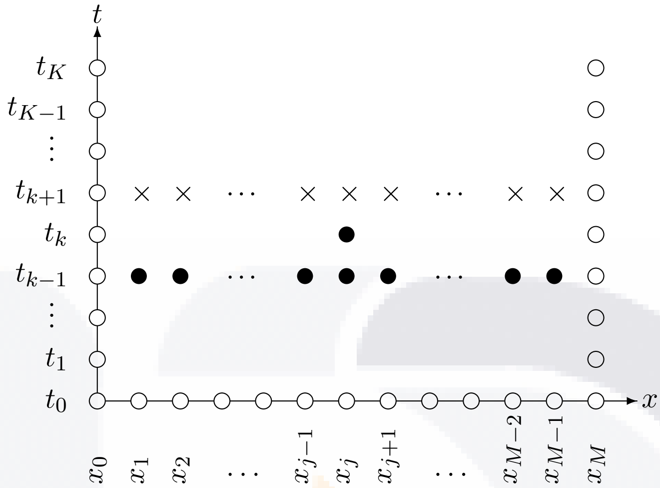


**Figure 3.1** Forward-difference stencil for the approximation to the exact solution of the one dimensional form of (3.2.13) at the time $t_k$, using the finite-difference scheme (3.3.18). The black circles represent the known approximations at the times $t_{k-1}$ and $t_k$, while the crosses denote the unknown approximations at the time $t_{k+1}$.

About the computer implementation, we just need to mention that the approximation at time $t = 0$ using the recursive formula of (3.3.18) with $k = 0$, makes use of the initial conditions $\delta_t^{(1)} \Phi_j^0 = \phi_1(x_j)$, for each $j \in J$. Indeed, from this expressions we readily obtain that $\Phi_j^{-1} = \Phi_j^1 - 2\tau^2 \phi_1(x_j)$, for each $j \in J$. This expressions are substituted then into the recursive formula, obtaining algebraic equations in which the only unknown is the vector $\Phi^1$. More precisely, the following identity must be satisfied, for each $j \in \overline{J}$:

$$\begin{aligned}\Phi_j^1 = \phi_0(x_j) + \tau^2 \phi_1(x_j) + \frac{\tau^2}{2}\left[\mu_t^{(1)} f(t_0)\right] \delta_x^{(\alpha)} \Phi_j^1 - \frac{\tau^4}{2} f(t_{-1}) \phi_1(x_j) - \frac{\tau^2 \gamma}{2} \phi_1(x_j) \\ - \frac{F(\Phi_j^1) - F(\Phi_j^1 - 2\tau^2 \phi_1(x_j))}{4\phi_1(x_j)}.\end{aligned} \tag{3.3.19}$$

**Definition 3.3.9.** Let $\Phi$ be a solution of the discrete model (3.3.18). For each $(j,k) \in J \times I_{K-2}$, we define the *discrete energy density* at the point $(x_j, t_k)$ through the formula

$$\begin{aligned}H_j^k = e^{\gamma t_{k-1}} \left[ \frac{1}{2}\left(\delta_t \Phi_j^{k-1}\right)^2 + \frac{\gamma}{2} (\delta_t \Phi_j^{k-1})(\mu_t \Phi_j^{k-1}) + \frac{f(t_{k-1})}{2} (\delta_x^{(\alpha/2)} \Phi_j^k)(\delta_x^{(\alpha/2)} \Phi_j^{k-1}) \right. \\ \left. + \mu_t F(\Phi_j^{k-1}) \right],\end{aligned} \tag{3.3.20}$$

meanwhile, the *discrete total energy* at the time $t_k$ is defined as

$$E^k = h_* \sum_{j \in \overline{J}} H_j^k$$
$$= e^{\gamma t_{k-1}} \left[ \frac{1}{2} \left\| \delta_t \Phi^{k-1} \right\|_2^2 + \frac{\gamma}{2} \langle \delta_t \Phi^{k-1}, \mu_t \Phi^{k-1} \rangle + \frac{f(t_{k-1})}{2} \langle \delta_x^{(\alpha/2)} \Phi^k, \delta_x^{(\alpha/2)} \Phi^{k-1} \rangle \right. \tag{3.3.21}$$
$$\left. + \mu_t \langle F(\Phi^{k-1}), 1 \rangle \right].$$

## 3.4 Structural properties

In this stage of our work, we will establish the most important structural properties of the finite-difference scheme (3.3.18). More precisely, we will show that our method and its discrete total energy operator (3.3.21) satisfy a discrete analogue of Theorem 3.2.6. However, we must establish firstly the solvability of the discrete system (3.3.18). The following result will be helpful to that end.

**Lemma 3.4.1** (Brouwer's fixed-point theorem)**.** *Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{V}$. Assume that $r : \mathcal{V} \to \mathcal{V}$ is continuous, and that there exists $\lambda > 0$ such that $\langle r(\Phi), \Phi \rangle \geqslant 0$, for all $\Phi \in \mathcal{V}$ with $\|\Phi\|_2 = \lambda$. Then there exists $\Phi \in \mathcal{V}$ with $\|\Phi\|_2 \leqslant \lambda$, such that $r(\Phi) = 0$.*

We establish next that the finite-difference method (3.3.18) is solvable. For convenience, additional notation will be required in the proof. Concretely, for each $\Phi, \Psi \in \mathcal{V}_h$ and $j \in J$, we define

$$\delta^{(1)}_{\Phi,\Psi,t} F(\Phi_j^k) = \begin{cases} \dfrac{F(\Psi_j) - F(\Phi_j^{k-1})}{\Psi_j - \Phi_j^{k-1}}, & \text{if } \Psi_j \neq \Phi_j^{k-1}, \\ F'(\Phi_j^k), & \text{otherwise.} \end{cases} \tag{3.4.1}$$

**Theorem 3.4.2** (Solvability)**.** *Suppose that there is $K_1 \geq 0$ such that $|f(t)| \leqslant K_1$, for all $t > 0$. If $F' \in L^\infty(\mathbb{R})$ and $2 + \gamma\tau - \tau^2 K_1 g_h^{(\alpha)} > 0$ holds then the discrete method (3.3.18) is solvable for any set of initial conditions.*

*Proof.* Notice that $\Phi^0$ and $\Phi^1$ are defined through the initial data. So let $k \in I_{K-1}$, and assume that $\Phi^{k-1}$ and $\Phi^k$ have been calculated already. Notice now that the assumptions on the regularity of $F$ imply that there exists a constant $K_2 \geq 0$, with the property that $\|\delta^{(1)}_{\psi,\phi,t} F(\Phi^k)\|_2 \leqslant K_2$. Let $r : \mathcal{V}_h \to \mathcal{V}_h$ be the continuous function whose $j$th component $r_j : \mathcal{V}_h \to \mathbb{R}$ is defined by

$$r_j(\Psi) = \frac{\Psi_j - 2\Phi_j^k + \Phi_j^{k-1}}{\tau^2} + \frac{f(t_{k+1})\delta_x^{(\alpha)} \Psi_j + f(t_{k-1}) \delta_x^{(\alpha)} \Phi_j^{k-1}}{2} + \gamma \frac{\Psi_j - \Phi_j^{k-1}}{2\tau} \tag{3.4.2}$$
$$+ \delta^{(1)}_{\Phi,\Psi,t} F(\Phi_j^k),$$

for each $\Psi \in \mathcal{V}_h$. Applying the Cauchy–Schwarz inequality, using the formulas in Lemma 3.3.8 along with the square-root properties of fractional-order centered differences, we obtain

$$\begin{aligned}
\langle r(\Psi), \Psi \rangle \geqslant{}& \frac{1}{\tau^2}\left(\|\Psi\|_2^2 - 2\|\Phi^k\|_2\|\Psi\|_2 - \|\Phi^{k-1}\|_2\|\Psi\|_2\right) - \frac{f(t_{k+1})}{2}\|\delta_x^{(\alpha/2)}\Psi\|_2^2 \\
&- \frac{f(t_{k-1})}{2}\langle \delta_x^{(\alpha/2)}\Phi^{k-1}, \delta_x^{(\alpha/2)}\Psi\rangle + \frac{\gamma}{2\tau}\left(\|\Psi\|_2^2 - \|\Phi^{k-1}\|_2\|\Psi\|_2\right) \\
&- \|\delta_{\Psi,\Phi,t}^{(1)} F(\Phi^k)\|_2\|\Psi\|_2 \\
\geqslant{}& \frac{1}{2\tau^2}\|\Psi\|_2\left[(2+\gamma\tau)\|\Psi\|_2 - 4\|\Phi^k\|_2 - (2+\gamma\tau)\|\Phi^{k-1}\|_2 - 2K_2\tau^2\right] \\
&- \frac{f(t_{k+1})}{2}\|\delta_x^{(\alpha/2)}\Psi\|_2^2 - \frac{f(t_{k-1})}{2}\langle \delta_x^{(\alpha/2)}\Phi^{k-1}, \delta_x^{(\alpha/2)}\Psi\rangle \\
\geqslant{}& \frac{\|\Psi\|_2}{2\tau^2}\Big[(2+\gamma\tau - \tau^2 K_1 g_h^{(\alpha)})\|\Psi\|_2 - 4\|\Phi^k\|_2 - (2+\gamma\tau+\tau^2 K_1 g_h^{(\alpha)})\|\Phi^{k-1}\|_2 \\
&\quad - 2K_2\tau^2\Big] \\
={}& \frac{2+\gamma\tau - \tau^2 K_1 g_h^{(\alpha)}}{2\tau^2}\|\Psi\|_2\left[\|\Psi\|_2 - \lambda\right],
\end{aligned} \tag{3.4.3}$$

for each $\Psi \in \mathcal{V}_h$. Here, we used the constant

$$\lambda = \frac{4\|\Phi^k\|_2 + (2+\gamma\tau+\tau^2 K_1 g_h^{(\alpha)})\|\Phi^{k-1}\|_2 + 2K_2\tau^2}{2+\gamma\tau - \tau^2 K_1 g_h^{(\alpha)}}. \tag{3.4.4}$$

Notice that $\lambda > 0$ by hypothesis. Also, $\langle r(\Psi), \Psi \rangle \geqslant 0$ is satisfied, for each $\Psi \in \mathcal{V}_h$ with $\|\Psi\|_2 = \lambda$. By Lemma 3.4.1, there exists $\Phi^{k+1} \in \mathcal{V}_h$ with $\|\Phi^{k+1}\|_2 \leqslant \lambda$, such that $r(\Phi^{k+1}) = 0$. Equivalently, $\Phi^{k+1}$ is a solution of the $k$th recursive equation in (3.3.18). The theorem follows now by induction. □

**Lemma 3.4.3.** *If $(\Psi^k)_{k\in\overline{I}_K}$ is any sequence in $\mathcal{V}_h$ then*

$$2\tau\langle \delta_t^{(2)}\Psi^k, \delta_t^{(1)}\Psi^k\rangle = \|\delta_t\Psi^k\|_2^2 - \|\delta_t\Psi^{k-1}\|_2^2, \qquad \forall k \in I_K. \tag{3.4.5}$$

*Proof.* Let $k \in I_K$. It is easy to see that

$$\begin{aligned}
\|\delta_t\Psi^k\|_2^2 - \|\delta_t\Psi^{k-1}\|_2^2 = \langle \delta_t\Psi^k, \delta_t\Psi^k\rangle - \langle \delta_t\Psi^k, \delta_t\Psi^{k-1}\rangle + \langle \delta_t\Psi^k, \delta_t\Psi^{k-1}\rangle \\
- \langle \delta_t\Psi^{k-1}, \delta_t\Psi^{k-1}\rangle.
\end{aligned} \tag{3.4.6}$$

The conclusion follows from the fact that the right-hand side is equal to $2\tau\langle \delta_t\delta_t\Psi^{k-1}, \mu_t\delta_t\Psi^{k-1}\rangle$. □

To establish the most important energy properties of the discrete model (3.3.18), we will prove now discrete analogues of the continuous identities (3.2.19)–(3.2.22).

**Lemma 3.4.4.** *If* $\Phi$ *is a solution of* (3.3.18) *then the following identities hold for each* $k \in I_{K-1}$*:*

$$\delta_t \left[ \frac{e^{\gamma t_{k-1}}}{2} \|\delta_t \Phi^{k-1}\|_2^2 \right] = \frac{\delta_t e^{\gamma t_{k-1}}}{2} \|\delta_t \Phi^k\|_2^2 + e^{\gamma t_{k-1}} \left\langle \delta_t^{(2)} \Phi^k, \delta_t^{(1)} \Phi^k \right\rangle, \tag{3.4.7}$$

$$\begin{aligned} \delta_t \left[ \frac{\gamma e^{\gamma t_{k-1}}}{2} \langle \delta_t \Phi_j^{k-1}, \mu_t \Phi_j^{k-1} \rangle \right] = \frac{\gamma e^{\gamma t_k}}{2} \langle \delta_t^{(1)} \Phi_j^k, \delta_t \Phi_j^k \rangle + \frac{\gamma e^{\gamma t_k}}{2} \langle \delta_t^{(2)} \Phi_j^k, \mu_t \Phi_j^{k-1} \rangle \\ + \frac{\gamma \delta_t e^{\gamma t_{k-1}}}{2} \langle \delta_t \Phi_j^{k-1}, \mu_t \Phi_j^{k-1} \rangle, \end{aligned} \tag{3.4.8}$$

$$\begin{aligned} \delta_t \left[ \frac{e^{\gamma t_{k-1}} f(t_{k-1})}{2} \langle \delta_x^{(\alpha/2)} \Phi_j^k, \delta_x^{(\alpha/2)} \Phi_j^{k-1} \rangle \right] = \frac{\delta_t e^{\gamma t_{k-1}}}{2} f(t_k) \langle \delta_x^{(\alpha/2)} \Phi_j^{k+1}, \delta_x^{(\alpha/2)} \Phi_j^k \rangle \\ + \frac{e^{\gamma t_{k-1}}}{2} \delta_t f(t_{k-1}) \langle \delta_x^{(\alpha/2)} \Phi_j^{k+1}, \delta_x^{(\alpha/2)} \Phi_j^k \rangle - e^{\gamma t_{k-1}} f(t_{k-1}) \langle \delta_x^{(\alpha)} \Phi_j^k, \delta_t^{(1)} \Phi_j^k \rangle, \end{aligned} \tag{3.4.9}$$

*and*

$$\delta_t \left[ e^{\gamma t_{k-1}} \mu_t \langle F(\Phi_j^{k-1}, 1) \rangle \right] = e^{\gamma t_k} \langle \delta_{\Phi,t}^{(1)} F(\Phi_j^k), \delta_t^{(1)} \Phi_j^k \rangle + \delta_t e^{\gamma t_{k-1}} \mu_t \langle F(\Phi_j^{k-1}), 1 \rangle. \tag{3.4.10}$$

*Proof.* Let $\Phi$ be a solution of (3.3.18), and let $k \in I_{K-1}$. Using the definition of the difference operator, adding and subtracting the term $\exp(\gamma t_{k-1}) \|\delta_t \Phi_j^k\|_2^2 / 2\tau$, using Lemma 3.4.3 and simplifying, we obtain

$$\delta_t \left[ \frac{e^{\gamma t_{k-1}}}{2} \|\delta_t \Phi_j^{k-1}\|_2^2 \right] = \frac{1}{2} \left( \frac{e^{\gamma t_k} - e^{\gamma t_{k-1}}}{\tau} \right) \|\delta_t \Phi_j^k\|_2^2 + \frac{e^{\gamma t_{k-1}}}{2\tau} \left[ \|\delta_t \Phi_j^k\|_2^2 - \|\delta_t \Phi_j^{k-1}\|_2^2 \right]. \tag{3.4.11}$$

This readily establishes (3.4.7), and the identity (3.4.8) is proved in similar fashion. On the other hand,

$$\begin{aligned} \delta_t \left[ \frac{e^{\gamma t_{k-1}} f(t_{k-1})}{2} \langle \delta_x^{(\alpha/2)} \Phi_j^k, \delta^{(\alpha/2)} \Phi_j^{k-1} \rangle \right] = \\ \frac{\delta_t e^{\gamma t_{k-1}}}{2} f(t_k) \langle \delta_x^{(\alpha/2)} \Phi_j^{k+1}, \delta_x^{(\alpha/2)} \Phi_j^k \rangle + \frac{e^{\gamma t_{k-1}}}{2} \delta_t f(t_{k-1}) \langle \delta_x^{(\alpha/2)} \Phi_j^{k+1}, \delta_x^{(\alpha/2)} \Phi_j^k \rangle \\ + \frac{e^{\gamma t_{k-1}}}{2\tau} f(t_{k-1}) \left[ \langle \delta_x^{(\alpha/2)} \Phi_j^{k+1}, \delta_x^{(\alpha/2)} \Phi_j^k \rangle - \langle \delta_x^{(\alpha/2)} \Phi_j^k, \delta_x^{(\alpha/2)} \Phi_j^{k-1} \rangle \right]. \end{aligned} \tag{3.4.12}$$

Applying the square-root property of Lemma 3.3.7 on the last term, we can readily reach (3.4.9). Finally, the identity (3.4.10) is proved in an analogous way. □

It is important to point out that the identities (3.4.7)–(3.4.10) may be expressed in alternative (though equivalent) forms. Some of those alternative expressions are provided in the Appendix A.2 for convenience. To prove them, one needs to employ arguments similar to those used in the proofs of Lemma 3.4.4. Using those results, one can readily reach alternative conclusions of the following theorem, which summarizes the most important energy properties of our numerical model.

**Theorem 3.4.5** (Discrete energy rate of change)**.** *If* $\Phi$ *is a solution of* (3.3.18) *then*

$$\begin{aligned}\delta_t E^k &= e^{\gamma t_k}\left\langle \delta^{(1)}_{\Phi,t}F(\Phi^k_j),\delta^{(1)}_t\Phi^k_j\right\rangle + \delta_t e^{\gamma t_{k-1}}\mu_t\left\langle F(\Phi^{k-1}_j),1\right\rangle \\ &+ \frac{\gamma e^{\gamma t_k}}{2}\langle\delta^{(1)}_t\Phi^k_j,\delta_t\Phi^k_j\rangle + \frac{\gamma e^{\gamma t_k}}{2}\langle\delta^{(2)}_t\Phi^k_j,\mu_t\Phi^{k-1}_j\rangle + \frac{\gamma\delta_t e^{\gamma t_{k-1}}}{2}\langle\delta_t\Phi^{k-1}_j,\mu_t\Phi^{k-1}_j\rangle \\ &+ \frac{\delta_t e^{\gamma t_{k-1}}}{2}f(t_k)\langle\delta^{(\alpha/2)}_x\Phi^{k+1}_j,\delta^{(\alpha/2)}_x\Phi^k_j\rangle + \frac{e^{\gamma t_{k-1}}}{2}\delta_t f(t_{k-1})\langle\delta^{(\alpha/2)}_x\Phi^{k+1}_j,\delta^{(\alpha/2)}_x\Phi^k_j\rangle \\ &- e^{\gamma t_{k-1}}f(t_{k-1})\langle\delta^{(\alpha)}_x\Phi^k_j,\delta^{(1)}_t\Phi^k_j\rangle + \frac{\delta_t e^{\gamma t_{k-1}}}{2}\|\delta_t\Phi^k_j\|_2^2 + e^{\gamma t_{k-1}}\left\langle\delta^{(2)}_t\Phi^k_j,\delta^{(1)}_t\Phi^k_j\right\rangle.\end{aligned} \tag{3.4.13}$$

*Proof.* The conclusion readily follows from the identities in Lemma 3.4.4. □

Before closing this section, it is worthwhile to notice that the formulas in Lemma 3.4.4 are consistent approximations of the formulas (3.2.19)–(3.2.22), respectively. This fact will be used in the following section in order to show the consistency properties in the energy domain of the discrete model (3.3.18).

## 3.5 Numerical properties

In this section, we will derive the most important numerical properties of the finite-difference model (3.3.18). More precisely, we will establish that consistency, stability and convergence of our numerical scheme. To prove the consistency properties of our methodology, we need to define the following continuous and discrete functionals:

$$\mathcal{L}\phi(x,t) = \frac{\partial^2\phi(x,t)}{\partial t^2} - f(t)\Delta^\alpha\phi(x,t) + \gamma\frac{\partial\phi(x,t)}{\partial t} + F'(\phi(x,t)), \qquad \forall(x,t)\in\Omega, \tag{3.5.1}$$

$$L\phi^k_j = \delta^{(2)}_t\phi^k_j - \mu^{(1)}_t(f(t_k)\delta^{(\alpha)}_x\phi^k_j) + \gamma\delta^{(1)}_t\phi^k_j + \delta^{(1)}_{\phi,t}F(\phi^k_j), \qquad \forall(j,k)\in J\times I_{K-1}. \tag{3.5.2}$$

**Theorem 3.5.1** (Consistency)**.** *Suppose that* $f:\overline{\mathbb{R}^+}\to\mathbb{R}$ *and* $F:\mathbb{R}\to\mathbb{R}$ *are differentiable functions. If* $\phi\in\mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$ *then exists a constant* $C, C'>0$ *such that, for any uniform partitions of* $\Omega$ *and* $[0,T]$ *with respective norms* $h=(h_1,h_2,\dots,h_p)$ *and* $\tau$*, the following are satisfied for each* $(j,k)\in J\times I_{K-2}$*:*

$$\left|\mathcal{L}\phi(x_j,t_k) - L\phi^k_j\right| \leqslant C(\tau^2+\|h\|_2^2) \quad \textit{and} \quad \left|\mathcal{H}(x_j,t_k) - H^k_j\right| \leqslant C'(\tau+\|h\|_2^2). \tag{3.5.3}$$

*Proof.* Using the regularity of $\phi$, $f$ and $F$, the Mean Value Theorem, Taylor's theorem and Lemma 3.3.5, there exist constants $C_1, C_3, C_4\geqslant 0$ and $C_{2,i}\geqslant 0$ for $i\in I_p$, such that

$$\left|\frac{\partial^2\phi(x_j,t_k)}{\partial t^2} - \delta^{(2)}_t\phi^k_j\right| \leqslant C_1\tau^2, \qquad \forall(j,k)\in J\times I_{K-2}, \tag{3.5.4}$$

$$\left|f(t_k)\frac{\partial^\alpha \phi(x_j,t_k)}{\partial|x_i|^\alpha} - \mu_t^{(1)}(f(t_k)\delta_{x_i}^{(\alpha)}\phi_j^k)\right| \leqslant C_{2,i}(\tau^2+h_i^2), \qquad \forall (j,k)\in J\times I_{K-2}, \tag{3.5.5}$$

$$\left|\frac{\partial \phi(x_j,t_k)}{\partial t} - \delta_t^{(1)}\phi_j^k\right| \leqslant C_3\tau^2, \qquad \forall (j,k)\in J\times I_{K-2}, \tag{3.5.6}$$

$$\left|F'(\phi(x_j,t_k)) - \delta_{\phi,t}^{(1)}F(\phi_j^k)\right| \leqslant C_4\tau^2, \qquad \forall (j,k)\in J\times I_{K-2}. \tag{3.5.7}$$

The first inequality of (3.5.3) is reached if we take $C$ as the maximum between $C_1$, $\gamma C_3$, $C_4$ and $C_{2,i}$ for $i\in I_p$, and after applying the triangle inequality. To prove the second inequality of (3.5.3), we use again the Mean Value Theorem, Taylor's theorem, Lemma 3.3.5 and the regularity assumptions to show that there exist a constant $C_5\geq 0$, such that for each $(j,k)\in J\times I_{K-2}$,

$$\begin{aligned}\left|e^{\gamma t_k}\left(\frac{\partial\phi(x_j,t_k)}{\partial t}\right)^2 - e^{\gamma t_{k-1}}(\delta_t\phi_j^{k-1})^2\right| &\leqslant \left|e^{\gamma t_k}\frac{\partial\phi(x_j,t_k)}{\partial t}\right|\left|\frac{\partial\phi(x_j,t_k)}{\partial t} - \delta_t\phi_j^{k-1}\right| \\ &+ \left|e^{\gamma t_k}\delta_t\phi_j^{k-1}\right|\left|\frac{\partial\phi(x_j,t_k)}{\partial t} - \delta_t\phi_j^{k-1}\right| + \left|e^{\gamma t_k} - e^{\gamma t_{k-1}}\right|\left|\delta_t\phi_j^{k-1}\right|^2 \leqslant C_5\tau.\end{aligned} \tag{3.5.8}$$

Similarly, there exist constants $C_6, C_{7,i}, C_8 \in \mathbb{R}^+\cup\{0\}$ for $i\in I_p$, with the property that

$$\left|\frac{\partial\phi(x_j,t_k)}{\partial t_k}\phi(x_j,t_k) - (\delta_t\phi_j^{k-1})(\mu_t\phi_j^{k-1})\right| \leqslant C_6\tau, \tag{3.5.9}$$

$$\left|f(t_k)\left(\frac{\partial^{\alpha/2}\phi(x_j,t_k)}{\partial|x_i|^{\alpha/2}}\right)^2 - f(t_{k-1})(\delta_{x_i}^{(\alpha/2)}\Phi_j^k)(\delta_{x_i}^{(\alpha/2)}\Phi_j^{k-1})\right| \leqslant C_{7,i}(\tau+h_i^2), \tag{3.5.10}$$

$$\left|F(\phi(x,t)) - \mu_t F(\Phi_j^{k-1})\right| \leqslant C_8\tau. \tag{3.5.11}$$

The conclusion is reached now letting $C'$ being the maximum of $C_5/2$, $\gamma C_6/2$, $C_8$ and $C_{7,i}/2$ for $i\in I_p$, and applying triangle inequality. □

In order to establish the remaining numerical properties of the finite-difference method (3.3.18), various crucial lemmas and hypotheses will be needed. The following are some of the hypotheses which will be required and their consequences.

- $\mathbf{H}_1$ $f\in\mathcal{C}^1([0,T])$. As a consequence of this condition and the Mean Value Theorem, there exists $C_0\geq 0$ such that $|\delta_t^{(1)}f(t_k)|\leq C_0$, for each $k\in I_{K-1}$.

- $\mathbf{H}_2$ $f$ is positive on $[0,T]$. This assumption together with the regularity of $f$ in $\mathbf{H}_1$ guarantee that there exists $C_1>0$ which depends on $f$, such that $f(t)\geq C_1$ for all $t\in[0,T]$.

It is worth pointing out that the function $f$ for the Higgs boson equation in the de Sitter space-time obviously satisfies these hypotheses. In addition, the following lemmas will be cornerstones to prove stability and convergence of (3.3.18).

**Lemma 3.5.2.** *If $f$ is a nonnegative function on $\mathbb{R}^+ \cup \{0\}$ and $(\Psi^k)_{k\in\overline{I}_K}$ is any sequence in $\mathcal{V}_h$ then*

$$\begin{aligned}\left\langle -\mu_t^{(1)}\left(f(t_k)\delta_x^{(\alpha)}\Psi^k\right), \delta_t^{(1)}\Psi^k\right\rangle = -\frac{1}{2}\left(\delta_t^{(1)}f(t_k)\right)\langle\delta_x^{(\alpha/2)}\Psi^{k+1}, \delta_x^{(\alpha/2)}\Psi^{k-1}\rangle \\ \frac{1}{2\tau}\left[\mu_t\left(f(t_k)\|\delta_x^{(\alpha/2)}\Psi^k\|_2^2\right) - \mu_t\left(f(t_{k-1})\|\delta_x^{(\alpha/2)}\Psi^{k-1}\|_2^2\right)\right], \qquad \forall k \in I_{K-1}.\end{aligned} \tag{3.5.12}$$

*Proof.* Using the definitions of the discrete operators, the distributivity property of the inner product and the square-root properties of the fractional-ordered centered differences, we obtain

$$\begin{aligned}&\left\langle -\mu_t^{(1)}\left(f(t_k)\delta_x^{(\alpha)}\Psi^k\right), \delta_t^{(1)}\Psi^k\right\rangle \\ &\quad = \frac{1}{4\tau}\Big[f(t_{k+1})\langle -\delta_x^{(\alpha)}\Psi^{k+1}, \Psi^{k+1}\rangle - f(t_{k+1})\langle -\delta_x^{(\alpha)}\Psi^{k+1}, \Psi^{k-1}\rangle \\ &\qquad + f(t_{k-1})\langle -\delta_x^{(\alpha)}\Psi^{k-1}, \Psi^{k+1}\rangle - f(t_{k-1})\langle -\delta_x^{(\alpha)}\Psi^{k-1}, \Psi^{k-1}\rangle\Big] \\ &\quad = \frac{1}{4\tau}\Big[f(t_{k+1})\|\delta_x^{(\alpha/2)}\Psi^{k+1}\|_2^2 - f(t_{k-1})\|\delta_x^{(\alpha/2)}\Psi^{k-1}\|_2^2 \\ &\qquad + \left(f(t_{k-1}) - f(t_{k+1})\right)\langle\delta_x^{(\alpha/2)}\Psi^{k+1}, \delta_x^{(\alpha/2)}\Psi^{k-1}\rangle\Big], \qquad \forall k \in I_{K-1}.\end{aligned} \tag{3.5.13}$$

The conclusion of this result readily follows after adding and subtracting the term $f(t_k)\|\delta_x^{(\alpha/2)}\Psi^k\|_2^2$ inside the parenthesis, and combining terms. □

**Lemma 3.5.3** (Macías-Díaz [72])**.** *Let $F \in \mathcal{C}^2(\mathbb{R})$ and $F'' \in L^\infty(\mathbb{R})$, and suppose that $(\Phi^k)_{k\in\overline{I}_K}$, $(\Psi^k)_{k\in\overline{I}_K}$ and $(R^k)_{k\in\overline{I}_K}$ are sequences in $\mathcal{V}_h$. Let $\varepsilon^k = \Phi^k - \Psi^k$ and $\widetilde{F}^k = \delta_{\Phi,t}^{(1)}F(\Phi^k) - \delta_{\Psi,t}^{(1)}F(\Psi^k)$, for each $k \in \overline{I}_K$. Then there exist constants $C_2, C_3 \in \mathbb{R}^+$ which depend only on $G$ such that for each $m \in I_{K-1}$,*

$$2\tau\left|\sum_{k=1}^{m}\langle R^k - \widetilde{F}^k, \delta_t^{(1)}\varepsilon^k\rangle\right| \le 2\tau\sum_{k=0}^{m}\|R^k\|_2^2 + C_2\|\varepsilon^0\|_2^2 + C_3\tau\sum_{k=0}^{m}\|\delta_t\varepsilon^k\|_2^2. \tag{3.5.14}$$

**Lemma 3.5.4** (Pen-Yu [84])**.** *Let $(\omega^k)_{k=0}^K$ and $(\rho^k)_{k=0}^K$ be finite sequences of nonnegative real numbers, and suppose that there exists $C \ge 0$ such that*

$$\omega^m \le \rho^m + C\tau\sum_{k=0}^{m-1}\omega^k, \qquad \forall m \in \overline{I}_K. \tag{3.5.15}$$

*Then $\omega^k \le \rho^k e^{Ck\tau}$, for each $k \in \overline{I}_K$.*

In the following result, the constants $C_0$, $C_1$, $C_2$ and $C_3$ will as Lemma 3.5.3 and its preceding remark. Moreover, we will let $\Phi_0 = (\phi_0, \phi_1)$ and $\Psi_0 = (\psi_0, \psi_1)$ denote two sets of initial conditions of (3.3.18), and the corresponding numerical solutions will be represented by $\Phi = (\Phi^k)_{k\in\overline{I}_K}$ and $\Psi = (\Psi^k)_{k\in\overline{I}_K}$.

**Theorem 3.5.5** (Stability). *Let $F \in \mathcal{C}^2(\mathbb{R})$ and $F'' \in L^\infty(\mathbb{R})$, and assume that $f \in \mathcal{C}^1([0,T])$ is positive on $[0,T]$. Let $\Phi$ and $\Psi$ be solutions of (3.3.18) corresponding to the initial conditions $\Phi_0$ and $\Psi_0$, respectively, and let $\varepsilon^k = \Phi^k - \Psi^k$ for each $k \in \overline{I}_K$. Define the nonnegative constants*

$$\rho = 2\left(C_2\|\varepsilon^0\|_2^2 + \|\delta_t \varepsilon^0\|_2^2 + \mu_t\left[f(t_0)\|\delta_x^{(\alpha/2)}\varepsilon^0\|_2^2\right]\right), \tag{3.5.16}$$

$$\omega^k = \|\delta_t\varepsilon^k\|_2^2 + \mu_t\left[f(t_k)\|\delta_x^{(\alpha/2)}\varepsilon^k\|_2^2\right], \qquad \forall k \in \overline{I}_K, \tag{3.5.17}$$

*and suppose that the following inequality is satisfied*

$$2\left(C_3 + \frac{2C_0}{C_1}\right)\tau < 1. \tag{3.5.18}$$

*Then there exists $C_4 \in \mathbb{R}^+ \cup \{0\}$ independent of $\Phi$ and $\Psi$, such that $\omega^k \leq \rho e^{C_4 k\tau}$ for each $k \in \overline{I}_K$.*

*Proof.* It is easy to see that the sequence $(\varepsilon^k)_{k\in\overline{K}}$ satisfies

$$\begin{array}{c} \delta_t^{(2)}\varepsilon_j^k - \mu_t^{(1)}\left(f(t_k)\delta_x^{(\alpha)}\varepsilon_j^k\right) + \gamma\delta_t^{(1)}\varepsilon_j^k + \widetilde{F}_j^k = 0, \qquad \forall (j,k) \in J \times I_{K-1}, \\ \text{such that} \left\{ \begin{array}{ll} \varepsilon_j^0 = \phi_0(x_j) - \psi_0(x_j), & \forall j \in J, \\ \delta_t^{(1)}\varepsilon_j^0 = \phi_1(x_j) - \psi_1(x_j), & \forall j \in J. \end{array}\right. \end{array} \tag{3.5.19}$$

Here, we are using the nomenclature of Lemma 3.5.3. On the other hand, according to Lemmas 3.4.3 and 3.5.2, the following identities are satisfied, for each $k \in I_K$:

$$\left\langle \delta_t^{(2)}\varepsilon^k, \delta_t^{(1)}\varepsilon^k \right\rangle = \frac{1}{2\tau}\left[\|\delta_t\varepsilon^k\|_2^2 - \|\delta_t\varepsilon^{k-1}\|_2^2\right] \tag{3.5.20}$$

and

$$\begin{aligned} \left\langle -\mu_t^{(1)}\left(f(t_k)\delta_x^{(\alpha)}\varepsilon^k\right),\, \delta_t^{(1)}\varepsilon^k \right\rangle = & -\frac{1}{2}\delta_t^{(1)} f(t_k)\langle \delta_x^{(\alpha/2)}\varepsilon^{k+1}, \delta_x^{(\alpha/2)}\varepsilon^{k-1}\rangle \\ & + \frac{1}{2\tau}\left[\mu_t\left(f(t_k)\|\delta_x^{(\alpha/2)}\varepsilon^k\|_2^2\right) - \mu_t\left(f(t_{k-1})\|\delta_x^{(\alpha/2)}\varepsilon^{k-1}\|_2^2\right)\right]. \end{aligned} \tag{3.5.21}$$

Let $k \in I_{K-1}$ and take the inner product of $\delta_t^{(1)}\varepsilon^k$ with both sides of the $k$th difference equation of (3.5.19). Substitute the identities (3.5.20) and (3.5.21). Fix $m \in I_{K-1}$, and take the sum over all indexes $k \in I_m$ and use the formula for telescoping sums. After multiplying both sides by $2\tau$, we obtain the identity

$$\|\delta_t\varepsilon^m\|_2^2 - \|\delta_t\varepsilon^0\|_2^2 + \mu_t\left[f(t_m)\|\delta_x^{(\alpha/2)}\varepsilon^m\|_2^2\right] - \mu_t\left[f(t_0)\|\delta_x^{(\alpha/2)}\varepsilon^0\|_2^2\right]$$
$$= \tau\sum_{k=1}^{m}\delta_t^{(1)}f(t_k)\langle\delta_x^{(\alpha/2)}\varepsilon^{k+1},\delta_x^{(\alpha/2)}\varepsilon^{k-1}\rangle - 2\tau\gamma\sum_{k=1}^{m}\|\delta_t^{(1)}\varepsilon^k\|_2^2 - 2\tau\sum_{k=1}^{m}\langle\widetilde{F}^k,\delta_t^{(1)}\varepsilon^k\rangle. \tag{3.5.22}$$

Rearranging terms, taking absolute value on both sides, using the upper bound $C_0$, applying then Lemma 3.5.3 with $R^k = 0$ for each $k \in \overline{I}_K$, and using Young's inequality, we see that

$$\begin{aligned}
\omega^m &\leq \frac{\rho}{2} + \frac{C_0}{2C_1}\tau\sum_{k=1}^{m}\left(C_1\|\delta_x^{(\alpha/2)}\varepsilon^{k+1}\|_2^2 + C_1\|\delta_x^{(\alpha/2)}\varepsilon^{k-1}\|_2^2\right) + C_3\tau\sum_{k=0}^{m}\|\delta_t\varepsilon^k\|_2^2 \\
&\leq \frac{\rho}{2} + \frac{C_0}{2C_1}\tau\sum_{k=1}^{m}\left(f(t_{k+1})\|\delta_x^{(\alpha/2)}\varepsilon^{k+1}\|_2^2 + f(t_{k-1})\|\delta_x^{(\alpha/2)}\varepsilon^{k-1}\|_2^2\right) + C_3\tau\sum_{k=0}^{m}\|\delta_t\varepsilon^k\|_2^2 \\
&\leq \frac{\rho}{2} + \frac{C_0}{C_1}\tau\sum_{k=1}^{m}\left[\mu_t\left(f(t_k)\|\delta_x^{(\alpha/2)}\varepsilon^k\|_2^2\right) + \mu_t\left(f(t_{k-1})\|\delta_x^{(\alpha/2)}\varepsilon^{k-1}\|_2^2\right)\right] + C_3\tau\sum_{k=0}^{m}\|\delta_t\varepsilon^k\|_2^2 \\
&\leq \frac{\rho}{2} + \frac{2C_0}{C_1}\tau\sum_{k=0}^{m}\mu_t\left(f(t_k)\|\delta_x^{(\alpha/2)}\varepsilon^k\|_2^2\right) + C_3\tau\sum_{k=0}^{m}\|\delta_t\varepsilon^k\|_2^2 \\
&\leq \frac{\rho}{2} + \left(C_3 + \frac{2C_0}{C_1}\right)\tau\sum_{k=0}^{m}\omega^k,
\end{aligned} \tag{3.5.23}$$

for each $m \in I_{K-1}$. Using this inequality and the hypothesis (3.5.18), we obtain

$$\omega^m = 2\left(\omega^m - \frac{1}{2}\omega^m\right) \leq 2\rho + 2\left(C_3 + \frac{2C_0}{C_1}\right)\tau\sum_{k=0}^{m}\omega^k - \omega^m = \rho + C_4\tau\sum_{k=0}^{m-1}\omega^k, \tag{3.5.24}$$

for each $m \in I_{K-1}$. Here, we employed the nonnegative constant $C_4 = 2(C_3 + \frac{2C_0}{C_1})$. The conclusion of the theorem readily follows now by Lemma 3.5.4. □

The uniqueness of the solutions of (3.3.18) is now a consequence of Theorem 3.5.5.

**Corollary 3.5.6** (Uniqueness). *Let $F \in \mathcal{C}^2(\mathbb{R})$ and $F'' \in L^\infty(\mathbb{R})$, and let $f \in \mathcal{C}^1([0,T])$ be positive on $[0,T]$. If (3.5.18) holds then any two solutions of (3.3.18) corresponding to the same initial conditions are equal.*

*Proof.* Let $\Phi$ and $\Psi$ be two solutions corresponding to the initial data $\Phi_0$, and define $\varepsilon^k = \Phi^k - \Psi^k$, for each $k \in \overline{I}_K$. By Theorem 3.5.5, there exists a constant $C_4 \geq 0$ which is independent of $\Phi$ and $\Psi$, such that $\omega^k \leq \rho e^{C_4 k\tau}$ holds for each $k \in \overline{I}_K$. As a consequence, note that

$$\|\delta_t\varepsilon^k\|_2^2 \leq \omega^k \leq 2e^{C_4T}\left(C_2\|\varepsilon^0\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 + \mu_t\left[f(t_0)\|\delta_x^{(\alpha/2)}\varepsilon^0\|_2^2\right]\right), \qquad \forall k \in \overline{I}_{K-1}. \tag{3.5.25}$$

But the assumption on the initial conditions guarantee that the Euclidean norms at the right-hand side of this chain of inequalities are all equal to zero. This means that $\|\delta_t \varepsilon^k\|_2 = 0$ or, equivalently, that $\varepsilon^{k+1} = \varepsilon^k$ is satisfied, for each $k \in \overline{I}_{K-1}$. Applying a recursive argument, we conclude that the two solutions $\Phi$ and $\Psi$ are the same, as desired. □

Finally, we establish the convergence property of the finite-difference scheme (3.3.18).

**Theorem 3.5.7** (Convergence)**.** *Let $F \in \mathcal{C}^2(\mathbb{R})$ and $F'' \in L^\infty(\mathbb{R})$, and let $f \in \mathcal{C}^1([0,T])$ be positive on $[0,T]$. If $\phi \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$ is solution of* (3.2.13) *corresponding to the initial data $\Phi_0$ and* (3.5.18) *is satisfied, then the solutions of* (3.3.18) *converge quadratically to $\phi$ in the $\|\cdot\|_2$-norm.*

*Proof.* Let $h$ and $\tau$ be computational parameters satisfying (3.5.18), and let $\Phi$ be a solution of (3.3.18) corresponding to the initial conditions $\Phi_0$. For each $k \in \overline{I}_K$, define $\epsilon^k = \Phi^k - \phi^k$, and let $R^k_j$ represent the local truncation error at the point $(x_j, t_k)$. Notice that the following problem is satisfied:

$$\begin{gathered}\delta_t^{(2)} \epsilon_j^k - \mu_t^{(1)} \left( f(t_k) \delta_x^{(\alpha)} \epsilon_j^k \right) + \gamma \delta_t^{(1)} \epsilon_j^k + \widetilde{F}_j^k = R_j^k, \qquad \forall (j,k) \in J \times I_{K-1}, \\ \text{such that } \left\{ \ \epsilon_j^0 = \epsilon_j^1 = 0, \quad \forall j \in J. \right.\end{gathered} \tag{3.5.26}$$

The argument is similar to the proof of Theorem 3.5.5. In a first stage, let $k \in I_{K-1}$ and take the inner product of $\delta_t^{(1)} \epsilon^k$ with both sides of the difference equation of (3.5.26). Substitute the equivalent forms of identities (3.5.20) and (3.5.21), and fix $m \in I_{K-1}$. Take the sum over all indexes $k \in I_m$ and use the formula for telescoping sums. After multiplying both sides by $2\tau$, we reach

$$\begin{aligned}\|\delta_t \epsilon^m\|_2^2 - \|\delta_t \epsilon^0\|_2^2 + \mu_t \left[ f(t_m) \|\delta_x^{(\alpha/2)} \epsilon^m\|_2^2 \right] - \mu_t \left[ f(t_0) \|\delta_x^{(\alpha/2)} \epsilon^0\|_2^2 \right] \\ = \tau \sum_{k=1}^m \left( \delta_t^{(1)} f(t_k) \right) \langle \delta_x^{(\alpha/2)} \epsilon^{k+1}, \delta_x^{(\alpha/2)} \epsilon^{k-1} \rangle \\ - 2\tau\gamma \sum_{k=1}^m \|\delta_t^{(1)} \epsilon^k\|_2^2 + 2\tau \sum_{k=1}^m \langle \widetilde{R}^k - F^k, \delta_t^{(1)} \epsilon^k \rangle.\end{aligned} \tag{3.5.27}$$

Rearranging terms, taking absolute value on both sides, using the upper bound $C_0$, applying then Lemma 3.5.3 with $R^k = 0$ for each $k \in \overline{I}_K$, and using Young's inequality, we obtain that

$$\omega^m = \rho^m + C_4 \tau \sum_{k=0}^{m-1} \omega^k, \qquad \forall m \in \overline{I}_{K-1}, \tag{3.5.28}$$

where $C_4$ is as in the proof of Theorem 3.5.5, and

$$\rho^m = 2 \left( C_2 \|\epsilon^0\|_2^2 + \|\delta_t \epsilon^0\|_2^2 + \mu_t \left[ f(t_0) \|\delta_x^{(\alpha/2)} \epsilon^0\|_2^2 \right] + 2\tau \sum_{k=0}^m \|R^k\|_2^2 \right), \quad \forall m \in \overline{I}_K, \tag{3.5.29}$$

$$\omega^m = \|\delta_t \epsilon^m\|_2^2 + \mu_t \left[ f(t_m) \|\delta_x^{(\alpha/2)} \epsilon^m\|_2^2 \right], \qquad \forall m \in \overline{I}_K. \tag{3.5.30}$$

Notice that a convenient simplification in the expressions (3.5.29) is readily at hand when we consider the initial conditions of (3.5.26). Also, the regularity on $\phi$ guarantees that there exists a constant $C \geq 0$ which is independent of $h$ and $\tau$, such that $\|R^k\|_2 \leq C(\tau^2 + \|h\|^2)$, for each $k \in \overline{I}_K$. Moreover, the conclusion of Lemma 3.5.4 yields now that

$$\|\delta_t \epsilon^m\|_2^2 \leq \omega^m \leq \rho^m e^{C_4 T} = 4 e^{C_4 T} \tau \sum_{k=0}^{m} \|R^k\|_2^2 \leq 4 C^2 e^{C_4 T} T (\tau^2 + \|h\|_2^2)^2, \qquad \forall m \in \overline{I}_K. \tag{3.5.31}$$

Taking square-root on both sides, using the triangle inequality and multiplying by $\tau$, it follows that

$$\|\epsilon^{m+1}\|_2 - \|\epsilon^m\|_2 \leq 2 C e^{C_4 T/2} \sqrt{T} \tau (\tau^2 + \|h\|_2^2), \qquad \forall m \in \overline{I}_{K-1}. \tag{3.5.32}$$

Let $k \in \overline{I}_{K-1}$, and take the sum on both ends of this inequalities, for $k \in \overline{I}_{K-1}$. Using the formula for telescoping sums, the initial data in (3.5.26) and bounding from above, we readily observe that for each $k \in \overline{I}_K$, it follows that $\|\epsilon^k\|_2 \leq C_5(\tau^2 + \|h\|_2^2)$, where the constant $C_5 = 2 C e^{C_4 T/2} T^{3/2}$ is independent of $h$ and $\tau$. We conclude that the solutions of (3.3.18) converge to $\phi$ in the $\|\cdot\|_2$-norm, as desired. □

## 3.6 Results

The present section is devoted to provide some illustrative computer simulations of the scheme (3.3.18). Moreover, a simpler algorithm will be introduced and theoretically analyzed in the second half of this section. Throughout, we will consider only the one-dimensional problem $p = 1$, and set $M = M_1$.

For our simulations, we will use the fully discrete discrete scheme (3.3.18), which uses an implementation of the Newton–Raphson method to approximate real roots of nonlinear systems of equations. To that end, a maximum number of iterations equal to 20 and a tolerance of $1 \times 10^{-8}$ in the infinity norm will be employed. Beforehand, we must mention that the maximum number of iterations was sufficient to achieve the desired level of tolerance. Indeed, most steps of Newton's method reached the desired tolerance in less than 10 iterations. It is important to point out also that our numerical experiments are motivated by the simulations obtained in [6]. Following ideas in that article, for each $x_0 \in \mathbb{R}$ and $R > 0$, we define the function $\varphi_{x_0,R} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{x_0,R}(x) = \begin{cases} \exp\left( \dfrac{1}{R^2} - \dfrac{1}{R^2 - (x - x_0)^2} \right), & \text{if } |x - x_0| < R, \\ 0, & \text{if } |x - x_0| \geq R. \end{cases} \tag{3.6.1}$$

*Example* 3.6.1. Consider the system (3.2.13) with $f(t)=e^{-2t}$, $F(\phi)=-\frac{1}{2}\mu^2\phi^2+\frac{1}{q+1}\lambda|\phi|^{q+1}\phi$ parameters $\gamma=1$, $q=3$, $\lambda=2$ and $\mu=3$. The diffusion and the potential functions are those corresponding to the Higgs boson equation in the de Sitter space-time. We restrict our attention to the spatial domain $B=(-1,2)$ and let $T=0.03$. Computationally, we let $h=0.01$ and $\tau=0.00005$, and set the initial conditions $\phi_0=\varphi_{0.5,0.3}$ and $\phi_1=0$. Under these circumstances, Figure 3.2 provides the approximate solutions obtained using our implementation of the discrete model (3.3.18). Various values of $\alpha$ were used, namely, $\alpha=1.6$ (left column), $\alpha=1.8$ (middle column) and $\alpha=2$ (right column). The top row provides the graphs of the approximate solution of the system as a function of $x$ and $t$, while the graphs on the bottom correspond to the respective discrete energy densities. The results are in good qualitative agreement with [6]. Particularly, it is important to point out the presence of one-dimensional "bubbles", as predicted by Theorem 3.2.1 when $\alpha=2$. □

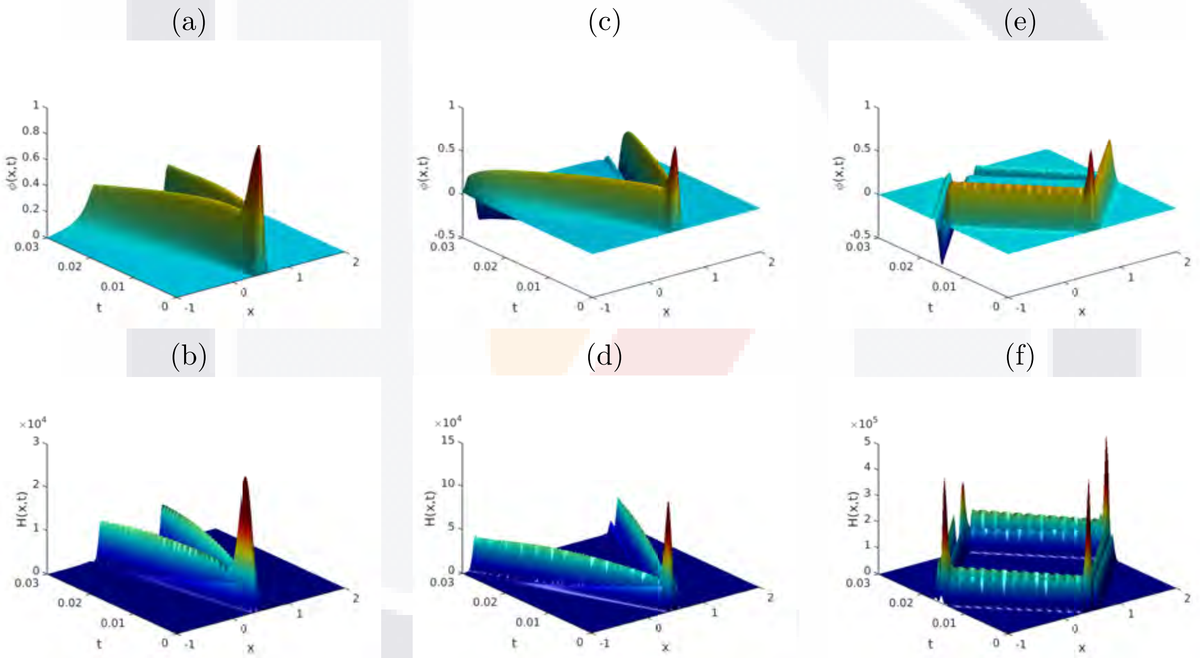


**Figure 3.2** Graphs of the approximate solution (top row) and the respective local energy density (bottom row) versus $x$ and $t$, of the system (3.2.13) with $\gamma=1$, $q=3$, $\lambda=2$, $\mu=3$, $F(\phi)=-\frac{1}{2}\mu^2\phi^2+\frac{1}{q+1}\lambda|\phi|^{q+1}\phi$, $f(t)=e^{-2t}$, $B=(-1,2)$ and $T=0.03$. Computationally, we let $h=0.01$ and $\tau=0.00005$. We employed $\alpha=1.6$ (left column), $\alpha=1.8$ (middle column) and $\alpha=2$ (right column). As initial data, we used $\phi_0=\varphi_{0.5,0.3}$ and $\phi_1=0$.

*Example* 3.6.2. Consider the same problem studied in Example 3.6.1, using now the initial data $\phi_0=-\varphi_{0.5,0.3}+\varphi_{0.55,0.3}$ and $\phi_1=0$. In this case, observe that the initial profile is the difference between two "bubbles". The results of our simulations are shown in Figure 3.3, for $\alpha=1.6$ (left column), $\alpha=1.8$ (middle column) and $\alpha=2$ (right column). The results again show the presence of moving “bubbles” as before. Again, this fact is in perfect agreement with the theoretical results available for the case when $\alpha = 2$. □

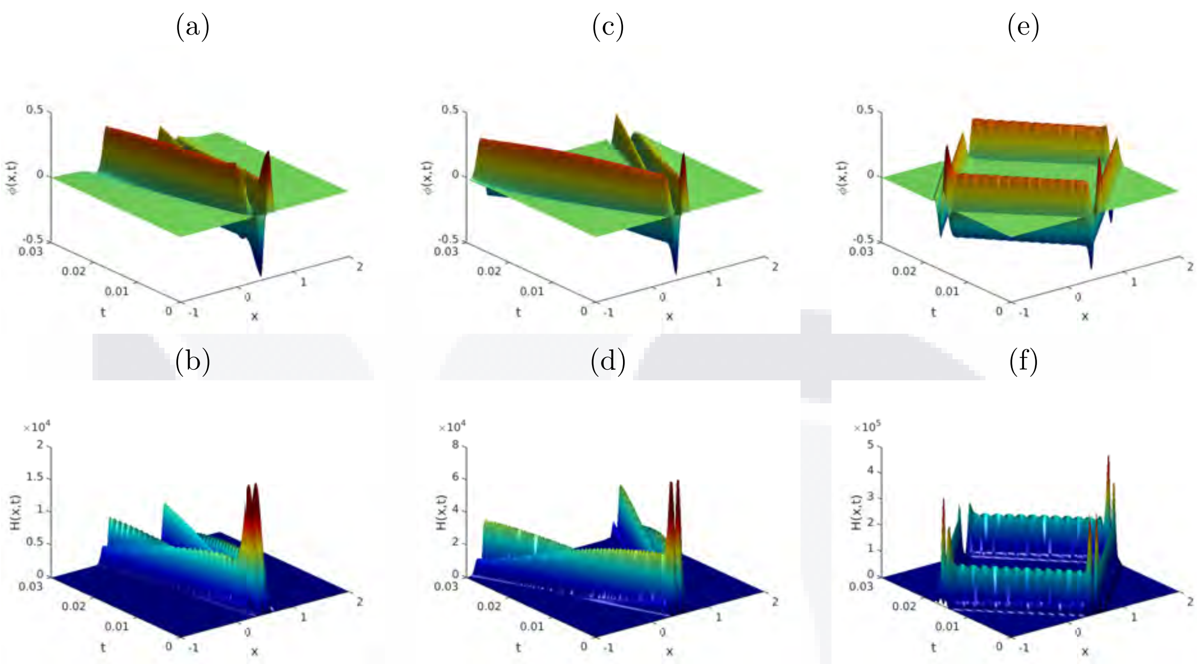


**Figure 3.3** Graphs of the approximate solution (top row) and the respective local energy density (bottom row) versus $x$ and $t$, of the system (3.2.13) with $\gamma = 1$, $q = 3$, $\lambda = 2$, $\mu = 3$, $F(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{q+1}\lambda|\phi|^{q+1}\phi$, $f(t) = e^{-2t}$, $B = (-1,2)$ and $T = 0.03$. Computationally, $h = 0.01$ and $\tau = 0.00005$. We used $\alpha = 1.6$ (left column), $\alpha = 1.8$ (middle column) and $\alpha = 2$ (right column). As initial data, we let $\phi_0 = -\varphi_{0.5,0.3} + \varphi_{0.55,0.3}$ and $\phi_1 = 0$.

Finally, we will detail a computer implementation of the finite-difference scheme in the case when the definition of the operator (3.3.10) reduces to $\delta^{(1)}_{\Psi,t}F(\Psi^k_j) = F'(\Psi^k_j)$. The method below will be easy to implement, but it will have no energy properties associated, as opposed to the variational scheme (3.3.18). With this convention, multiply the $k$th difference equation of (3.3.18) by $2\tau^2$ and rearrange terms algebraically. It is easy to see then that the equation is equivalent to the expression

$$\zeta^{k+1}\Phi^{k+1}_j + R^{k+1}\sum_{\substack{l=0\\ l\neq j}}^{M} g_{j-l}\Phi^{k+1}_l = 4\Phi^k_j - 2\tau^2 F'(\Phi^k_j) - \eta^{k+1}\Phi^{k-1}_j - R^{k-1}\sum_{\substack{l=0\\ l\neq j}}^{M} g^{(\alpha)}_{j-l}\Phi^{k-1}_l, \quad (3.6.2)$$

for each $j \in \overline{I}_M$, where

$$\zeta^k = 2 + \gamma\tau + R^{k+1}g^{(\alpha)}_0, \qquad \forall k \in \overline{I}_K, \quad (3.6.3)$$

$$\eta^k = 2 - \gamma\tau + R^{k-1}g^{(\alpha)}_0, \qquad \forall k \in \overline{I}_K. \quad (3.6.4)$$

Here, we agree that $R^k = \tau^2 h^{-\alpha} f(t_k)$, for each $k \in \overline{I}_K$. In turn, the set of equations (3.6.2), for all $j$ ranging in $\overline{I}_M$, can be rewritten equivalently in vector form as

$$\left[(2+\gamma\tau)I + R^{k+1}H^{(\alpha)}\right]\Phi^{k+1} = 4\Phi^k - 2\tau^2 G_\Phi^k - \left[(2-\gamma\tau)I + R^{k-1}H^{(\alpha)}\right]\Phi^{k-1}, \tag{3.6.5}$$

for each $k \in \overline{I}_{K-1}$, such that $\Phi^0 = \overline{\phi}_0$ and $\delta_t^{(1)}\Phi^0 = \overline{\phi}_1$. For the sake of convenience, a Matlab implementation of an even simpler form of this scheme will be provided in A.3. It is worth pointing out once more that the simulations obtained in Examples 3.6.1 and 3.6.2 were obtained using an implementation of (3.3.18), which is more complicated. The computations were performed in Matlab 8.5.0.197613 (R2015a) on a ©Sony Vaio PCG-5L1P laptop computer with Fedora 31 as operating system.

In the expression (3.6.5), $I$ is the identity matrix of size $(M+1)\times(M+1)$, and we let

$$\Phi^k = \left(\Phi_0^k, \Phi_1^k, \ldots, \Phi_M^k\right)^\top, \qquad \forall k \in \overline{I}_K, \tag{3.6.6}$$

$$G_\Phi^k = \left(F'(\Phi_0^k), F'(\Phi_1^k), \ldots, F'(\Phi_M^k)\right)^\top, \qquad \forall k \in \overline{I}_K, \tag{3.6.7}$$

$$\overline{\phi}_0 = (\phi_0(x_0), \phi_0(x_1), \ldots, \phi_0(x_M))^\top, \tag{3.6.8}$$

$$\overline{\phi}_1 = (\phi_1(x_0), \phi_1(x_1), \ldots, \phi_1(x_M))^\top. \tag{3.6.9}$$

Obviously, the symbol $^\top$ in these expressions represents the matrix operation of transposition. Meanwhile, the matrix $H^{(\alpha)}$ has size $(M+1)\times(M+1)$. Using the properties summarized in Lemma 3.3.4, the matrix $H^{(\alpha)}$ can be presented as

$$H^{(\alpha)} = \begin{pmatrix} g_0^{(\alpha)} & g_1^{(\alpha)} & g_2^{(\alpha)} & \cdots & g_M^{(\alpha)} \\ g_1^{(\alpha)} & g_0^{(\alpha)} & g_1^{(\alpha)} & \cdots & g_{M-1}^{(\alpha)} \\ g_2^{(\alpha)} & g_1^{(\alpha)} & g_0^{(\alpha)} & \cdots & g_{M-2}^{(\alpha)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_M^{(\alpha)} & g_{M-1}^{(\alpha)} & g_{M-2}^{(\alpha)} & \cdots & g_0^{(\alpha)} \end{pmatrix}. \tag{3.6.10}$$

**Theorem 3.6.3** (Solvability)**.** *The recursive scheme* (3.6.5) *has a unique solution for any set of initial conditions.*

*Proof.* Notice that $\Phi^0$ and $\Phi^1$ are defined through the initial conditions, so suppose that $\Phi^{k-1}$ and $\Phi^k$ have been calculated already, for some $k \in \overline{I}_{K-1}$, and let $A^{k+1} = (2+\gamma\tau)I + R^{k+1}H^{(\alpha)}$. Let us represent $A^{k+1} = (a_{jl})$, where $j, l \in \overline{I}_M$. Notice that the properties stated in Lemma 3.3.4 imply that

$$\sum_{\substack{l=0 \\ l\neq j}}^M |a_{jl}| = R^{k+1}\sum_{\substack{l=0 \\ l\neq j}}^M |g_{j-l}^{(\alpha)}| = -R^{k+1}\sum_{\substack{l=0 \\ l\neq j}}^M g_{j-l}^{(\alpha)} \le R^{k+1}g_0^{(\alpha)} < a_{jj}, \qquad \forall j \in \overline{I}_M. \tag{3.6.11}$$

This means that the matrix $A^{k+1}$ is strictly diagonally dominant, so invertible. As a consequence, it follows that the $k$th vector equation in (3.6.5) has a unique solution for $\Phi^{k+1}$. The conclusion follows now using induction. □

**Definition 3.6.4.** A real matrix is a *$Z$-matrix* if all its off-diagonal entries are non-positive.

**Definition 3.6.5.** We say that a square real matrix $A$ is a *Minkowski matrix* if the following are satisfied:

(i) $A$ is a $Z$-matrix,

(ii) all the diagonal entries of $A$ are positive, and

(iii) there is a diagonal matrix $D$ with positive diagonal entries, such that $AD$ is strictly diagonally dominant.

As a side note, the matrices $A^k$ of Theorem 3.6.3 are Minkowski matrices. Minkowski matrices are important in numerical analysis because they are nonsingular matrices, and all the entries of their inverses are positive numbers. Several characterizations of Minkowski matrices can be found in [86]. Moreover, it is important to remark that the matrices $A^k$ do not depend on $\Phi$.

**Lemma 3.6.6** (Chen *et al.* [17])**.** *Let $A$ be a real matrix of size $M\times M$ that satisfies*

$$\sum_{\substack{j=1\\ j\neq i}}^{M} |a_{ij}| \leq |a_{ii}| - 1, \qquad \forall i \in I_M. \tag{3.6.12}$$

*Then $\|v\|_\infty \leq \|Av\|_\infty$ is satisfied for all $v\in\mathbb{R}^M$.*

In the following results, $A^k$ will denote the matrix introduced in the proof of Theorem 3.6.3. Moreover, $B^k$ will be the matrix of size $(M+1)\times(M+1)$ defined by $B^k = (2-\gamma\tau)I + R^k H^{(\alpha)}$, for each $k\in \overline{I}_K$.

**Lemma 3.6.7.** *If $k\in\overline{I}_K$ and $\Phi\in\mathcal{V}_h$ then $\|\Phi\|_\infty \leq \|A^k\Phi\|_\infty$.*

*Proof.* Using the same argument as in the proof of Theorem 3.6.3, we readily note that

$$\sum_{\substack{l=0\\ l\neq j}}^{M} |a_{jl}| \leq R^{k+1} g_0^{(\alpha)} < 1 + \gamma\tau + R^{k+1} g_0^{(\alpha)} = |a_{jj}| - 1, \qquad \forall j \in \overline{I}_M. \tag{3.6.13}$$

The conclusion is now a straightforward consequence of Lemma 3.6.6. □

**Definition 3.6.8.** For each real matrix $B$ of size $M\times M$, the *infinity norm* of $B$ is defined by

$$\|B\|_\infty = \sup\left\{ \|Bv\|_\infty : v\in\mathbb{R}^M \text{ such that } \|v\|_\infty = 1\right\} = \max_{1\leq i\leq M} \sum_{j=1}^{M} |b_{ij}|. \tag{3.6.14}$$

**Lemma 3.6.9.** *Let $C' \geq 0$ satisfy $|f(t)| \leq C'$, for all $t \in [0,T]$. If $\gamma\tau < 2$ then*

$$\|B^k\|_\infty < 2 - \gamma\tau + 2\tau^2 h^{-1} C' g_0^{(\alpha)}, \qquad \forall k \in \overline{I}_K. \tag{3.6.15}$$

*Proof.* Beforehand, notice that $B^k$ is a symmetric matrix. Now, let $j \in \overline{I}_M$ and observe that

$$\begin{aligned}\sum_{l=0}^{M} |b_{jl}| &= 2 - \gamma\tau + R^k g_0^{(\alpha)} + R^k \sum_{\substack{l=0 \\ l\neq j}}^{M} |g_{j-l}^{(\alpha)}| \leq 2 - \gamma\tau + R^k g_0^{(\alpha)} - R^k \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} g_l^{(\alpha)} \\ &= 2 - \gamma\tau + 2R^k g_0^{(\alpha)}.\end{aligned} \tag{3.6.16}$$

Obviously, $R^k \leq \tau^2 h^{-1} C' g_0^{(\alpha)}$, whence the conclusion of the theorem readily follows. □

The following result establishes the stability of the linear scheme (3.6.5). The quadratic convergence of this method will be obviously a consequence of Lax's equivalence theorem.

**Theorem 3.6.10** (Stability)**.** *Fix $h, \tau \in \mathbb{R} \cup \{0\}$ such that $\gamma\tau < 2$. If $f$ is bounded on $[0,T]$ and $F'$ is Lipschitz on $\mathbb{R}$ then there exist constants $C_1, C_2 \in \mathbb{R}^+ \cup \{0\}$ such that, for any two solutions $\Phi$ and $\Psi$ of (3.3.18) corresponding to the initial conditions $\Phi_0$ and $\Psi_0$, respectively, the following is satisfied:*

$$\|\varepsilon^{k+1}\|_\infty \leq C_1 \|\varepsilon^0\|_\infty + C_2 \|\varepsilon^1\|_\infty, \qquad \forall k \in I_{K-1}. \tag{3.6.17}$$

*Proof.* Beforehand, observe that the hypotheses guarantee that there exists $C' \geq 0$ such that (3.6.15) is satisfied. On the other hand, let $C'' \geq 0$ be a Lipschitz constant for the function $F'$ on $\mathbb{R}$. Then

$$\begin{aligned}\|\varepsilon^{k+1}\|_\infty &\leq \|A^{k+1}(\Phi^{k+1} - \Psi^{k+1})\|_\infty \leq 4\|\varepsilon^k\|_\infty + 2\tau^2 \|G_\Phi^k - G_\Psi^k\|_\infty + \|B^{k-1}\varepsilon^{k-1}\|_\infty \\ &\leq (4 + 2\tau^2 C'')\|\varepsilon^k\|_\infty + (2 - \gamma\tau + 2\tau^2 h^{-1} C' g_0^{(\alpha)})\|\varepsilon^{k-1}\|_\infty, \qquad \forall k \in I_K.\end{aligned} \tag{3.6.18}$$

Using a recursive argument, it is easy to show that there exist constants $C_1, C_2 \geq 0$ which are independent of $\Phi$ and $\Psi$, such that the inequality (3.6.17) is satisfied. □

# Conclusions

## Conclusions from Chapter 1

In this work, we considered a multi-dimensional equation with an arbitrary potential, general time-dependent diffusion and an energy function. Motivated by these facts, we proposed a finite-difference discretization of our model. The scheme considers an associated discrete energy and we established a formula for the discrete rate of change of energy. The methodology was implemented computationally, and some simulations were produced.

The simulations are in qualitative agreement with some results available in the literature on the presence of bubble-like solutions. We confirmed numerically that the energy of Higgs boson equation in the de Sitter space-time is a decreasing function of time, in agreement with the theory in the literature. On the other hand, various avenues of research open after the completion of this work. To start with, the rigorous analysis of the stability and the convergence of the scheme proposed here is a task that must be tackled in a future work. Moreover, extending the present approach to the three-dimensional scenario would be a physically meaningful direction of investigation. The use of such methodology could be helpful in elucidating novel results on the solutions of Higgs boson equation in the de Sitter space-time.

## Conclusions from Chapter 2

In this work, we investigated the radially symmetric solutions of a generalization of the Higgs boson equation in the de Sitter space-time. The model investigated here is a three-dimensional system with initial-boundary data on a sphere with center at the origin of $\mathbb{R}^3$. The system has an energy functional which is dissipated with respect to time, and we expressed the mathematical model, the total energy and the rate of change of energy in terms of the new radial variable. Using a finite-difference methodology, we proposed an implicit discrete model to approximate the solutions of the radially symmetric Higgs equation of interest. At the same time, a discrete form of the total energy of the system was provided, and we showed that the discrete model is a dissipative technique. The existence of solutions for any set of initial conditions was established using Brouwer's fixed-point theorem, and we provided a formula for the dissipation of energy. The numerical properties of quadratic consistency, stability and quadratic convergence were theoretically proved. The uniqueness of discrete solutions was obtained as a consequence of stability. It is worth pointing out that the properties of stability and convergence were also proved using a discrete form of the energy method. Some numerical simulations were provided to show the effectiveness of our approach. In particular, we provided illustrations of radially symmetric solutions of the mathematical model, and exhibited the presence of bubble-like solutions of the mathematical model. For the convenience of the reader and for reproducibility purposes, we also provided Matlab® functions to obtain the approximations to the solutions of the continuous model. In various senses, this manuscript improves some previous reports available in the literature to solve numerically the Higgs boson equation in the de Sitter space-time.

After the completion of this work, there are various avenues of research which remain open. From the physical point of view, the present methodology and the code available at the end of this work may be helpful simulation tools in the investigation of radially symmetric solutions of the Higgs boson equation in the de Sitter space time. However, from the theoretical point of view, the applicability of the present approach to other different problems may be also a topic of interest in various areas of applied mathematics. As one of the reviewers pointed out, the present approach and the theoretical analysis may find applications in the investigation of the traffic dynamics on transportation networks [58, 38, 83, 24, 22], general urban networks [20, 12, 15] and urban networks and supply chains [21, 23]. Moreover, another topic of interest may be the feasibility to extend the present approach using differential quadrature methods [73, 105, 106]. In addition, a possible extension of the present methodology to fractional forms of variants of the Higgs boson equation in the de Sitter space-time. Some progress has been reported already [63], but the design of fully Hamiltonian schemes for dissipative wave equations is still a question whose answer is missing in the literature. There are already some reports available in the literature in those scenarios [64, 70, 65], but most of the schemes proposed in those works are Hamiltonian discretizations in the regimes without damping.

## Conclusions from Chapter 3

For the first time in the literature, the present work reports on a finite-difference discretization for the Higgs boson equation in the De Sitter space-time, which is capable of preserving the energy properties of the continuous model. The model investigated in this manuscript is actually a general form of the Higgs boson equation. Indeed, the mathematical model investigated in this work considers the presence of fractional diffusion of the Riesz type, a finite (though arbitrary) number of spatial dimensions, an extended time-dependent diffusion factor, a generalized potential function and an arbitrary constant damping coefficient. We noted that the mathematical model has an associated energy functional, and we calculated its dissipation of energy. The finite-difference model proposed to solve the continuous system also considers a discretized form of the energy functions, and the discrete system satisfies energy properties which resemble the continuous counterpart. In that sense, the numerical integrator is a three-step implicit Hamiltonian methodology. We proved thoroughly the consistency properties of both the discrete equations of motion and the discrete energy densities. The method is a stable technique which is quadratically convergent. Some simulations were conducted in order to assess the applicability of the methodology. In particular, our simulations exhibited the existence of "bubble" solutions in the non-fractional scenario, when the analytical apparatus available in the literature predicted them. It is worth pointing out that these structures were also found in the fractional case, for which no analytical theory is available to this day.

Before closing this work, we must mention that there are various avenues of research which still remain open, not mentioning the potential physical applications [66, 68, 67]. For example, the methodology reported in this work is an implicit technique which is difficult to be implemented computationally for dimensions higher than 1. A natural question is whether it is possible to provide an explicit discretization for the model (3.2.13) which can alleviate this shortcoming. Moreover, in view of the physical relevance of the three-dimensional scenario and the computational cost for that case, it would be expected to wonder if parallel implementations of such explicit techniques. Such computational implementation would be a helpful tool in the investigation of the solutions of fractional forms of Higgs boson equation in the de Sitter space-time and, in particular, in the search for more analytical conditions which guarantee the existence of "bubble" solutions. Also, it is important to point out that the discrete methodology (3.3.18) has interesting variational properties. However, the expression for the discrete rate of change of energy provided in Theorem 3.4.5 cannot be simplified to agree with the conclusion of its continous counterpart, Corollary 3.2.6. This is perhaps the most important analytical limitation of our approach. Various discretizations for the discrete energy can be proposed, but the present is the expression that provides the most interesting structural and numerical properties.

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# A Appendices

## A.1

In this appendix, we will provide a Matlab® implementation of the finite-difference scheme (2.3.11) to approximate the solutions of the problem (2.2.10). The code may be modified and improved freely by the interested reader. Various input and output parameters are employed in this implementation of the numerical model, all of them related to the mathematical model and the computational setting described in the previous sections. The relation between the computational variables and the model/numerical parameters is described next.

- **Input parameters.** All the input parameters may be modified within the Matlab® code.
  - **Model parameters.** Those parameters correspond to the expression of the problem (2.5.1).
    - `lambda`= $\lambda$.
    - `mu`= $\mu$.
  - **Space-time domain parameters.**
    - `T`= $T$.
    - `L`= $L$.
  - **Computational parameters.**
    - `h`= $h$.
    - `tau`= $\tau$.
  - **Initial conditions.**
    - `phi0`= $\phi_0$.

* phi1= $\phi_1$.

- **Output variables.**

  * r= $(r_k)_{k=0}^{K}$.
  * PSI= $(\Psi_k^N)_{k=0}^{K}$.
  * H= $(H_k^N)_{k=0}^{K}$.
  * t= $(t_n)_{n=0}^{N}$.
  * E= $(E^n)_{n=0}^{N}$.

```
function [r,PSI,H,E] = HiggsRadial

 lambda = 2;
 mu = 3;

 L = 1;
 T = 7;
 tau = 0.002;
 h = 0.002;
 r = (h:h:L)';
 t = (0:tau:T)';
 M = size(r,1);
 N = size(t,1);

 phi0 = 3*B(r,0.5,0.3)';
 phi1 = 0*phi0;
 psi0 = r.*phi0;
 psi1 = r.*phi1;

 PSI1 = zeros(M,1);
 H    = zeros(M,N-2);
 E    = zeros(N-2,1);
 J    = zeros(M,M);
 F_x  = zeros(M,1);

 S = ones(M,1);
 l = 1;
 PSI1 = (2*psi0-2*tau*psi1
  -f(t(1))*(tau/h)^2*([psi0(2:M);0]-2*psi0+[0;psi0(1:M-1)]) ...
  -3*tau*psi0+tau^2*r.*Fprim(phi0))/(2+3*tau);

 PSI = zeros(M,N-2);
 PSI(:,1:3) = [PSI1-2*tau*r.*phi1,r.*phi0, PSI1];

 for n=3:N-1
   PSI(:,n+1) = (2*PSI(:,n)-PSI(:,n-1)...
    + f(t(n-1))*(tau/h)^2.*([PSI(2:M,n); 0]-2*PSI(:,n)+[0;PSI(1:M-1,n)])...
    + 3*tau*PSI(:,n)-tau^2*r.*Fprim(PSI(:,n)./r))/(1+3*tau);
   l = 1;
   S = ones(M,1);
   while l<100 && norm(S,inf)>10^(-9)
```

```
            l = l+1;
            F_x = PSI(:,n+1) - 2*PSI(:,n) + PSI(:,n-1)...
             - f(t(n-1))*(tau/h)^2.*([PSI(2:M,n); 0]-2*PSI(:,n)+[0; PSI(1:M-1,n)]) ...
             + 3*tau*( PSI(:,n+1)-PSI(:,n) ) + tau^2*r.*Fprim(PSI(:,n)./r);
            J = zeros(M,M);
            J((0:M-1)*M+(1:M)) = 1 + 3*tau;
            S = linsolve(J,-F_x);
            PSI(:,n+1) = PSI(:,n+1) + S;
        end
        H(:,n-2) = 4*pi*exp(3*t(n-2))*(...
          0.5/tau*( PSI(:,n+1)-PSI(:,n) ).^2 ...
          +1.5/(2*tau)*(PSI(:,n+1)-PSI(:,n)).*(PSI(:,n+1)+PSI(:,n))...
+(0.5/h^2)*f(t(n-2))*([0;PSI(2:M,n+1)]-PSI(:,n+1)).*([0;PSI(2:M,n)]-PSI(:,n))
         +r.^2.*F(PSI(:,n)./r));
        E(n-2) = h*sum(H(:,n-2));
    end
end

function Fp_k = Fprim(phik)
  mu = 3; lambda = 2;
  Fp_k = -mu^2*phik + lambda*abs(phik).^2.*phik;
end

function F_k = F(phik)
  mu = 3; lambda = 2;
  F_k = -0.5.*mu^2*phik.^2 + 0.25*lambda*abs(phik).^4;
end

function y = f(t)
  y = exp(-2*t);
end

function Bb = B(r,C,R)
  Bb = exp( (1/R^2) - 1./( R^2-(r-C).^2 ) ).*(abs(r-C)<R);
  Bb( isnan( Bb ) ) = 0;
  Bb = Bb';
end
```

# A.2

This section is devoted record some alternative expressions of the equations (3.4.7)–(3.4.10). With such expressions, one can establish alternative forms of the discrete rate of change of energy given in Theorem 3.4.5. Those expressions are provided in the following lemma, which is stated without proof in view of the similarity to Lemma 3.4.4.

**Lemma A.2.1.** *Let $\Phi \in \mathcal{V}_h$ be a solution of* (3.3.18)*, and $(j,k) \in J \times I_{K-2}$. The following identities hold:*

$$\frac{1}{2}\delta_t\left[e^{\gamma t_{k-1}}\|\delta_t\Phi_j^{k-1}\|_{x,2}^2\right] = \frac{\delta_t e^{\gamma t_{k-1}}}{2}\|\delta_t\Phi_j^{k-1}\|_{x,2}^2 + e^{\gamma t_k}\left\langle \delta_t^{(2)}\Phi_j^k, \delta_t^{(1)}\Phi_j^k\right\rangle_x,$$

$$\frac{1}{2}\delta_t\left[\gamma e^{\gamma t_{k-1}}\left\langle \delta_t\Phi_j^{k-1},\mu_t\Phi_j^{k-1}\right\rangle_x\right]$$
$$=\frac{\gamma e^{\gamma t_{k-1}}}{2}\left\langle \delta_t^{(1)}\Phi_j^k,\delta_t\Phi_j^{k-1}\right\rangle_x+\frac{\gamma e^{\gamma t_{k-1}}}{2}\left\langle \delta_t^{(2)}\Phi_j^k,\mu_t\Phi_j^k\right\rangle_x+\frac{\gamma\delta_t e^{\gamma t_{k-1}}}{2}\left\langle \delta_t\Phi_j^k,\mu_t\Phi_j^k\right\rangle_x$$
$$=\frac{\gamma e^{\gamma t_{k-1}}}{2}\left\langle \delta_t^{(1)}\Phi_j^k,\delta_t\Phi_j^k\right\rangle_x+\frac{\gamma e^{\gamma t_{k-1}}}{2}\left\langle \delta_t^{(2)}\Phi_j^k,\mu_t\Phi_j^{k-1}\right\rangle_x+\frac{\gamma\delta_t e^{\gamma t_{k-1}}}{2}\left\langle \delta_t\Phi_j^k,\mu_t\Phi_j^k\right\rangle_x,$$

$$\frac{1}{2}\delta_t\left[e^{\gamma t_{k-1}}f(t_{k-1})\left\langle \delta_x^{(\alpha/2)}\Phi_j^k,\delta_x^{(\alpha/2)}\Phi_j^{k-1}\right\rangle_x\right]$$
$$=\frac{\delta_t e^{\gamma t_{k-1}}}{2}f(t_{k-1})\left\langle \delta_x^{(\alpha/2)}\Phi_j^k,\delta_x^{(\alpha/2)}\Phi_j^{k-1}\right\rangle_x+\frac{e^{\gamma t_k}}{2}\delta_t f(t_{k-1})\left\langle \delta_x^{(\alpha/2)}\Phi_j^k,\delta_x^{(\alpha/2)}\Phi_j^{k-1}\right\rangle_x$$
$$-e^{\gamma t_k}f(t_k)\left\langle \delta_x^{(\alpha)}\Phi_j^k,\delta_t^{(1)}\Phi_j^k\right\rangle_x$$
$$=\frac{\delta_t e^{\gamma t_{k-1}}}{2}f(t_k)\left\langle \delta_{(x)}^{(\alpha/2)}\Phi_j^k,\delta_x^{(\alpha/2)}\Phi_j^{k-1}\right\rangle_x+\frac{e^{\gamma t_{k-1}}}{2}\delta_t f(t_{k-1})\left\langle \delta_x^{(\alpha/2)}\Phi_j^k,\delta_x^{(\alpha/2)}\Phi_j^{k-1}\right\rangle_x$$
$$-e^{\gamma t_k}f(t_k)\left\langle \delta_x^{(\alpha)}\Phi_j^k,\delta_t^{(1)}\Phi_j^k\right\rangle_x$$

*and*

$$\delta_t\left(e^{\gamma t_{k-1}}\mu_t\left\langle F(\Phi_j^{k-1},1)\right\rangle_x\right)=e^{\gamma t_{k-1}}\left\langle \delta_{t,\Phi}^{(1)}F(\Phi_j^k),\delta_t^{(1)}\Phi_j^k\right\rangle_x+\delta_t e^{\gamma t_{k-1}}\mu_t\left\langle F(\Phi_j^k),1\right\rangle_x.$$

# A.3

The following is a Matlab implementation of the numerical model (3.6.5). It is important to mention that the code considers exact initial data. Some straightforward changes readily lead to the implementation of the full finite-difference scheme (3.6.5).

```
function [x,t,phi3,LocalE,TotalE]=higgs

  function y=ball(x,C,R)
    L=length(x);
    y=zeros(1,L);
    for l=1:L
      if abs(x(l)-C)<R
        y(l)=exp(1./R.^2-1./(R.^2-(x(l)-C).^2));
      end
    end
  end

  a=-1;
  b=2;
  T=0.03;
  h=0.01;
  tau=0.00005;

  mu=3;
  alpha=2;
  gama=1;
  lambda=2;

  x=a:h:b;
```

```
t=0:tau:T;

M=length(x);
K=length(t);

g=zeros(1,M);
g(1)=gamma(alpha+1)/gamma(0.5*alpha+1)^2/h^alpha;
for k=1:M-1
  g(k+1)=(1-(alpha+1)/(0.5*alpha+k))*g(k);
end

I=eye(M);
H=zeros(M,M);
for i=1:M
  for j=1:M
    H(i,j)=g(abs(i-j)+1);
  end
end

R=tau.^2*exp(-2.*t)./h^alpha;

phi1=ball(x,0.5,0.3)';
phi2=phi1;

TotalE=zeros(1,K);

frac1=H*phi1;
F1=-0.5.*mu.^2.*phi1.^2+0.25.*lambda.*abs(phi1).^4;

for k=2:K-1
  A=(2+gama*tau).*I+R(k+1).*H;
  B=(2-gama*tau).*I+R(k-1).*H;
  G=lambda.*phi2.*abs(phi2).^2-mu.^2.*phi2;
  v=4*phi2-2.*tau.^2.*G-B*phi1;

  frac2=H*phi2;
  F2=-0.5.*mu.^2.*phi2.^2+0.25.*lambda.*abs(phi2).^4;
  LocalE=exp(gama*t(k-1)).*(0.5.*(phi2-phi1).^2./tau/tau...
    +0.25.*gama.*(phi2-phi1).*(phi2+phi1)./tau...
    +0.5.*exp(-2*t(k-1)).*frac2.*frac1+0.5.*(F1+F2));
  TotalE(k)=tau*sum(LocalE);

  phi3=linsolve(A,v);
  phi1=phi2;
  phi2=phi3;
  frac1=frac2;
  F1=F2;
  end
end
```

In light of this code, the method (3.6.5) is obviously a simple scheme in the sense that its computational implementation is relatively easy. Indeed, once that the matrices `A` and `B` of the code have been constructed at each iteration, the resolution of the linear system describing (3.6.5) is straightforward. In our case, we have used the Matlab function `linsolve` to solve the linear system. It is important to point out that the matrix `A` is a full matrix in the fully fractional case. This fact could result in a slow performance of the scheme for relatively large matrices. This method could be combined with the full nonlinear scheme (3.3.18) to provide the initial approximations needed by Newton's method.