UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

ANALYSIS OF CONTINUOUS AND DISCRETE MODELS FOR FRACTIONAL KLEIN–GORDON–ZAKHAROV SYSTEMS

PRESENTA

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PARA OPTAR POR EL GRADO DE DOCTOR EN CIENCIAS APLICADAS Y TECNOLOGÍA

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TITULO: ANALYSIS OF CONTINUOUS AND DISCRETE MODELS FOR FRACTIONAL KLEIN–GORDON–ZAKHAROV SYSTEMS

IMPACTO SOCIAL (señalar el impacto logrado): Ciencia básica orientada a la aplicación de derivadas espaciales fraccionarias en una extensión fraccionaria del sistema Klein–Gordon-Zakharov demostrando que la energía total del sistema se conserva y que las soluciones globales del sistema están acotadas. Tiene impacto en aplicaciones físicas.

INDICAR SI NO N.A. (NO APLICA) SEGÚN CORRESPONDA:

	Elementos para la revisión académica del trabajo de tesis o trabajo práctico:
SI	El trabajo es congruente con las LGAC del programa de posgrado
SI	La problemática fue abordada desde un enfoque multidisciplinario
SI	Existe coherencia, continuidad y orden lógico del tema central con cada apartado
SI	Los resultados del trabajo dan respuesta a las preguntas de investigación o a la problemática que aborda
SI	Los resultados presentados en el trabajo son de gran relevancia científica, tecnologíca o profesional según el área
SI	El trabajo demuestra más de una aportación original al conocimiento de su área
N.A.	Las aportaciones responden a los problemas prioritarios del país
NO	Generó transferecia del conocimiento o tecnológica
SI	Cumple con la ética para la investigación (reporte de la herramienta antiplagio)
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SI	Cumple con los requisitos señalados en el plan de estudios (créditos curriculares, optativos, actividades complementarias,estancia, predoctoral, etc)
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SI	Tiene congruencia con cuerpos académicos
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SI	Tiene el artículo aceptado o publicado y cumple con los requisitos institucionales (en caso que proceda)
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SI	Aceptación o Publicación de los articulos según el nivel del programa
SI	El estudiante es el primer autor
SI	El autor de correspondencia es el Tutor del Núcleo Académico Básico
SI	En los artículos se ven reflejados los objetivos de la tesis, ya que son producto de este trabajo de investigación.
SI	Los artículos integran los capítulos de la tesis y se presentan en el idioma en que fueron publicados
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Theoretical analysis of an explicit energy-conserving scheme for a fractional Klein–Gordon–Zakharov system

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A B S T R A C T

Departing from an initial-boundary-value problem governed by a Klein–Gordon–Zakarov system with fractional derivatives in the spatial variable, we provide an explicit finite-difference scheme to approximate its solutions. In agreement with the continuous system, the method proposed in this work is also capable of preserving the energy of the system, and the energy quantities are nonnegative under flexible parameter conditions. The boundedness, the consistency, the stability and the convergence of the technique are also established rigorously. The method is easy to implement, and the computer simulations confirm the main analytical and numerical properties of the new model.

1. Introduction

Let $B = (x_L, x_R)$ be a nonempty interval in $\mathbb{R}$, let $T > 0$ and define $\Omega = B \times (0, T)$. In this work, for each $S \subseteq \mathbb{R}^p$, we will use the notation $\overline{S}$ to represent the closure of S with respect to the standard topology of $\mathbb{R}^p$, for any $p \in \mathbb{N}$. Assume that u and m are a complex- and a real-valued functions, respectively, whose domains are both equal to $\overline{\Omega}$. Moreover, let $u_0, u_1 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions, and let $v, w : \overline{\Omega} \to \mathbb{R}$ be such that

$$v(x,t) = -\frac{\partial w(x,t)}{\partial x}, \tag{1.1}$$

$$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial m(x,t)}{\partial t}. \tag{1.2}$$

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 < \alpha \leq n$. The *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is defined (when it exists) as

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Letter to the editor

Corrigendum to “A numerically efficient and conservative model for a Riesz space-fractional Klein–Gordon–Zakharov system”

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ABSTRACT

In the article [Commun Nonlinear Sci Numer Simul 2019;71:22–37], the authors provided an erroneous proof for the existence of solutions of a finite-difference model for a fractional Klein–Gordon–Zakharov equation. This letter is intended to provide a correct proof of that theorem.

In the article [1], the authors established a theorem on the existence of solutions of a finite-difference model for a fractional Klein–Gordon–Zakharov (KGZ) equation [2,3]. The proof made use of the Leray–Schauder fixed-point theorem. However, the authors committed an involuntary mistake in the proof, for which they sincerely apologize. The mistake were committed in Eqs. (5.6) and (5.7) of that work. Indeed, the second and third terms of the right-hand sides of those equations had to be multiplied by $\lambda \in [0, 1]$, and they were not. It turns out that the corrected proof becomes much harder, and we provide it in this letter. To that end, let us recall that the mathematical model under investigation in [1] is the space-

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An energy-preserving and efficient scheme for a double-fractional conservative Klein–Gordon–Zakharov system

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Numerical efficiency analysis

ABSTRACT

In this work, we consider a fractional extension of the Klein–Gordon–Zakharov system which describes the propagation of strong turbulences on the Langmuir wave in a high-frequency plasma. Both components consider space-fractional derivatives of the Riesz type, and initial-boundary conditions are imposed on a closed and bounded interval of the real numbers. In a first stage, we show that the total energy of the system is conserved, and that the global solutions of the system are bounded. Motivated by these results, we propose a finite-difference scheme to approximate the solutions, and a discrete form of the energy functional. The advantage of the discretization proposed in this work lies in that the difference equations to solve the component equations are decoupled. This implies that the numerical schemes can be solved separately at each temporal step. We establish rigorously the existence of solutions, as well as the capability of the scheme to conserve the discrete energy. The method has a second-order consistency in both space and time. Moreover, using a discrete form of the energy method, we establish mathematically that the finite-difference scheme is stable and quadratically convergent. We provide some simulations to show that the proposed methodology is quadratically convergent and that it preserves the total energy of the system.

1. Introduction

The well-known Zakharov equations are partial differential equations that appear in the investigation of the propagation of Langmuir waves in plasma physics [53]. In his seminal work, V.E. Zakharov [65] showed that arbitrary Langmuir turbulences of sufficient intensity are unstable, and that the instability leads to the development of low density regions in plasma which collapse in finite time. These regions were called *caverns*, and they are the mechanism of energy dissipation of long-wave Langmuir oscillations. These conclusions were the result of the quantitative analysis on the equations describing the interaction between the fast-time scale component of an electric field and the deviation of ion density from equilibrium. This system was called the *Zakharov system*, and it motivated the development and analysis of further models that provided more realistic descriptions of similar physical phenomena [18,38], including the Klein–Gordon–Zakharov system

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An implicit semi-linear discretization of a bi-fractional Klein–Gordon–Zakharov system which conserves the total energy

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A B S T R A C T

In this work, we propose an implicit finite-difference scheme to approximate the solutions of a generalization of the well-known Klein–Gordon–Zakharov system. More precisely, the system considered in this work is an extension to the spatially fractional case of the classical Klein–Gordon–Zakharov model, considering two different orders of differentiation and fractional derivatives of the Riesz type. The numerical model proposed in this work considers fractional-order centered differences to approximate the spatial fractional derivatives. The energy associated to this discrete system is a non-negative invariant, in agreement with the properties of the continuous fractional model. We establish rigorously the existence of solutions using fixed-point arguments and complex matrix properties. To that end, we use the fact that the two difference equations of the discretization are decoupled, which means that the computational implementation is easier than for other numerical models available in the literature. We prove that the method has square consistency in both time and space. In addition, we prove rigorously the stability and the quadratic convergence of the numerical model. As a corollary of stability, we are able to prove the uniqueness of numerical solutions. Finally, we provide some illustrative simulations with a computer implementation of our scheme.

1. Introduction

The field of fractional calculus has witnessed a vertiginous development in recent years, partly due to the vast amount of potential applications in the sciences [17]. It is worth pointing out that many different fractional derivatives and integrals have been proposed. For example, some of the first fractional derivatives introduced historically in mathematics were the

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Two energy-preserving numerical models for a multi-fractional extension of the Klein–Gordon–Zakharov system

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ABSTRACT

This manuscript is devoted to studying approximations of a coupled Klein–Gordon–Zakharov system where different orders of fractional spatial derivatives are utilized. The fractional derivatives involved are in the Riesz sense. It is understood that such a modeling system possesses an energy functional which is conserved throughout the period of time considered, and that its solutions are uniformly bounded. Motivated by these facts, we propose two numerical models to approximate the underlying continuous system. While both approximations remain to be nonlinear, one of them is implicit and the other is explicit. For each of the discretized models, we introduce a proper discrete energy functional to estimate the total energy of the continuous system. We prove that such a discrete energy is conserved in both cases. The existence of solutions of the numerical models is established via fixed-point theorems. Continuing explorations of intrinsic properties of the numerical solutions are carried out. More specifically, we show rigorously that the two schemes constructed are capable of preserving the boundedness of the approximations and that they yield consistent estimates of the true solution. Numerical stability and convergence are likewise proved theoretically. As one of the consequences, the uniqueness of numerical solutions is shown rigorously for both discretized models. Finally, comparisons of the numerical solutions are provided, in order to evaluate the capabilities of these discrete methods to preserve the discrete energy of underlying systems.

1. Introduction

The field of fractional calculus has witnessed a vertiginous development in recent years, partly due to the vast amount of potential applications in the physical sciences [1]. To-day, many different fractional derivatives and integrals have been proposed. For example, some of the first fractional derivatives introduced historically in mathematics were the

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Original articles

A nonlinear discrete model for approximating a conservative multi-fractional Zakharov system: Analysis and computational simulations

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Abstract

A system of two partial differential equations with fractional diffusion is considered in this study. The system extends the conventional Zakharov system with unknowns being nonlinearly coupled complex- and real-valued functions. The diffusion is understood in the Riesz sense, and suitable initial-boundary conditions are imposed on an open and bounded domain of the real numbers. It is shown that the mass and Higgs' free energy of the system are conserved. Moreover, the total energy is proven to be dissipated, and that both the free and the total energy are non-negative. As a corollary from the conservation of energy, we find that the solutions of the system are bounded throughout time. Motivated by these properties on the solutions of the system, we propose a numerical model to approximate the fractional Zakharov system via finite-difference approaches. Along with this numerical model for solving the continuous system, discrete analogues for the mass, the Higgs' free energy and the total energy are we provided. Furthermore, utilizing Browder's fixed-point theorem, we establish the solubility of the discrete model. It is shown that the discrete total mass and the discrete free energy are conserved, in agreement with the continuous case. The discrete energy functionals (both the discrete free energy and the discrete total energy) are proven to be non-negative functions of the discrete time thoroughly the boundedness of the numerical solutions. Properties of consistency, stability and convergence of the scheme are also studied rigorously. Numerical simulations illustrate some of the anticipated theoretical features of our finite-difference solution procedure.

Keywords: Fractional Zakharov systems; Conservation of Higgs' free energy; Riesz space-fractional equations; Energy-conserving method; Fractional-order central differences; Numerical efficiency analysis

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Resumen

En esta tesis, investigamos una extensión fraccionaria del sistema Klein–Gordon-Zakharov donde se utilizan diferentes órdenes de derivadas espaciales fraccionarias en el sentido de Riesz. Demostramos que la energía total del sistema se conserva y que las soluciones globales del sistema están acotadas. Motivados por estos hechos, proponemos modelos numéricos para aproximar el sistema continuo subyacente. Para cada uno de los modelos discretizados, introducimos un funcional discreto de energía adecuado para estimar la energía total del sistema continuo. Probamos que tal energía discreta se conserva en todos los casos. La existencia de soluciones de los modelos numéricos se establece mediante teoremas de punto fijo. Mostramos rigurosamente que los esquemas construidos son capaces de preservar la acotación de las aproximaciones y que producen estimaciones consistentes de la solución real. La estabilidad numérica y la convergencia también se prueban teóricamente. Como una de las consecuencias, se muestra rigurosamente la unicidad de las soluciones numéricas para todos modelos discretizados. Finalmente, se proporcionan comparaciones de las soluciones numéricas para evaluar las capacidades de estos métodos discretos para preservar la energía discreta de sus sistemas.

Abstract

In this thesis, we investigate a fractional extension of the Klein–Gordon–Zakharov system where different orders of fractional spatial derivatives are utilized in the Riesz sense. We show that the total energy of the system is conserved, and that the global solutions of the system are bounded. Motivated by these facts, we propose numerical models to approximate the underlying continuous system. For each of the discretized models, we introduce a proper discrete energy functional to estimate the total energy of the continuous system. We prove that such a discrete energy is conserved in all cases. The existence of solutions of the numerical models is established via fixed-point theorems. We show rigorously that the schemes constructed are capable of preserving the boundedness of the approximations and that they yield consistent estimates of the true solution. Numerical stability and convergence are likewise proved theoretically. As one of the consequences, the uniqueness of numerical solutions is shown rigorously for all discretized models. Finally, comparisons of the numerical solutions are provided, in order to evaluate the capabilities of these discrete methods to preserve the discrete energy of their systems.

Introduction

Background

The field of fractional calculus has witnessed a vertiginous development in recent years, partly due to the vast amount of potential applications in the physical sciences [31]. To-day, many different fractional derivatives and integrals have been proposed. For example, some of the first fractional derivatives introduced historically in mathematics were the Riemann–Liouville fractional derivatives [68], which generalized the classical integer-order derivatives with respect to some specific analytical properties [77]. The fractional derivatives in the senses of Caputo, Riesz and Grünwald–Letnikov are also extensions of the traditional derivatives of integer order. It is worth pointing out here that these fractional operators are nonequivalent in general, and various applications of all of them have been proposed to science and engineering [19, 87]. For instance, some reports have provided theoretical foundations for the application of fractional calculus to the theory of viscoelasticity [4], while others have proposed possible applications of fractional calculus to dynamic problems of solid mechanics [80], financial economics [25, 83], Earth system dynamics [104], mathematical modeling of biological phenomena [37] and the modeling of two-phase gas/liquid flow systems [69], just to mention some potential applications. However, it is important to recall that Riesz-type derivatives may be the only fractional derivatives which have real physical applications [107]. This is due to a well-known result by Tarasov which establishes that Riesz fractional derivatives result from systems with long-range interactions in a continuum-limit case [88]. In turn, systems consisting of particles with long-range interactions are useful nowadays in statistical mechanics, thermostatics [57, 17] and the theory of biological oscillator networks [10].

From the mathematical point of view, many interesting avenues of investigation have been opened by the progress in fractional calculus. Indeed, the different fractional derivatives have found discrete analogues which have been extensively in the literature. As examples, Riesz fractional derivatives have been discretized consistently in various fashions using fractional-order centered differences [72, 73] and weighted-shifted Grünwald differences [33, 43]. Obviously, those discrete approaches have been studied to determine their analytical properties, and they have been used extensively to provide discrete models to solve Riesz space-fractional conservative/dissipative space-fractional wave equations [34], a Hamiltonian fractional nonlinear elastic string equation [56], an energy-preserving double fractional Klein–Gordon–Zakharov system [62] and even a Riesz space-fractional generalization with generalized time-dependent diffusion coefficient and potential of the Higgs boson equation in the de Sitter space-time [56], among other complex systems [51]. On the other hand, Caputo fractional derivatives have been discretized consistently using various criteria. For instance, some high-order $L2$-compact difference approaches have used to that end [99], as well as $L1$ formulas [23, 70] and $L1$-2 methodologies [29]. Using those approaches, various numerical schemes have been proposed to solve efficiently some Caputo time-fractional diffusive and wave differential equations [101, 52, 66]. Also, various potential applications of these systems have been reported in the sciences [47, 57].

From the analytical point of view, the literature offers a wide range of reports which focus on the extension of integer-order methods and results to the fractional case. For example, there are various articles which tackle the existence, uniqueness, regularity and asymptotic behavior of the solution for the fractional porous medium equation [20], nonlinear fractional diffusion equations [93], nonlinear fractional heat equations [9], the Fisher–Kolmogorov–Petrovskii–Piscounov equation with nonlinear fractional diffusion [85], fractional thin-film equations [84] and the fractional Schrödinger equation with general nonnegative potentials [22]. From a more particular point of view, the fractional generalization of the classical vector calculus operators (that is, the gradient, divergence, curl and Laplacian operators) has been also an active topic of research which has been developed from different approaches. Some of the first attempts to extend those operators to the fractional scenario were proposed in [1, 2] using the Nishimoto fractional derivative. These operators were used later on in [64] to provide a physical interpretation for the fractional advection-dispersion equation for flow in heterogeneous porous media (see [89] and references therein for a historic account of the efforts to formulate a fractional form of vector calculus). More recently, a new generalization of the Helmholtz decomposition theorem for both fractional time and space was proposed in [71, 74] using the discrete Grünwald–Letnikov fractional derivative. The authors of those works consider different derivative orders, assuming non-homogeneous models and non-isotropic spaces.

Aims and scope

In this work, we introduce two numerical schemes to solve a double-fractional Klein–Gordon–Zakharov system of equations. We establish thoroughly the existence and the uniqueness of solutions, prove the conservation of discrete energy and prove that the numerical schemes are consistent, stable and convergent. It is worthwhile to recall that the well-known Zakharov equations are partial differential equations that appear in the investigation of the propagation of Langmuir waves in plasma physics [90]. Zakharov [103] showed that arbitrary Langmuir turbulences of sufficient intensity are unstable, and that the instability leads to the development of low density regions in plasma which collapse in finite time. These regions were called *caverns*, and they are the mechanism of energy dissipation of long-wave Langmuir oscillations. These conclusions were the result of the quantitative analysis on the equations describing the interaction between the fast-time scale component of an electric field and the deviation of ion density from equilibrium. This system was called the *Zakharov system*, and it motivated the development and analysis of further models that provided more realistic descriptions of similar physical phenomena [30, 59], including the Klein–Gordon–Zakharov system [91]. To this day, the Zakharov system is still considered as one of the best systems to describe the coupling between high-frequency Langmuir waves and low-frequency ion-acoustic waves. Needless to mention that this model has been applied to the description of shallow-water waves [11] and nonlinear optics [18], among other physical applications. In the present work, we will consider a coupled Riesz space-fractioinal Klein–Gordon–Zakharov system with two (not necessarily equal) fractional derivatives. The system is capable of conserving the energy, whence the development of conservative schemes to approximate the solutions of this system is numerically justified.

Summary

The present thesis is organized as follows. In Chapter 1, we introduce an explicit method to approximate the solutions of Klein–Gordon–Zakharov system. In Chapter 2, we show the existence of solutions for an implicit model. In Chapter 3, we introduce a method with double fractional derivative. Later, we propose a semi-linear model in Chapter 4. In Chapter 5, we analyze two different methods and discuss pros and cons of each one. Finally, in Chapter 6, we tackle a rigorous study of Zakharov equation, both continuous and discrete case.

1. An explicit method

1.1 Introduction

Let $B = (x_L, x_R)$ be a nonempty interval in $\mathbb{R}$, let $T > 0$ and define $\Omega = B \times (0, T)$. In this work, for each $S \subseteq \mathbb{R}^p$, we will use the notation $\overline{S}$ to represent the closure of S with respect to the standard topology of $\mathbb{R}^p$, for any $p \in \mathbb{N}$. Assume that u and m are a complex- and a real-valued functions, respectively, whose domains are both equal to $\overline{\Omega}$. Moreover, let $u_0, u_1 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions, and let $v, w : \overline{\Omega} \to \mathbb{R}$ be such that

$$v(x,t) = -\frac{\partial w(x,t)}{\partial x}, \tag{1.1}$$

$$\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial m(x,t)}{\partial t}. \tag{1.2}$$

Definition 1.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 < \alpha \leq n$. The *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is defined (when it exists) as

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{1.3}$$

Here, Γ denotes the usual Gamma function. For the sake of briefness, we will sometimes use the nomenclature $d^\alpha_{|x|} f(x)$ to represent the Riesz fractional derivative of f of order α at x.

In the present work, we will investigate an extension of the Klein–Gordon–Zakharov equations which considers the presence of Riesz space-fractional derivatives [100]. In the following, all the relevant functions will be defined on $\overline{\Omega}$. However, for the sake of simplicity, we will extend their definitions to the set $\mathbb{R} \times [0, T]$, by letting them be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$. For the remainder of this work and unless we mention otherwise, we will fix $\alpha \in (1, 2]$. Under these circumstances, the problem investigated in this work is given by the following system, where $(x, t) \in \Omega$.

$$\begin{gathered}
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^2 m(x,t)}{\partial x^2} - \frac{\partial^2 \left(|u(x,t)|^2\right)}{\partial x^2} = 0, \\
\text{such that } \begin{cases} u(x,0) = u_0(x), \ m(x,0) = m_0(x), & \forall x \in \overline{B}, \\ \dfrac{\partial u(x,0)}{\partial t} = u_1(x), \ \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, m(x_L,t) = m(x_R,t) = 0, & \forall t \in [0,T]. \end{cases}
\end{gathered} \tag{1.4}$$

It is important to recall that the Klein–Gordon–Zakharov equations appears in the investigation of the propagation of Langmuir waves in plasma physics [90, 12, 91]. In his seminal work, V. E. Zakharov [103] showed that arbitrary Langmuir turbulences of sufficient intensity are unstable, and that the instability leads to the development of low density regions in plasma which collapse in finite time. These regions were called *caverns*, and they are the mechanism of energy dissipation of long-wave Langmuir oscillations. These conclusions were the result of the quantitative analysis on the equations describing the interaction between the fast-time scale component of an electric field and the deviation of ion density from equilibrium. This system was called the *Zakharov system*, and it motivated the development and analysis of further models that provided more realistic descriptions of similar physical phenomena [30, 59, 11, 18, 32].

Example 1.1.2. The system of partial differential equations of (1.4) possess some known exact traveling-wave solution defined on $B = (-\infty, \infty)$ when $\alpha = 2$, one solution being the following system of functions (see [45, 44]):

$$u(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right) \exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \tag{1.5}$$

$$m(x,t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \tag{1.6}$$

for every $(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}$.

Definition 1.1.3. Let $p \in [1,\infty)$, and let $F = \mathbb{R}, \mathbb{C}$. We let $L^p_x(\overline{\Omega})$ denote the set of all functions $f : \overline{\Omega} \to F$ such that $f(\cdot,t) \in L^p(\overline{B})$, for each $t \in [0,T]$. For any function $f \in L^p_x(\overline{\Omega})$, the *norm* of f is the function of $t \in [0,T]$ given by

$$\|f\|_{x,p} = \left(\int_{\overline{B}} |f(x,t)|^p dx\right)^{1/p}, \quad \forall t \in [0,T]. \tag{1.7}$$

Moreover, for each pair $f, g \in L^2_x(\overline{\Omega})$, the *inner product* of f and g is the function of t defined by

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x,t)\overline{g(x,t)}dx, \quad \forall t \in [0,T]. \tag{1.8}$$

From the mathematical point of view, the model (1.4) is interesting in light of the fact that it has conserved quantities. To show this fact, recall that the additive inverse of the Riesz fractional derivative of order α has a unique square-root operators over the space of sufficiently integrable functions with compact support. This unique square-root operator is denoted by $\partial^{\alpha/2}/\partial|x|^{\alpha/2}$, and it satisfies

$$\left\langle u, -\frac{\partial^\alpha v}{\partial|x|^\alpha}\right\rangle_x = \left\langle -\frac{\partial^\alpha u}{\partial|x|^\alpha}, v\right\rangle_x = \left\langle \frac{\partial^{\alpha/2} u}{\partial|x|^{\alpha/2}}, \frac{\partial^{\alpha/2} v}{\partial|x|^{\alpha/2}}\right\rangle_x, \quad \forall t \in [0,T], \tag{1.9}$$

for any two functions u and v (see [27]).

Definition 1.1.4. Let u, m be a pair of functions satisfying the initial-boundary-value problem (1.4). We define the Hamiltonian of that fractional system as

$$\mathcal{H}(x,t) = \left|\frac{\partial u}{\partial t}\right|^2 - u\frac{\partial^\alpha u}{\partial|x|^\alpha} + |u|^2 + m|u|^2 + \frac{1}{2}v^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \tag{1.10}$$

for each $(x,t) \in \Omega$. Here, v satisfies (1.1) and (1.2), and we obviated again the dependence of all the functions on the right-hand side of this identity with respect to (x,t). The associated total energy of the system at the time $t \in [0,T]$ is given then by the function

$$\begin{aligned}\mathcal{E}(t) &= \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \left\langle u, -\frac{\partial^\alpha u}{\partial |x|^\alpha} \right\rangle_x + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\|v\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4. \\ &= \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\|v\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4.\end{aligned} \tag{1.11}$$

Theorem 1.1.5 (Hendy and Macías-Díaz [35])**.** *If u and m satisfy* (1.4) *then $\mathcal{E}$ is nonnegative and constant.* □

The development of fractional calculus has opened new perspectives in sciences [39, 5, 3]. In particular, the family of Zakharov equations has been extended to the fractional case, and the solutions of those systems have been examined from the mathematical and numerical points of view. Indeed, some studies report on fractional extensions of the $(3+1)$-dimensional Kurteweg-de Vries–Zakharov–Kuznetsov equation [82], the $(3+1)$-dimensional time-fractional Zakharov–Kuznetsov equation [41], the Zakharov–Kuznetsov–Benjamin–Bona–Mahony equation [6] and the Klein–Gordon–Zakharov system [79]. Most of these fractional extensions of the Zakharov equations also possess conserved quantities that resemble an energy functional, whence the need to propose energy-conserving schemes for those models is an interesting topic of research in numerical mathematics. There are various numerical methods in the literature that approximate the solutions of fractional systems [7, 14, 21, 65]. For instance, some schemes have been proposed to solve a time-space fractional Fokker–Planck equation with variable force field and diffusion [76], nonlinear fractional-order Volterra integro-differential equations [106], variable-order fractional Schrödinger equations [8] and the fractional two-dimensional heat equation [40]. However, there are few reports on methods that preserve the energy of these fractional systems. Indeed, most of them refer to the non-fractional case [96].

Inspired by previous successful energy-preserving discretizations for fractional wave equations [49, 46, 58] and their physical applications [47, 50, 48, 57], we propose a nonlinear finite-difference model to approximate the solutions of (1.4). The new numerical model is based on the use of fractional-order centered differences to approximate the Riesz fractional derivatives [73]. The scheme will have an associated discrete energy that resembles its continuous counterpart (1.11), and which is preserved also at each temporal step. It is worth pointing out that the numerical model proposed in this work has also many other interesting features. For example, we will show here that the model is uniquely solvable, and we will show that the solutions are bounded. The technique is an explicit technique and, as opposed to some implicit approaches reported in the literature [35], its implementation is relatively simple [96]. The method is a consistent technique with a second order of consistency under suitable assumptions on the smoothness of the continuous solutions. Moreover, the technique is quadratically convergent.

The present manuscript is organized as follows. In Section 1.2, we present the discrete nomenclature, and recall the definition and properties of the fractional-order centered differences, which are crucial in our investigation. The numerical method is provided therein, together with the associated discrete energy quantities. Section 1.3 is devoted to establish the main physical properties of the method. More precisely, we show that, as the continuous counterparts, the discrete energy quantities are preserved. Moreover, we establish the most important physical properties of the technique, namely, its capability to preserve the discrete energy and the positivity of the energy functionals. The main numerical properties of the method will be proved in Section 1.4, namely, the consistency, the stability and the convergence of the numerical model. Some numerical simulations are provided afterwards, in order to confirm the energy and convergence properties. Finally, we close this manuscript with a section of concluding remarks.

1.2 Numerical model

In this section, we will propose a numerical model to solve the initie-boundary-value problem (1.4). The discretization will hinge on the use of fractional-order centered differences which are defined below.

Definition 1.2.1 (Ortigueira [73])**.** For any function $f : \mathbb{R} \to \mathbb{R}$, $h > 0$ and $\alpha > -1$, we define the *fractional-order centered difference* of order α of f at the point x as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{1.12}$$

whenever the right-hand side of this expression converges. The coefficients of the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ are defined by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{1.13}$$

When $\alpha \in (0,1) \cup (1,2]$, $h > 0$ and $f \in \mathcal{C}^5(\mathbb{R})$ has its derivatives up to order five which belong to $L^1(\mathbb{R})$ then [97]

$$-\frac{1}{h^\alpha} \Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{1.14}$$

Let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$, for each $q \in \mathbb{N}$. Throughout, we let $J, N \in \mathbb{N}$, and define $h = (x_R - x_L)/J$ and $\tau = T/N$. We consider uniform partitions of the intervals $[x_L, x_R]$ and $[0, T]$, of the forms

$$x_L = x_0 < x_1 < \ldots < x_j < \ldots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{1.15}$$

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T, \quad \forall n \in \overline{I}_N, \tag{1.16}$$

respectively. For convenience, let $I = I_{J-1} \times I_{N-1}$ and $\overline{I} = \overline{I}_J \times \overline{I}_N$. For each $(j,n) \in \overline{I}$, we will use U_j^n and M_j^n to represent numerical approximations to the values of $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, respectively. In this manuscript, we let $R_h = \{x_j : j \in \overline{I}_J\}$, and represent the set of all complex functions on R_h by $\mathcal{V}_h$. If $V \in \mathcal{V}_h$ then we set $V_j = V(x_j)$ for each $j \in \overline{I}_J$. Let $U^n = (U_j^n)_{j \in \overline{I}_J}$ and $M^n = (M_j^n)_{j \in \overline{I}_J}$, and set $U = (U^n)_{n \in \overline{I}_N}$ and $M = (M^n)_{n \in \overline{I}_N}$.

Let V represent any of the functions U or M. In our work, we will employ the first-order difference operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j,n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{1.17}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{1.18}$$

the second-order difference operators

$$\delta_t^{(1)} V_j^n = \frac{V_j^{n+1} - V_j^{n-1}}{2\tau}, \qquad \forall (j,n) \in \overline{I}_J \times I_{N-1}, \tag{1.19}$$

$$\delta_x^{(2)} V_j^n = \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{h^2}, \qquad \forall (j,n) \in I_{J-1} \times \overline{I}_N, \tag{1.20}$$

$$\delta_t^{(2)} V_j^n = \frac{V_j^{n+1} - 2V_j^n + V_j^{n-1}}{\tau^2}, \qquad \forall (j,n) \in \overline{I}_J \times I_{N-1}, \tag{1.21}$$

and the average operators

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{1.22}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j,n) \in \overline{I}_J \times I_{N-1}, \tag{1.23}$$

It is well known that all of these quantities provide consistent estimations of suitable functions under appropriate smoothness conditions. Moreover, using the notation introduced in Definition 1.2.1, we consider the following consistent estimate of the fractional partial derivative of order α of u with respect to x at (x_j, t_n):

$$\delta_x^{(\alpha)} U_j^n = -\frac{1}{h^\alpha} \sum_{k \in \overline{I}_J} g_{j-k}^{(\alpha)} U_k^n. \tag{1.24}$$

Definition 1.2.2. Let $p \in [1, \infty)$. The *inner product* $\langle \cdot, \cdot \rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{C}$ and the *norms* $\| \cdot \|_p, \| \cdot \|_\infty : \mathcal{V}_h \to \mathbb{R}$ are defined, respectively, by

$$\langle U, V \rangle = h \sum_{j \in \overline{I}_J} U_j \overline{V_j}, \quad \forall U, V \in \mathcal{V}_h, \tag{1.25}$$

$$\|U\|_p^p = h \sum_{j \in \overline{I}_J} |U_j|^p, \quad \forall U \in \mathcal{V}_h \tag{1.26}$$

$$\|U\|_\infty = \max \left\{ |U_j| : j \in \overline{I}_J \right\}, \quad \forall U \in \mathcal{V}_h. \tag{1.27}$$

Moreover, for each $V = (V^n)_{n \in \overline{I}_N} \subseteq \mathcal{V}_h$, we define

$$|||V|||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}. \tag{1.28}$$

It is useful to note that the following identity holds for any $U, V \in \mathcal{V}_h$:

$$\langle U, \delta_x^{(2)} V \rangle = -\langle \delta_x U, \delta_x V \rangle - U_0 \delta_x V_0 + U_J \delta_x V_{J-1}. \tag{1.29}$$

A convenient reduction is obtained when $U_0 = U_J = 0$, in which case $\langle U, \delta_x^{(2)} V \rangle = -\langle \delta_x U, \delta_x V \rangle$. Under these circumstances, the discrete function δ_x is called the *square-root operator* of $\delta_x^{(2)}$. Moreover, the existence of square roots has been extended to account for fractional-order centered differences. Indeed, let $\mathring{\mathcal{V}}_h$ be the subspace of $\mathcal{V}_h$ consisting of those grid functions V such that $V_0 = V_J = 0$.

Lemma 1.2.3 (Wang and Huang [95], Macías-Díaz [49])**.** *For each $\alpha \in (1, 2]$ there exists is a unique, self-adjoint, positive, linear operator $\Lambda_x^{(\alpha)} : \mathring{\mathcal{V}}_h \to \mathring{\mathcal{V}}_h$ such that, if $u, v \in \mathring{\mathcal{V}}_h$ then $\langle -\delta_x^{(\alpha)} u, v \rangle = \langle \Lambda_x^{(\alpha)} u, \Lambda_x^{(\alpha)} v \rangle$.* □

Using the nomenclature of this section, the finite-difference model employed in this work to approximate the solutions of (1.4) is described by the following algebraic system, for each $\forall (j,n) \in I$:

$$\begin{gathered}
\delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} U_j^n + \mu_t^{(1)} U_j^n + M_j^n \mu_t^{(1)} U_j^n + \left(\mu_t^{(1)} |U_j^n|^2 \right) \left(\mu_t^{(1)} U_j^n \right) = 0, \\
\delta_t^{(2)} M_j^n - \delta_x^{(2)} M_j^n - \delta_x^{(2)} |U_j^n|^2 = 0, \\
\text{such that} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), & \forall j \in \overline{I}_J, \\ \delta_t^{(1)} U_j^0 = u_1(x_j) & \delta_t^{(1)} M_j^0 = m_1(x_j), & \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, & \forall n \in \overline{I}_N. \end{cases}
\end{gathered} \tag{1.30}$$

Note that the numerical model (1.30) is a three-step explicit technique. Indeed, note that the first equation of that system yields an expression with complex parameters in which the only unknown is U_j^{n+1}. On the other hand, the second equation of (1.30) is a fully explicit difference equation which can be easily solved for M_j^{n+1}, for each $(j,n) \in I$. Obviously, this explicit character of the method has computational advantages over the implicit schemes [35]

Note that the initial conditions of the numerical model (1.30) requires the knowledge of the fictitious U^{-1} and M^{-1}. In order to eliminate them, we require for the difference equations of (1.30) to hold also for $n = 0$. Using then the initial data, we readily obtain that for each $j \in I_{J-1}$, the following identities hold:

$$\frac{2U_j^1 - 2u_0(x_j) - 2\tau u_1(x_j)}{\tau^2} = \delta_x^{(\alpha)} U_j^0 - (U_j^1 - \tau u_1(x_j)) \left[1 + M_j^0 + \frac{1}{2}\left(|U_j^1|^2 + |U_j^1 - 2\tau u_1(x_j)|^2 \right) \right], \tag{1.31}$$

$$M_j^1 = m_0(x_j) + \tau m_1(x_j) + \frac{\tau^2}{2}\delta_x^{(2)}\left(M_j^0 + |U_j^0|^2 \right). \tag{1.32}$$

In what follows, we will let $(V^n)_{n\in\overline{I}_N}$ be a sequence in $\mathring{\mathcal{V}}_h$ such that

$$\delta_x^{(2)} V_j^n = \delta_t M_j^n, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{1.33}$$

Under these circumstances, (U, M) will denote a solution of (1.30), and $V = (V^n)_{n\in\overline{I}_N}$ will satisfy (1.33).

Definition 1.2.4. Let (U, M) be a solution of (1.30). The *discrete energy density* at the time t_n is given by

$$\begin{aligned} H_j^n = {} & |\delta U_j^n|^2 - U_j^n \delta_x^{(\alpha)} U_j^{n+1} + \mu_t |U_j^n|^2 + \frac{1}{2}|\delta_x V_j^n|^2 \\ & + \frac{1}{2} M_j^{n+1} M_j^n + \frac{1}{2}\mu_t |U_j^n|^4 + \frac{1}{2}\left[M_j^n |U_j^{n+1}|^2 + M_j^{n+1}|U_j^n|^2 \right], \quad \forall j \in I_{J-1}. \end{aligned} \tag{1.34}$$

The *total discrete energy* at the time t_n is defined, for each $n \in \overline{I}_{N-1}$, by

$$\begin{aligned} E^n &= \|\delta_t U^n\|_2^2 + \mathrm{Re}\langle -\delta_x^{(\alpha)} U^{n+1}, U^n \rangle + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x V^n\|_2^2 \\ &\quad + \frac{1}{2}\langle M^{n+1}, M^n\rangle + \frac{1}{2}\mu_t \|U^n\|_4^4 + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2 \rangle \right] \\ &= \|\delta_t U^n\|_2^2 + \mathrm{Re}\langle \Lambda_x^{(\alpha)} U^{n+1}, \Lambda_x^{(\alpha)} U^n \rangle + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x V^n\|_2^2 \\ &\quad + \frac{1}{2}\langle M^{n+1}, M^n\rangle + \frac{1}{2}\mu_t \|U^n\|_4^4 + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2 \rangle \right], \end{aligned} \tag{1.35}$$

Here, we employ the notation $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$, for each $n \in \overline{I}_N$.

1.3 Physical properties

In this section we prove the main physical properties of the finite-difference method, namely, its capability to conserve both the energy and the positive character of the energy functionals.

Definition 1.3.1. Given any $U, V \in \mathring{\mathcal{V}}_h$, we define their product point-wisely, that is, $UV = (U_j V_j)_{j\in\overline{I}_J}$. Moreover, if $F = \mathbb{R}, \mathbb{C}$ and if $U \in \mathring{\mathcal{V}}_h$ is a function which takes on values in F, then $f(U) = (f(U_j))_{j\in\overline{I}_J}$, for any function $f : F \to F$.

Lemma 1.3.2 (Hendy and Macías-Díaz [35]). *The following are satisfied for each $n \in I_{N-1}$,*

(a) $2\,\mathrm{Re}\langle \delta_t^2 U^n, \delta_t^{(1)} U^n\rangle = \delta_t \|\delta_t U^{n-1}\|_2^2$,

(b) $2\,\mathrm{Re}\langle -\delta_x^{(\alpha)} U^n, \delta_t^{(1)} U^n\rangle = \delta_t\,\mathrm{Re}\langle \Lambda_x^{(\alpha)} U^n, \Lambda_x^{(\alpha)} U^{n-1}\rangle$,

(c) $2\,\mathrm{Re}\langle \mu_t^{(1)} U^n, \delta_t^{(1)} U^n\rangle = \delta_t \mu_t \|U^{n-1}\|_2^2$,

(d) $2\,\mathrm{Re}\langle M^n \mu_t^{(1)} U^n, \delta_t^{(1)} U^n\rangle = \langle M^n, \delta_t^{(1)} |U^n|^2\rangle$,

(e) $4\,\mathrm{Re}\langle (\mu_t^{(1)} |U^n|^2)(\mu_t^{(1)} U^n), \delta_t^{(1)} U^n\rangle = \delta_t \mu_t \|U^{n-1}\|_4^4$,

(f) $2\langle M^n, \delta_t^{(1)} |U^n|^2\rangle + 2\langle |U^n|^2, \delta_t^{(1)} M^n\rangle = \delta_t \left[\langle M^{n-1}, |U^n|^2\rangle + \langle M^n, |U^{n-1}|^2\rangle\right]$,

(g) $\mathrm{Re}\langle U^{n+1}, U^n\rangle = \mu_t \|U^n\|_2^2 - \frac{1}{2}\tau^2 \|\delta_t U^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}$,

(h) $|\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2\rangle| \leq \mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4$.

(i) $-2\langle \delta_t^{(2)} M^n, \mu_t V^{n-1}\rangle = \delta_t \|\delta_x V^{n-1}\|_2^2$,

(j) $2\langle \delta_x^{(2)} M^n, \mu_t V^{n-1}\rangle = \delta_t \langle M^n, M^{n-1}\rangle$,

(k) $\langle \delta_x^{(2)} |U^n|^2, \mu_t V^{n-1}\rangle = \langle |U^n|^2, \delta_t^{(1)} M^n\rangle$.

Proof. The identities (a)–(c) can be established following arguments similar to those in [49]. On the other hand, (d) and (e) make use of the product defined in 1.3.1, though the proofs are also straightforward. The identity (f) is also easy. Meanwhile, (g) readily follows after rewriting the left-hand side as a sum of conjugates, adding and subtracting the quantity $\mu_t \|U^n\|_2^2$, and performing some algebraic simplifications, namely,

$$\begin{aligned}\mathrm{Re}\langle U^{n+1}, U^n\rangle =& \frac{1}{2}\left[\langle U^{n+1}, U^n\rangle + \langle U^n, U^{n+1}\rangle\right] + \mu_t \|U^n\|_2^2 - \frac{1}{2}\left[\langle U^{n+1}, U^{n+1}\rangle + \langle U^n, U^n\rangle\right] \\ =& \mu_t \|U^n\|_2^2 - \frac{1}{2}\langle U^{n+1} - U^n, U^{n+1} - U^n\rangle = \mu_t \|U^n\|_2^2 - \frac{1}{2}\tau^2 \|\delta_t U^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}.\end{aligned} \tag{1.36}$$

To prove (h), first note that two applications of Young's inequality yields $|\langle M^n, |U^{n+1}|^2\rangle| \leq \frac{1}{2}\|M^n\|_2^2 + \frac{1}{2}\|U^{n+1}\|_2^2$ and $|\langle M^{n+1}, |U^n|^2\rangle| \leq \frac{1}{2}\|M^{n+1}\|_2^2 + \frac{1}{2}\|U^n\|_2^2$, whence the inequality follows. To establish now (i) and (j), assume that $(V^n)_{n\in\overline{I}_N}$ satisfies condition (1.33), and note that

$$\langle \delta_t^{(2)} M^n, \mu_t V^{n-1}\rangle = \langle \delta_t \delta_x^{(2)} V^{n-1}, \mu_t V^{n-1}\rangle = \langle \delta_x^{(2)} \delta_t V^{n-1}, \mu_t V^{n-1}\rangle = -\frac{1}{2\tau}\left[\|\delta_x V^n\|_2^2 - \|\delta_x V^{n-1}\|_2^2\right], \tag{1.37}$$

and

$$\langle \delta_x^{(2)} M^n, \mu_t V^{n-1}\rangle = \langle M^n, \delta_x^{(2)} \mu_t V^{n-1}\rangle = \langle M^n, \delta_t \mu_t M^n\rangle = \frac{1}{2\tau}\left[\langle M^n, M^{n+1}\rangle - \langle M^n, M^{n-1}\rangle\right]. \tag{1.38}$$

The identity (k) is finally proved in a similar fashion. □

An alternative expression of the energy constants is already available if the identity (g) of Lemma 1.3.2 is applied to the sequences $(\Lambda_x^{(\alpha)} U^n)_{n\in\overline{I}_N}$ and $(M^n)_{n\in\overline{I}_N}$. Indeed, it is easy to check that

$$\begin{aligned}E^n =& \|\delta_t U^n\|_2^2 + \mu_t \|\Lambda_x^{(\alpha)} U^n\|_2^2 - \frac{1}{2}\tau^2 \|\Lambda_x^{(\alpha)} \delta_t U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x V^n\|_2^2 - \frac{1}{4}\tau^2 \|\delta_t M^n\|_2^2 \\ &+ \frac{1}{2}\mu_t \|M^n\|_2^2 + \frac{1}{2}\mu_t \|U^n\|_4^4 + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2\rangle\right], \quad \forall n \in \overline{I}_{N-1}.\end{aligned} \tag{1.39}$$

Theorem 1.3.3 (Energy conservation)**.** *If* (U, M) *is solution of* (1.30) *then the quantities* (1.35) *are constant. Moreover, if* $g_0^{(\alpha)}\tau^2 h^{1-\alpha} < 1$ *then the constants* (1.30) *are nonnegative.*

Proof. Let $n \in I_{N-1}$. Compute the inner product of the left- and the right-hand sides of the first vector equation of (1.30) with $\delta_t^{(1)} U^n$. Take then the real part, multiply by 2 and use the identities (a)–(e) of Lemma 1.3.2 to obtain

$$0 = \delta_t \|\delta_t U^{n-1}\|_2^2 + \delta_t \operatorname{Re}\langle \Lambda_x^{(\alpha)} U^n, \Lambda_x^{(\alpha)} U^{n-1}\rangle + \delta_t \mu_t \|U^{n-1}\|_2^2 + \langle M^n, \delta_t^{(1)} |U^n|^2\rangle + \frac{1}{2}\delta_t \mu_t \|U^{n-1}\|_4^4. \tag{1.40}$$

On the other hand, take the inner product of the left- and the right-hand sides of the second vector equation of (1.30) with $\mu_t V^{n-1}$ and use the last three identities of Lemma 1.3.2 to obtain that

$$\frac{1}{2}\delta_t \|\delta_x V^{n-1}\|_2^2 + \frac{1}{2}\delta_t \langle M^n, M^{n-1}\rangle + \langle |U^n|^2, \delta_t^{(1)} M^n\rangle = 0. \tag{1.41}$$

We add then these last two identities and use Lemma 1.3.2(j) to simplify algebraically the resulting expression, obtaining thus that $\delta_t E^{n-1} = 0$ for each $n \in I_{N-1}$. The first part of this result follows now by induction. To show that the quantities E^n are nonnegative under the condition $g_0^{(\alpha)}\tau^2 h^{1-\alpha} < 1$, note firstly that $\|\Lambda_x^{(\alpha)} \delta_t U^n\|_2^2 \leq 2 g_0^{(\alpha)} h^{1-\alpha} \|\delta_t U^n\|_2^2$ and that $\|\delta_t M^n\|_2^2 = \|\delta_x \delta_x V^n\|_2^2 \leq 2 g_0^{(\alpha)} h^{1-\alpha} \|\delta_x V^n\|_2^2$. Using these inequalities, Lemma 1.3.2(g), the identity (1.39) and simplifying algebraically, we obtain that

$$E^n \geq \left(1 - \frac{g_0^{(\alpha)}\tau^2}{h^{\alpha-1}}\right) \|\delta_t U^n\|_2^2 + \mu_t \|\Lambda_x^{(\alpha)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\left(1 - \frac{g_0^{(\alpha)}\tau^2}{h^{\alpha-1}}\right) \|\delta_x V^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{1.42}$$

We conclude that $E^n \geq 0$ for each $n \in \overline{I}_{N-1}$, as desired. □

1.4 Numerical properties

The aim of this section is to establish the most important numerical properties of the finite-difference model (1.30). More precisely, we will prove that the model is second-order consistent, stable and quadratically convergent.

Let $(U^n)_{n \in I_{N-1}}, (M^n)_{n \in I_{N-1}} \subseteq \mathring{\mathcal{V}}_h$. For convenience, we define $L = L_U \times L_M : \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h \to \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h$ by

$$L_U(U_j^n, M_j^n) = \delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} U_j^n + \mu_t^{(1)} U_j^n + M_j^n \mu_t^{(1)} U_j^n + \left(\mu_t^{(1)} |U_j^n|^2\right)\left(\mu_t^{(1)} U_j^n\right), \quad \forall (j, n) \in I, \tag{1.43}$$

$$L_M(U_j^n, M_j^n) = \delta_t^{(2)} M_j^n - \delta_x^{(2)} M_j^n - \delta_x^{(2)} |U_j^n|^2, \quad \forall (j, n) \in I. \tag{1.44}$$

We define $L(U^n, M^n) = (L(U_j^n, M_j^n))_{j \in \overline{I}_J}$ for each $n \in I_{N-1}$, and let $L(U, M) = (L(U^n, M^n))_{n \in I_{N-1}}$. Let us introduce also the continuous operator $\mathcal{L} = \mathcal{L}_u \times \mathcal{L}_m$, defined for each (u, m) by

$$\mathcal{L}_u(u(x,t), m(x,t)) = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t) u(x,t) + |u(x,t)|^2 u(x,t), \tag{1.45}$$

$$\mathcal{L}_m(u(x,t), m(x,t)) = \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^2 m(x,t)}{\partial x^2} - \frac{\partial^2 \left(|u(x,t)|^2\right)}{\partial x^2}, \tag{1.46}$$

for each $(x, t) \in \Omega$. Also, for each $x \in \{x_L, x_R\}$ and $t \in [0, T]$, we let $\mathcal{L}(u(x,t), m(x,t)) = 0$. Let $\mathcal{L}(u^n, m^n) = (\mathcal{L}(u_j^n, m_j^n))_{j \in \overline{I}_J}$ for each $n \in I_{N-1}$, and define $\mathcal{L}(u, v) = (\mathcal{L}(u^n, m^n))_{n \in I_{N-1}}$.

Similarly, consider $L(u^n, m^n) = (L(u_j^n, m_j^n))_{j\in\overline{I}_J} \in \mathring{\mathcal{V}}_h$ for each $n \in I_{N-1}$, and introduce $L(u, m) = (L(u^n, m^n))_{n\in I_{N-1}}$.

Theorem 1.4.1 (Consistency)**.** *Suppose that $u, m \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$. Then there exists a constant C which is independent of τ and h, such that $|||(\mathcal{L} - L)(u, m)|||_\infty \leq C(\tau^2 + h^2)$ and $|||(\mathcal{H} - H)(u, m)|||_\infty \leq C(\tau + h^2)$.*

Proof. The proof uses the classical argument based on Taylor approximations, the mean value theorem and the smoothness of u. Indeed, note that there exists constants $C_i \in \mathbb{R}^+$ for each $i \in I_5$, such that

$$\left|\frac{\partial^2 u(x_j, t_n)}{\partial t^2} - \delta_t^{(2)} u_j^n\right| \leq C_1\tau^2, \tag{1.47}$$

$$\left|\frac{\partial^\alpha u(x_j, t_n)}{\partial |x|^\alpha} - \delta_x^{(\alpha)} u_j^n\right| \leq C_2 h^2, \tag{1.48}$$

$$\left|u(x_j, t_n) - \mu_t^{(1)} u_j^n\right| \leq C_3\tau^2, \tag{1.49}$$

$$\left|m(x_j, t_n)u(x_j, t_n) - m_j^n \mu_t^{(1)} u_j^n\right| \leq C_4\tau^2, \tag{1.50}$$

$$\left||u(x_j, t_n)|^2 u(x_j, t_n) - \left(\mu_t^{(1)}|u_j^n|\right)\left(\mu_t^{(1)} u_j^n\right)\right| \leq C_5\tau^2, \tag{1.51}$$

for $(j, n) \in \overline{I}_J \times I_{N-1}$. Using the triangle inequality and algebraic simplifications, it follows that $|||(\mathcal{L}_u - L_U)(u, m)|||_\infty \leq C_1'(\tau^2 + h^2)$, where $C_1' = \max\{C_i : i \in I_5\}$. Similarly, it is readily verified that there exist $C_6, C_7, C_8 \in \mathbb{R}^+$ such that

$$\left|\frac{\partial^2 m(x_j, t_n)}{\partial t^2} - \delta_t^{(2)} m_j^n\right| \leq C_6\tau^2, \tag{1.52}$$

$$\left|\frac{\partial^2 m(x, t)}{\partial x^2} - \delta_x^{(2)} m_j^n\right| \leq C_7 h^2, \tag{1.53}$$

$$\left|\frac{\partial^2 \left(|u(x, t)|^2\right)}{\partial x^2} - \delta_x^{(2)} |u_j^n|^2\right| \leq C_8 h^2, \tag{1.54}$$

for $(j, n) \in \overline{I}_J \times I_{N-1}$. Again, using the triangle inequality, it is easy to see that $|||(\mathcal{L}_m - L_M)(u, m)|||_\infty \leq C_2'(\tau^2 + h^2)$, with $C_2' = \max\{C_6, C_7, C_8\}$. The first inequality of the theorem is reached when we let $C = \max\{C_1', C_2'\}$. The second is obtained analogously. □

Definition 1.4.2. Let $\sigma \in [0, 1]$. Define the *fractional Sobolev norm* and *semi-norm* $\|\cdot\|_{H^\sigma}, |\cdot|_{H^\sigma} : \mathring{\mathcal{V}}_h \to \mathbb{R}$ by

$$\|U\|^2_{H^\sigma} = \int_{-\pi/h}^{\pi/h} (1 + |k|^{2\sigma})|\widehat{U}(k)|^2 dk, \quad \forall U \in \mathring{\mathcal{V}}_h, \tag{1.55}$$

$$|U|^2_{H^\sigma} = \int_{-\pi/h}^{\pi/h} |k|^{2\sigma}|\widehat{U}(k)|^2 dk, \quad \forall U \in \mathring{\mathcal{V}}_h, \tag{1.56}$$

respectively. Alternatively, $\|U\|^2_{H^\sigma} = \|U\|_2^2 + |U|^2_{H^\sigma}$ and $|U|^2_{H^0} = \|U\|_2^2$, for each $U \in \mathring{\mathcal{V}}_h$. The *fractional Sobolev space* H^σ is the set of all $U \in \mathring{\mathcal{V}}_h$ such that $\|U\|_{H^\sigma} < \infty$.

The following lemmas will be employed to establish the boundedness, the stability and the convergence of (1.30). For the sake of convenience, we will let $\gamma = \tau/h$, and we will assume that θ is a number in $[0, \frac{1}{2}]$.

Lemma 1.4.3. *Let $\gamma\sqrt{1-2\theta}<1$ and $\beta=(1+\gamma^2(1-2\theta))/(1-\gamma^2(1-2\theta))>1$. If $(U^n)_{n=0}^N\subseteq\mathring{\mathcal{V}}_h$ is a sequence of complex functions then $R^n\leq\beta Q^n$, for each $n\in\overline{I}_{N-1}$. Here, for each $n\in\overline{I}_{N-1}$,*

$$Q^n=\|\delta_t U^n\|_2^2+(1-2\theta)\operatorname{Re}\langle\Lambda_x^{(\alpha)}U^n,\Lambda_x^{(\alpha)}U^{n+1}\rangle, \tag{1.57}$$

$$R^n=\|\delta_t U^n\|_2^2+\tfrac{1}{2}(1-2\theta)\left(\|\Lambda_x^{(\alpha)}U^{n+1}\|_2^2+\|\Lambda_x^{(\alpha)}U^n\|_2^2\right). \tag{1.58}$$

Proof. Note beforehand that $\beta>1$. Using this fact, it is easy to check that

$$\begin{aligned}Q^n=&(1-2\theta)\left[\langle\operatorname{Re}(\Lambda_x^{(\alpha)}U^n),\operatorname{Re}(\Lambda_x^{(\alpha)}U^{n+1})\rangle+\langle\operatorname{Im}(\Lambda_x^{(\alpha)}U^n),\operatorname{Im}(\Lambda_x^{(\alpha)}U^{n+1})\rangle\right]\\&+\|\operatorname{Re}(\delta_t U^n)\|_2^2+\|\operatorname{Im}(\delta_t U^n)\|_2^2\\\geq&\beta^{-1}\left[\tfrac{1}{2}(1-2\theta)\left(\|\operatorname{Re}(\Lambda_x^{(\alpha)}U^n)\|_2^2+\|\operatorname{Im}(\Lambda_x^{(\alpha)}U^n)\|_2^2\right)+\|\operatorname{Re}(\delta_t U^n)\|_2^2\right.\\&\left.+\|\operatorname{Im}(\delta_t U^n)\|_2^2+\|\operatorname{Re}(\Lambda_x^{(\alpha)}U^{n+1})\|_2^2+\|\operatorname{Im}(\Lambda_x^{(\alpha)}U^{n+1})\|_2^2\right]=\beta^{-1}R^n,\end{aligned} \tag{1.59}$$

for each $n\in\overline{I}_{N-1}$, which is what we wanted to prove. □

Lemma 1.4.4 (Chang *et al.* [15, 16])**.** *Suppose that $\gamma\sqrt{1-2\theta}<1$ and $\beta=(1+\gamma^2(1-2\theta))/(1-\gamma^2(1-2\theta))>1$. If $(M^n)_{n\in\overline{I}_N},(V^n)_{n\in\overline{I}_N}\subseteq\mathring{\mathcal{V}}_h$ are real sequences then $\widetilde{R}^n\leq\beta\widetilde{Q}^n$, for each $n\in\overline{I}_{N-1}$. Here,*

$$\widetilde{Q}^n=\|\delta_t V^n\|_2^2+(1-2\theta)\langle M^n,M^{n+1}\rangle,\quad\forall n\in\overline{I}_{N-1}, \tag{1.60}$$

$$\widetilde{R}^n=\|\delta_t V^n\|_2^2+\tfrac{1}{2}(1-2\theta)\left(\|M^{n+1}\|_2^2+\|M^n\|_2^2\right),\quad\forall n\in\overline{I}_{N-1}. \tag{1.61}$$

Lemma 1.4.5 (Wang *et al.* [94])**.**

(a) *For each $\frac{1}{2}<\sigma\leq 1$, there exists $C_\sigma>0$ independent of h such that if $U\in H^\sigma$ then $\|U\|_\infty\leq C_\sigma\|U\|_{H^\sigma}$.*

(b) *For each $\frac{1}{4}<\sigma_0\leq 1$, there is $C_{\sigma_0}>0$ independent of h such that $\|U\|_\infty\leq C_{\sigma_0}\|U\|_{H^\sigma}^{\frac{\sigma_0}{\sigma}}\|U\|_2^{1-\frac{\sigma_0}{\sigma}}$ if $\sigma_0\leq\sigma\leq 1$ and $U\in H^\sigma$.*

(c) *For each $\alpha\in(1,2)$ there exists a constant $C>0$ such that $C|U|_{H^{\alpha/2}}^2\leq|\langle\delta_x^{(\alpha)}U,U\rangle|\leq|U|_{H^{\alpha/2}}^2$, for each $U\in H^\alpha$.*

(d) *For each $\alpha\in(1,2)$ and $U,V\in H^\alpha$, there exists $C>0$ such that $C|U|_{H^{\alpha/2}}|V|_{H^{\alpha/2}}\leq|\langle\delta_x^{(\alpha)}U,V\rangle|\leq|U|_{H^{\alpha/2}}|V|_{H^{\alpha/2}}$.*

Lemma 1.4.6 (Gronwall's inequality [105])**.** *Assume that $N\in\mathbb{N}$ with $N>1$. Let $(\omega^n)_{n\in\overline{I}_N}$ and $(C_n)_{n\in I_N}$ be sequences of nonnegative numbers, and let A, B and C_n be nonnegative, for each $n\in I_N$. Suppose that $\tau\in\mathbb{R}^+$, and that*

$$\omega^n-\omega^{n-1}\leq A\tau\omega^n+B\tau\omega^{n-1}+C_n\tau,\quad\forall n\in I_N. \tag{1.62}$$

If $(A+B)\tau\leq(N-1)/(2N)$ then

$$\max_{n\in I_N}|\omega^n|\leq\left(\omega^0+\tau\sum_{k\in I_N}C_k\right)e^{2(A+B)N\tau}. \tag{1.63}$$

Theorem 1.4.7 (Boundedness)**.** *Let $u_0,m_0\in H^1$ and $u_1,m_1\in L^2(\overline{B})$, and suppose that (U,M) is the solution of (1.30) corresponding to the initial data u_0, m_0, u_1 and m_1. If $g_0^{(\alpha)}\tau^2h^{1-\alpha}<1$ and $\gamma<1$ then there is a common bound $C\in\mathbb{R}^+$ for $(\|\delta_t U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|\Lambda_x^{(\alpha)}U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|U^n\|_\infty)_{n\in\overline{I}_{N-1}}$, $(\|\delta_x V^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|M^n\|_2)_{n\in\overline{I}_{N-1}}$ and $(\|U^n\|_4)_{n\in\overline{I}_{N-1}}$.*

Proof. Let $n \in \overline{I}_{N-1}$. The quantities (1.35) are equal to some $C_0 \geq 0$ by Theorem 1.3.3. Young's inequality yield

$$|\langle M^n, |U^{n+1}|^2\rangle| \leq \|M^n\|_2^2 + \tfrac{1}{4}\|U^{n+1}\|_4^4, \tag{1.64}$$
$$|\langle M^{n+1}, |U^n|^2\rangle| \leq \tfrac{1}{4\beta}\|M^{n+1}\|_2^2 + \beta\|U^n\|_4^4, \tag{1.65}$$

for each $n \in \overline{I}_{N-1}$. Let $C_0^n \geq 0$ be a common bound of $\|M^n\|_2^2$ and $\|U^n\|_4^4$. Using (1.35), letting $C_1^n = C_0 + \frac{1}{2}(\beta+1)C_0^n$ and using Lemma 1.4.4 with $\theta = 0$,

$$\begin{aligned} C_1^n \geq & \tfrac{1}{\beta}\left[\|\delta_t U^n\|_2^2 + \mu_t\|\Lambda_x^{(\alpha)}U^n\|_2^2\right] + \mu_t\|U^n\|_2^2 + \tfrac{1}{4}\mu_t\|U^n\|_4^4 \\ & - \tfrac{1}{8\beta}\|M^{n+1}\|_2^2 + \tfrac{1}{2}\left[\|\delta_x V^n\|_2^2 + \langle M^n, M^{n+1}\rangle\right] \\ \geq & \tfrac{1}{\beta}\left[\|\delta_t U^n\|_2^2 + \mu_t\|\Lambda_x^{(\alpha)}U^n\|_2^2\right] + \mu_t\|U^n\|_2^2 + \tfrac{1}{4}\mu_t\|U^n\|_4^4 \\ & + \tfrac{1}{2\beta}\left[\|\delta_x V^n\|_2^2 + \tfrac{1}{2}\mu_t\|M^n\|_2^2\right], \end{aligned} \tag{1.66}$$

for each $n \in \overline{I}_{N-1}$. As a consequence, there exists $C_2^n \geq 0$ such that

$$\|\delta_t U^n\|_2^2, \|\Lambda_x^{(\alpha)}U^{n+1}\|_2^2, \|U^{n+1}\|_2^2, \|\delta_x V^n\|_2^2, \|M^{n+1}\|_2^2, \|U^{n+1}\|_4^4 \leq C_2^n, \tag{1.67}$$

for each $n \in \overline{I}_{N-1}$. Moreover, using an argument similar to that in [35], there exists $C_3^n \geq 0$ such that $\|U^{n+1}\|_\infty \leq C_3^n$. Note now that the hypotheses assure that there exists a common bound $C_0^0 \geq 0$ for $\|\delta_t U^{-1}\|_2^2$, $\|\Lambda_x^{(\alpha)}U^0\|_2^2$, $\|U^0\|_2^2$, $\|\delta_x V^{-1}\|_2^2$, $\|M^0\|_2^2$, $\|U^0\|_4^4$ and $\|U^0\|_2^2$. Let $n \in \overline{I}_{N-1}$ and suppose that

$$\|\delta_t U^{n-1}\|_2^2, \|\Lambda_x^{(\alpha)}U^n\|_2^2, \|U^n\|_2^2, \|\delta_x V^{n-1}\|_2^2, \|M^n\|_2^2, \|U^n\|_4^4, \|U^n\|_2^2 \leq C_0^n, \tag{1.68}$$

for some $C_0^n \geq 0$. It follows that

$$\|\delta_t U^n\|_2^2, \|\Lambda_x^{(\alpha)}U^{n+1}\|_2^2, \|U^{n+1}\|_2^2, \|\delta_x V^n\|_2^2, \|M^{n+1}\|_2^2, \|U^{n+1}\|_4^4, \|U^{n+1}\|_2^2 \leq C_0^{n+1}, \tag{1.69}$$

where $C_0^{n+1} = \max\{C_2^n, C_3^n\}$. The proof follows by induction using $C = C_0^N$. □

Next, we wish to prove the stability and convergence properties of (1.4). In the following, (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ will represent two sets of initial conditions of (1.4), and we will assume that the initial data for (1.30) are provided exactly. Moreover, if $f : F \to F$ and $V \in \mathring{\mathcal{V}}_h$ then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$ and $F = \mathbb{R}, \mathbb{C}$. In the next result, we will use the identities of Lemma 1.3.2 without mentioning them explicitly. Some of the arguments will be similar to those in the proof of Theorem 1.3.3.

Theorem 1.4.8 (Stability). *The method* (1.30) *is stable under the hypotheses of Theorem 1.4.7.*

Proof. Let (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ be sets of initial data of (1.4), and observe that the assumptions of Theorem 1.4.7 hold for both (U, V) and $(\widetilde{U}, \widetilde{V})$. On the other hand, note that (ε, ζ) satisfies the following with $(j, n) \in I$:

$$\begin{gathered} \delta_t^{(2)}\varepsilon_j^n - \delta_x^{(\alpha)}\varepsilon_j^n + \mu_t^{(1)}\varepsilon_j^n + \widetilde{\delta}\left[M_j^n\left(\mu_t^{(1)}U_j^n\right) + \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0, \\ \delta_t^{(2)}\xi_j^n - \delta_x^{(2)}\mu_t^{(1)}\xi_j^n - \widetilde{\delta}\left(\delta_x^{(2)}|U_j^n|^2\right) = 0, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \xi_0^n = \xi_J^n = 0, \quad \forall n \in \overline{I}_N. \end{gathered} \tag{1.70}$$

Using the discrete inequalities for Sobolev spaces, there exists $C \in \mathbb{R}^+$ with the property that, for all $n \in I_{N-1}$:

$$\left|\left\langle \widetilde{\delta}\left[M^n\left(\mu_t^{(1)}U^n\right)\right], \delta_t^{(1)}\varepsilon^n\right\rangle\right| \leq C\left(\mu_t^{(1)}\|\xi^n\|_2^2 + \mu_t^{(1)}\|\Lambda_x^{(\alpha)}\varepsilon^n\|_2^2 + \mu_t^{(1)}\|\varepsilon^n\|_2^2 + \mu_t\|\delta_t\varepsilon^{n-1}\|_2^2\right), \tag{1.71}$$

τ	h	$T = 5$		$T = 10$	
		$\epsilon_{\tau,h}$	$\rho^x_{\tau,h}$	$\epsilon_{\tau,h}$	$\rho^x_{\tau,h}$
0.04	1×2^{-1}	3.1259×10^{-2}	–	3.3702×10^{-2}	–
	1×2^{-2}	9.3107×10^{-3}	1.7473	1.0490×10^{-2}	1.6837
	1×2^{-3}	2.7017×10^{-3}	1.7850	3.1681×10^{-3}	1.7274
	1×2^{-4}	7.5927×10^{-4}	1.8312	9.0755×10^{-4}	1.8036
	1×2^{-5}	2.0884×10^{-4}	1.8622	2.5309×10^{-4}	1.8423
0.02	1×2^{-1}	8.6470×10^{-3}	–	9.4516×10^{-3}	–
	1×2^{-2}	2.4595×10^{-3}	1.8138	2.7793×10^{-3}	1.7658
	1×2^{-3}	6.6965×10^{-4}	1.8769	7.8130×10^{-4}	1.8308
	1×2^{-4}	1.7416×10^{-4}	1.9430	2.0982×10^{-4}	1.8967
	1×2^{-5}	4.4584×10^{-5}	1.9658	5.5110×10^{-5}	1.9288
0.01	1×2^{-1}	2.3217×10^{-3}	–	2.5746×10^{-3}	–
	1×2^{-2}	6.3033×10^{-4}	1.8810	7.0669×10^{-4}	1.8652
	1×2^{-3}	1.6345×10^{-4}	1.9472	1.8529×10^{-4}	1.9313
	1×2^{-4}	4.1435×10^{-5}	1.9800	4.7430×10^{-5}	1.9659
	1×2^{-5}	1.0557×10^{-5}	1.9726	1.2104×10^{-5}	1.9703

Table 1.1: Table of absolute errors and standard convergence rates in space when approximating the solution m of (1.4) with $\alpha = 2$, using the method (1.30). We employed the spatial domain $B = (-20, 20)$ and two periods of time, namely, $T = 5$ and $T = 10$. The initial conditions were prescribed by the functions (1.76)–(1.79). Various sets of computational parameters were employed.

$$\left|\left\langle \widetilde{\delta}\left[\left(\mu_t^{(1)}|U^n|^2\right)\left(\mu_t^{(1)}U^n\right)\right], \delta_t^{(1)}\varepsilon^n\right\rangle\right| \leq C\left(\mu_t^{(1)}\|\varepsilon^n\|_2^2 + \mu_t\|\delta_t\varepsilon^{n-1}\|_2^2\right), \tag{1.72}$$

and

$$\left|\left\langle \widetilde{\delta}\left(\delta_x^{(2)}|U_j^n|^2\right), \mu_t\zeta^{n-1}\right\rangle\right| \leq C\left(\mu_t\|\delta_x\xi^{n-1}\|_2^2 + \|\Lambda_x^{(\alpha)}\varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2\right). \tag{1.73}$$

These inequalities are obtained in the same way that they were calculated in reference [35]. The details are omitted to avoid redundancy. It suffices to mention that the only difference in the proofs lies in the calculation of the terms $\mathrm{Re}\langle\Lambda_x^{(\alpha)}\varepsilon^n, \Lambda_x^{(\alpha}\varepsilon^{n+1}\rangle_x$ and $\langle\xi^n, \xi^{n+1}\rangle_x$, which are performed using Lemmas 1.4.3 and 1.4.4, respectively. Let $\delta_x^{(2)}\zeta^n = \delta_t\xi^n$, for each $n \in \overline{I}_{N-1}$. Multiply now the first difference equation of (1.70) by $2\delta_t^{(1)}\varepsilon^n$, take the real part and then the absolute value, use the inequalities (1.71)–(1.72) and simplify algebraically. At the same time, multiply the second difference equation of (1.70) by $2\mu_t\zeta^{n-1}$, take the absolute value, use the inequality (1.73) and simplify algebraically. The argument to obtain the conclusion uses then the discrete Gronwall's inequality. □

The argument of the proof of our next proposition is similar to that of stability.

Theorem 1.4.9 (Convergence)**.** *If $u, m \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$ solves (1.4) then the solution of (1.30) converges to that of the continuous problem with order $\mathcal{O}(\tau^2 + h^2)$ in L^∞ for $(U^n)_{n\in\overline{I}_N}$, and in L^2 for $(M^n)_{n\in\overline{I}_N}$.* □

1.5 Simulations

The purpose of this section is to provide some examples to illustrate the performance of the numerical model (1.30). The simulations were obtained using an implementation of our method in ©Matlab 8.5.0.197613 (R2015a) on a ©Hewlett-Packard 6005 Pro Microtower desktop computer

h	τ	$T=5$		$T=10$	
		$\epsilon_{\tau,h}$	$\rho^t_{\tau,h}$	$\epsilon_{\tau,h}$	$\rho^t_{\tau,h}$
0.04	0.01×2^{-1}	2.7837×10^{-5}	–	2.9019×10^{-5}	–
	0.01×2^{-2}	7.7555×10^{-6}	1.8437	8.1926×10^{-6}	1.8246
	0.01×2^{-3}	2.1145×10^{-6}	1.8749	2.2669×10^{-6}	1.8536
	0.01×2^{-4}	5.6512×10^{-7}	1.9037	6.1158×10^{-7}	1.8901
	0.01×2^{-5}	1.4921×10^{-7}	1.9212	1.6078×10^{-7}	1.9274
0.02	0.01×2^{-1}	7.4758×10^{-6}	–	8.0692×10^{-6}	–
	0.01×2^{-2}	1.9994×10^{-6}	1.9026	2.1747×10^{-6}	1.8916
	0.01×2^{-3}	5.1832×10^{-7}	1.9477	5.7915×10^{-7}	1.9088
	0.01×2^{-4}	1.3275×10^{-7}	1.9651	1.4997×10^{-6}	1.9492
	0.01×2^{-5}	3.3568×10^{-8}	1.9836	3.8139×10^{-8}	1.9754
0.01	0.01×2^{-1}	1.9615×10^{-6}	–	2.1985×10^{-6}	–
	0.01×2^{-2}	5.1037×10^{-7}	1.9424	5.7903×10^{-7}	1.9248
	0.01×2^{-3}	1.2875×10^{-7}	1.9869	1.4963×10^{-7}	1.9522
	0.01×2^{-4}	3.2084×10^{-8}	2.0047	3.8187×10^{-8}	1.9703
	0.01×2^{-5}	8.0506×10^{-9}	1.9947	9.6558×10^{-9}	1.9836

Table 1.2: Table of absolute errors and standard convergence rates in time when approximating the solution m of (1.4) with $\alpha = 2$, using the method (1.30). We employed the spatial domain $B = (-20, 20)$ and two periods of time, namely, $T = 5$ and $T = 10$. The initial conditions were prescribed by the functions (1.76)–(1.79). Various sets of computational parameters were employed.

with Linux Mint 18 "Sylvia" Cinnamon edition. The purpose of the next example is to verify the rate of convergence of (1.30). To that end, we will consider the absolute error at the time T between the exact solution u of (1.4) and the corresponding approximations U, which is given by

$$\epsilon_{\tau,h} = |||u - U|||_\infty, \tag{1.74}$$

and consider the standard rates

$$\rho^t_{\tau,h} = \log_2\left(\frac{\epsilon_{2\tau,h}}{\epsilon_{\tau,h}}\right), \qquad \rho^x_{\tau,h} = \log_2\left(\frac{\epsilon_{\tau,2h}}{\epsilon_{\tau,h}}\right). \tag{1.75}$$

In our first example, we provide an analysis of convergence of the finite-difference method (1.30) using the exact solution (1.5)–(1.6). It is important to point out that the literature lacks reports of exact solutions for the fully fractional form of (1.30). For that reason, the present example considers the non-fractional form of the mathematical model. To that end, we will consider a bounded open interval B and impose homogeneous Dirichlet boundary data, as required by the problem (1.4). Clearly, the exact solution does not satisfy the boundary conditions of problem (1.4), which means that the numerical comparisons will yield only upper bounds for the actual error committed by our numerical methodology. Unfortunately, these are some of the inherent shortcomings of not having at hand an exact analytical solution satisfying the fully fractional problem (1.4).

Example 1.5.1. Consider (1.4) with $\alpha = 2$, let $B = (-200, 200)$ and define the functions u_0, m_0, u_1 and m_1 by

$$u_0(x) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \tag{1.76}$$

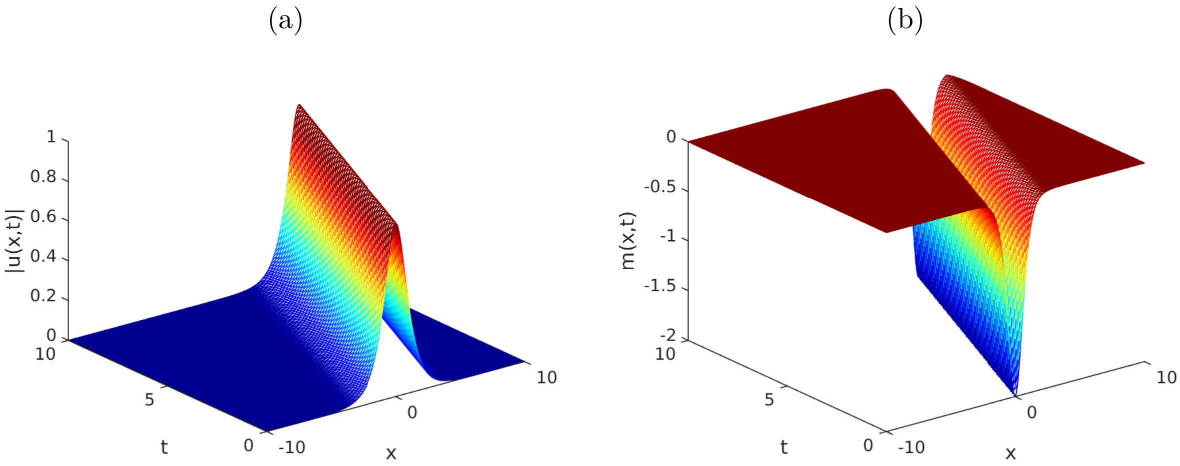

Figure 1.1: Approximate solution of the problem (1.4) versus x and t, using $\Omega = (-20, 20) \times (0, 10)$ and $\alpha = 2$. The graphs correspond to (a) $|u(x,t)|$ and (b) $m(x,t)$, and they were obtained using the initial data (1.76)–(1.79). Computationally, we used $h = 0.05$ and $\tau = 0.1$.

$$m_0(x) = -2\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \tag{1.77}$$

$$u_1(x) = \frac{\sqrt{10}-\sqrt{2}}{2}(\tanh x - 1)\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \tag{1.78}$$

$$m_1(x) = -4\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \tag{1.79}$$

where $i^2 = -1$ and $x \in B$. The corresponding solution of the initial-value problem (1.4) is given by the functions (1.5)–(1.6). For comparison purposes, consider the discrete model (1.30) with initial data (1.76)–(1.79). Under these circumstances, Tables 1.1 and 1.2 provide a numerical study of the convergence of the method. The results confirm the quadratic order of convergence of the scheme (1.30), agreeing with Theorem 1.4.9. For illustration purposes, Figure 1.1 provides graphs of the approximate solutions for $|u|$ and m as functions of x and t. □

In our second example, we will consider the fully fractional form of (1.4).

Example 1.5.2. Consider now $B = (-200, 200)$, along with the initial conditions used in Example 1.5.1, and set $T = 200$.

(a) Figure 1.2 shows a graph of the dynamics of the total energy for various values of $\alpha \in (1, 2]$, using $h = 0.05$ and $\tau = 0.1$. The results show that the total energy of the system is approximately conserved with respect to time, but that it depends on the value of α. This remark is in agreement with the theoretical results of this work.

(b) Next, we investigate the numerical accuracy of the finite-difference method (1.4) for different values of α. To that end, we will follow the methodology and notation used by [98]. More precisely, let $\Phi(h,\tau)$ and $\Psi(h,\tau)$ be, respectively, the numerical solutions U and M at the final time T, obtained using the method (1.30) with the parameters h and τ. We will employ also the notations

$$R^\alpha_\Phi(\tau) = \|\Phi(h,\tau) - \Phi(h,\tau/2)\|_\infty, \qquad F^\alpha_\Phi(h) = \|\Phi(h,\tau) - \Phi(h/2,\tau)\|_\infty, \tag{1.80}$$

$$R^\alpha_\Psi(\tau) = \|\Psi(h,\tau) - \Psi(h,\tau/2)\|_2, \qquad F_\Psi(h)^\alpha = \|\Psi(h,\tau) - \Psi(h/2,\tau)\|_2. \tag{1.81}$$

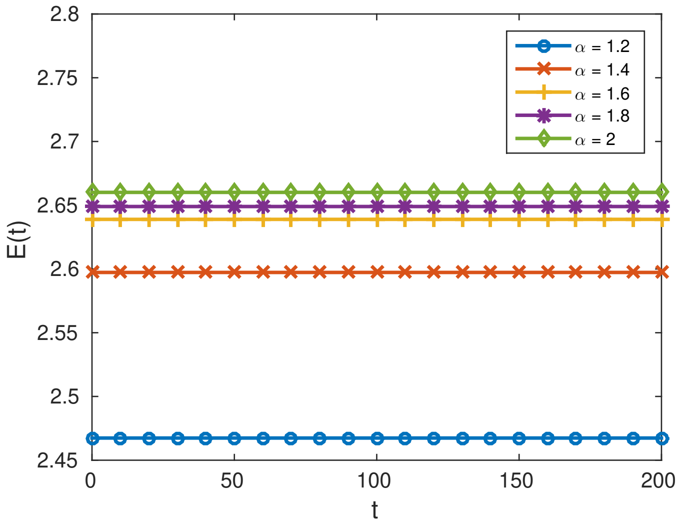

Figure 1.2: Dynamics of the total energy of the approximate solution of system (1.4), using $\Omega = (-200, 200) \times (0, 200)$, and the values of α indicated in the legend. We employed the initial data (1.76)–(1.79). Computationally, we used $h = 0.05$ and $\tau = 0.1$.

With this notation, Figures 1.3(a) and (b) show the graphs of R_Φ and F_Φ as functions of τ and h, respectively. The graphs are plotted in log-log scale, fixing $h = 0.2$ and $\tau = 0.01$, respectively. The results indicate that the scheme is convergent, with second-order accuracy in both time and space. The values of $\alpha = 1.2$, 1.4, 1.6 and 1.8 were employed. In turn, Figures 1.3(c) and (d) provide the graphs of R_Ψ and F_Ψ as functions of τ and h, respectively. The results again confirm the quadratic order of temporal and spatial convergence of the numerical scheme (1.30). □

In this work, we investigated the numerical solution of a fractional extension of the Klein–Gordon–Zakharov equations from plasma physics. The model considers the presence of space-fractional derivatives of the Riesz type, together with homogeneous Dirichlet data at the boundary and initial conditions. The fractional model has an invariant energy functional, and we propose an explicit numerical model to approximate the solutions using fractional-order centered differences. A discrete energy functional is also proposed in this work and we prove rigorously that, as its continuous counterpart, it is preserved at each iteration and, in that sense, the present work reports on a conservative finite-difference scheme to approximate the solutions of hyperbolic systems [36, 42, 38]. Among the most important numerical properties established in this work, we show that the model is a consistent, stable and convergent technique. Additionally, we propose some bounds for the numerical solutions, and provide some computer simulations which illustrate the fact that the numerical model is quadratically convergent.

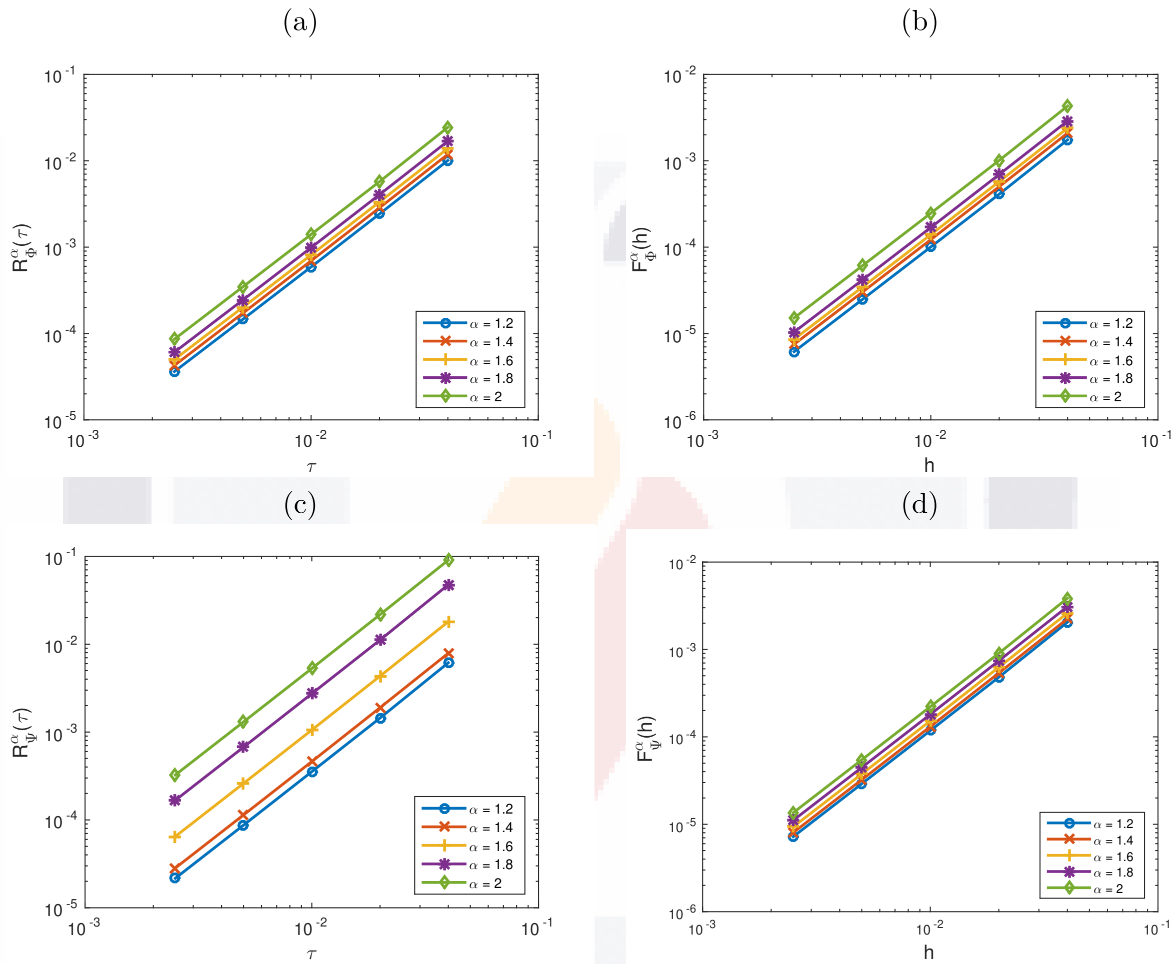

Figure 1.3: Errors committed when approximating the exact solution of (1.4) using the finite-difference method (1.30), using $B = (-200, 200)$ and $T = 200$. We used the initial approximations (1.76)–(1.79), and we fixed $h = 0.2$ for the results of the left column, and $\tau = 0.01$ for the right column. Various values of α were used (see the legends). The error quantities used in these experiments are provided by equations (1.80) and (1.81).

2. Existence of solutions for an implicit method

2.1 Introduction

In the published article [35], the authors established a theorem on the existence of solutions of a finite-difference model for a space-fractional Klein–Gordon–Zakharov (KGZ) equation [103, 86]. The proof made use of the well-known Leray–Schauder fixed-point theorem. However, the authors committed an involuntary mistake in the proof, for which they sincerely apologize. Concretely, the mistakes were committed in Equations (5.6) and (5.7) of that work. Indeed, the second and third terms of the right-hand sides of those equations had to be multiplied by $\lambda \in [0, 1]$, and they were not. It turns out that the corrected proof becomes much harder, and we provide it in this short note. To that end, let us recall that the mathematical model under investigation in [35] is the space-fractional KGZ system

$$
\begin{gathered}
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \qquad \forall (x,t) \in \Omega, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^2 m(x,t)}{\partial x^2} = \frac{\partial^2 \left(|u(x,t)|^2\right)}{\partial x^2}, \qquad \forall (x,t) \in \Omega, \\
\text{subject to} \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), \qquad \forall x \in \overline{B}, \\ \dfrac{\partial u(x,0)}{\partial t} = u_1(x), & \dfrac{\partial m(x,0)}{\partial t} = m_1(x), \qquad \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, \quad \forall t \in [0,T]. \end{cases}
\end{gathered}
\tag{2.1}
$$

In model (2.1), the fractional derivatives are understood in the Riesz sense [76, 47, 50]. Moreover, the authors of [35] proposed the following finite-difference model to solve the continuous problem:

$$
\begin{gathered}
\delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n + \left(\mu_t^{(1)} M_j^n\right)\left(\mu_t^{(1)} U_j^n\right) + \left(\mu_t^{(1)} |U_j^n|^2\right)\left(\mu_t^{(1)} U_j^n\right) = 0, \qquad \forall (j,n) \in I, \\
\delta_t^{(2)} M_j^n - \delta_x^{(2)} \mu_t^{(1)} M_j^n = \delta_x^{(2)} \mu_t^{(1)} |U_j^n|^2, \qquad \forall (j,n) \in I, \\
\text{subject to} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), \qquad \forall j \in \overline{I}_J, \\ \delta_t U_j^0 = u_1(x_j) & \delta_t M_j^0 = m_1(x_j), \quad \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, \qquad \forall n \in \overline{I}_N. \end{cases}
\end{gathered}
\tag{2.2}
$$

For the definitions of the discrete notations, we refer to that article. In the following stage, we will provide the correct proof of Theorem 5.3 of that manuscript. Additionally, we will consider an explicit and inequivalent reformulation of the finite-difference model (2.2). In agreement with the purpose of the present study, Section 2.3 provides a theorem on the existence of solutions for the numerical model proposed therein.

2.2 Corrigendum

Throughout this section, we will employ extensively the discrete nomenclature introduced in [35, Section 3]. In the way, we will also employ Lemma 4.1 of that paper. For the sake of convenience, recall that

$$\mu_{t,\Upsilon}^{(1)}V^n = \frac{\Upsilon + V^{n-1}}{2}, \qquad \text{for any } \Upsilon \in \mathring{\mathcal{V}}_h \text{ and } V = U, M. \tag{2.3}$$

Lemma 2.2.1 (Young's inequality). *Let $a, b \in \mathbb{R}^+ \cup \{0\}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For each $\epsilon > 0$, the following inequality holds:*

$$ab \leq \frac{|a|^p}{p\epsilon} + \frac{\epsilon |b|^q}{q}. \tag{2.4}$$

In the proof of the next lemma, we will occasionally use the following elementary facts:

i. If $V, W \in \mathring{\mathcal{V}}_h$ then $\|V \pm W\|_2^2 \leq 2\|V\|_2^2 + 2\|W\|_2^2$.

ii. It is also well known that $\|\delta_x W\|_2^2 \leq \frac{4}{h}\|W\|_2^2$ for each $\mathring{\mathcal{V}}_h$ (see [49]).

Lemma 2.2.2. *Let $U = (U^n)_{n\in\bar{I}_N}$ and $M = (M^n)_{n\in\bar{I}_N}$ be sequences in $\mathring{\mathcal{V}}_h$, and assume that U is a sequence of complex functions while the functions of M are real. The following identities are satisfied, for each $n \in I_{N-1}$ and $\Phi, \Psi \in \mathring{\mathcal{V}}_h$:*

(a) $4\,\mathrm{Re}\left\langle (\mu_{t,\Phi}^{(1)}|U^n|^2)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\right\rangle = \|\Phi\|_4^4 - \|U^{n-1}\|_4^4$.

(b) $4\,\mathrm{Re}\left\langle (\mu_{t,\Psi}^{(1)}M^n)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\right\rangle = \langle \Psi + M^{n-1}, |\Phi|^2 - |U^{n-1}|^2\rangle$.

(c) $2\,\mathrm{Re}\left\langle \mu_{t,\Phi}^{(1)}U^n, \Phi - U^{n-1}\right\rangle = \|\Phi\|_2^2 - \|U^{n-1}\|_2^2$.

(d) $2\,\mathrm{Re}\left\langle -\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)}U^n, \Phi - U^{n-1}\right\rangle = \|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2$.

(e) $2\left\langle -\delta_x^{(2)}\mu_{t,\Psi}^{(1)}M^n, \Psi - M^{n-1}\right\rangle = \|\delta_x\Psi\|_2^2 - \|\delta_x M^{n-1}\|_2^2$.

Additionally, the following inequalities are satisfied for each $\lambda \in [0, 1]$:

(f) $\mathrm{Re}\langle -2\lambda U^n, \Phi - U^{n-1}\rangle \geq -\frac{\lambda}{3}\|\Phi\|_2^2 - C_1$.

(g) $\mathrm{Re}\langle \Phi + \lambda U^{n-1}, \Phi - U^{n-1}\rangle \geq \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 - C_2$.

(h) $\langle \Psi - 2\lambda M^n + \lambda M^{n-1}, \Psi - M^{n-1}\rangle \geq \left[1 - \frac{1}{20}(3\lambda + 1)\right]\|\Psi\|_2^2 - C_3$.

(i) $\lambda\tau^2\langle -\delta_x^{(2)}\mu_{t,\Phi}^{(1)}|U^n|^2, \Psi - M^{n-1}\rangle \geq -\frac{2\lambda\tau^2}{\epsilon_1 h}\|\Phi\|_4^4 - \frac{2\lambda\tau^2\epsilon_1}{h}\|\Psi\|_2^2 - C_4$, *for each $\epsilon_1 > 0$.*

(j) $\lambda\tau^2\,\mathrm{Re}\langle (\mu_{t,\Psi}^{(1)}M^n)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\rangle \geq -\frac{\lambda\tau^2}{8}\left(1 + \frac{1}{\epsilon_2}\right)\|\Psi\|_2^2 - \frac{\lambda\tau^2}{8}\left(\epsilon_2 + \frac{1}{2}\right)\|\Phi\|_4^4 - C_5$, *for each $\epsilon_2 > 0$.*

Here, the constants $C_1, \ldots, C_5 \in \mathbb{R}^+$ depend only on U^n, U^{n-1}, M^n and M^{n-1}. Additionally, C_4 depends on ϵ_1, and C_5 depends on ϵ_2.

Proof. The proofs of the identities (a)–(e) are similar to some of those in [35, Lemma 4.3]. To prove (f), we use Young's inequality with $p = q = 2$ and $\epsilon = 3$. Using the fact that $\lambda \in [0, 1]$, we obtain

$$\operatorname{Re}\left\langle -2\lambda U^n, \Phi - U^{n-1} \right\rangle \geq -2\lambda \left|\langle \Phi, U^n \rangle\right| - 2\lambda \left|\langle U^n, U^{n-1} \rangle\right| \geq -\frac{\lambda}{3}\|\Phi\|_2^2 - 3\|U^n\|_2^2 - 2\left|\langle U^n, U^{n-1}\rangle\right|, \tag{2.5}$$

and the inequality follows with $C_1 = 3\|U^n\|_2^2 + 2|\langle U^n, U^{n-1}\rangle|$. Inequality (g) was obtained similarly using Lemma 2.2.1 with $\epsilon = 3$ and $C_2 = 4\|U^{n-1}\|_2^2$. Meanwhile, a double application of Young's inequality with $\epsilon = 10$ was needed to reach (h). Concretely, note that the following are satisfied:

$$\begin{aligned}\langle \Psi - 2\lambda M^n + \lambda M^{n-1}, \Psi - M^{n-1}\rangle &\geq \|\Psi\|_2^2 - (\lambda+1)\left|\langle \Psi, M^{n-1}\rangle\right| - \lambda\|M^{n-1}\|_2^2 \\ &\quad - 2\lambda\left|\langle M^n, \Psi\rangle\right| - 2\lambda\left|\langle M^n, M^{n-1}\rangle\right| \\ &\geq \|\Psi\|_2^2 - \frac{\lambda+1}{20}\|\Psi\|_2^2 - 5(\lambda+1)\|M^{n-1}\|_2^2 - \lambda\|M^{n-1}\|_2^2 \\ &\quad - \frac{\lambda}{10}\|\Psi\|_2^2 - 10\lambda\|M^n\|_2^2 - 2\lambda\left|\langle M^n, M^{n-1}\rangle\right|.\end{aligned} \tag{2.6}$$

The inequality (h) is obtained then recalling that $\lambda \in [0, 1]$, bounding from below the third, fourth, sixth and seventh terms at the right-hand side of these inequalities, regrouping and letting $C_3 = 11\|M^{n-1}\|_2^2 + 10\|M^n\|_2^2 + 2\left|\langle M^n, M^{n-1}\rangle\right|$. In turn, the relation (i) was obtained using the elementary facts that precede this lemma, Young's inequality, and bounding from below for $\lambda \in [0, 1]$, namely,

$$\begin{aligned}\lambda\tau^2\langle -\delta_x^{(2)}\mu_{t,\Phi}^{(1)}|U^n|^2, \Psi - M^{n-1}\rangle &\geq -\tfrac{\lambda\tau^2}{2}\left|\langle \delta_x(|\Phi|^2 + |U^{n-1}|^2), \delta_x(\Psi - M^{n-1})\rangle\right| \\ &\geq -\tfrac{\lambda\tau^2}{4\epsilon_1}\|\delta_x(|\Phi|^2 + |U^{n-1}|^2)\|_2^2 - \tfrac{\lambda\tau^2\epsilon_1}{4}\|\delta_x(\Psi - M^{n-1})\|_2^2 \\ &\geq -\tfrac{\lambda\tau^2}{2}\left[\frac{\|\delta_x|\Phi|^2\|_2^2 + \|\delta_x|U^{n-1}|^2\|_2^2}{\epsilon_1} + \epsilon_1\left(\|\delta_x\Psi\|_2^2 + \|\delta_x M^{n-1}\|_2^2\right)\right] \\ &\geq -\tfrac{2\lambda\tau^2}{\epsilon_1 h}\|\Phi\|_4^4 - \tfrac{2\lambda\tau^2\epsilon_1}{h}\|\Psi^{n-1}\|_4^4 - C_4,\end{aligned} \tag{2.7}$$

where $C_4 = \frac{2\tau^2}{\epsilon_1 h}\|U^{n-1}\|_4^4 + \frac{2\tau^2\epsilon_1}{h}\|M^{n-1}\|_2^2$. Finally, using (b), Young's inequality with $\epsilon_2 > 0$, $\epsilon_3 = 1$ and $\epsilon_4 = 2$, and regrouping terms, we obtain

$$\begin{aligned}\lambda\tau^2\operatorname{Re}\langle(\mu_{t,\Psi}^{(1)}M^n)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\rangle &\geq -\frac{\lambda\tau^2}{4}\left(\left|\langle\Psi, |\Phi|^2\rangle\right| + \left|\langle\Psi, |U^{n-1}|^2\rangle\right|\right. \\ &\quad \left. + \left|\langle M^{n-1}, |\Phi|^2\rangle\right| + \left|\langle M^{n-1}, |U^{n-1}|^2\rangle\right|\right) \\ &\geq -\frac{\lambda\tau^2}{8}\left(\frac{\|\Psi\|_2^2}{\epsilon_2} + \epsilon_2\|\Phi\|_4^4 + \frac{\|\Psi\|_2^2}{\epsilon_3} + \epsilon_3\|U^{n-1}\|_4^4\right. \\ &\quad \left. + \frac{\|\Phi\|_4^4}{\epsilon_4} + \epsilon_4\|M^{n-1}\|_2^2\right) - \frac{\tau^2}{4}\left|\langle M^{n-1}, |U^{n-1}|^2\rangle\right| \\ &\geq -\frac{\lambda\tau^2}{8}\left(1 + \frac{1}{\epsilon_2}\right)\|\Psi\|_2^2 - \frac{\lambda\tau^2}{8}\left(\epsilon_2 + \frac{1}{2}\right)\|\Phi\|_4^4 \\ &\quad - \frac{\tau^2}{8}\|U^{n-1}\|_4^4 - \frac{\tau^2}{4}\|M^{n-1}\|_4^4 - \frac{\tau^2}{4}\left|\langle M^{n-1}, |U^{n-1}|^2\rangle\right|.\end{aligned} \tag{2.8}$$

The inequality follows now with C_5 being the term in parenthesis at the right end of these inequalities. □

Lemma 2.2.3 (Leray–Schauder fixed-point theorem)**.** *Let X be a Banach space, and let $F : X \to X$ be continuous and compact. If the set $S = \{x \in X : \lambda F(x) = x$ for some $\lambda \in [0,1]\}$ is bounded then F has a fixed point.*

The following is the correct statement of [35, Theorem 5.3], and its corresponding corrected proof.

Theorem 2.2.4 (Solubility)**.** *The numerical model* (2.2) *is solvable for any set of initial conditions whenever*

$$\tau^2 < \min\left\{\frac{4}{5}, \frac{h^2}{320}\right\}. \tag{2.9}$$

Proof. The approximations (U^0, M^0) and (U^1, M^1) are defined through the initial data, so assume that (U^{n-1}, M^{n-1}) and (U^n, M^n) have been already obtained, for some $n \in I_{N-1}$. Let $X = \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h$ and define the function $F : X \to X$ as $F = G \times H$, where $G, H : X \to \mathring{\mathcal{V}}_h$. In turn, for each $j \in I_{J-1}$ and $\Phi, \Psi \in \mathring{\mathcal{V}}_h$, we let

$$G_j(\Phi,\Psi) = 2U_j^n - U_j^{n-1} + \tau^2 \delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U_j^n - \tau^2 \mu_{t,\Phi}^{(1)} U_j^n - \tau^2 \left(\mu_{t,\Phi}^{(1)} U_j^n\right)\left[\mu_{t,\Psi}^{(1)} M_j^n + \mu_{t,\Phi}^{(1)} |U_j^n|^2\right], \tag{2.10}$$

$$H_j(\Phi,\Psi) = 2M_j^n - M_j^{n-1} + \tau^2 \delta_x^{(2)} \mu_{t,\Psi}^{(1)} M_j^n + \tau^2 \delta_x^{(2)} \mu_{t,\Phi}^{(1)} |U_j^n|^2. \tag{2.11}$$

In the case when $j \in \{0, J\}$, we let $G_j(\Phi,\Psi) = H_j(\Phi,\Psi) = 0$. It is obvious that F is a continuous and compact map from the Banach space X into itself. We will prove next that S of Lemma 2.2.3 is a bounded subset of X. Let $(\Phi,\Psi) \in X$ and $\lambda \in [0,1]$ satisfy $\lambda F(\Phi,\Psi) = (\Phi,\Psi)$. Equivalently, the following identities hold for each $n \in I_{N-1}$:

$$0 = \Phi - 2\lambda U^n + \lambda U^{n-1} - \lambda\tau^2 \delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U^n + \lambda\tau^2 \mu_{t,\Phi}^{(1)} U^n + \lambda\tau^2 \left(\mu_{t,\Phi}^{(1)} U^n\right)\left[\mu_{t,\Psi}^{(1)} M^n + \mu_{t,\Phi}^{(1)} |U^n|^2\right], \tag{2.12}$$

$$0 = \Psi - 2\lambda M^n + \lambda M^{n-1} - \lambda\tau^2 \delta_x^{(2)} \mu_{t,\Psi}^{(1)} M^n - \lambda\tau^2 \delta_x^{(2)} \mu_{t,\Phi}^{(1)} |U^n|^2, \tag{2.13}$$

Note now that (2.9) assures that $\frac{32}{h} < \frac{h}{10\tau^2}$ and $\frac{5\tau^2}{8} < \frac{1}{2}$. Take the real part of the inner product of both sides of the equation $0 = \Phi - \lambda G(\Phi,\Psi)$ with $\Phi - U^{n-1}$. At the same time, take the inner product of both sides of the equation $0 = \Psi - \lambda H(\Phi,\Psi)$ with $\Psi - M^{n-1}$. Add both results, use the identities and inequalities of Lemma 2.2.2 with $\epsilon_1 \in \left(\frac{32}{h}, \frac{h}{10\tau^2}\right)$ and $\epsilon_2 \in \left(\frac{5\tau^2}{8}, \frac{1}{2}\right)$, rearrange terms and simplify to obtain

$$\begin{aligned} 0 \geq{}& \frac{\lambda\tau^2}{4}\left(\|\Phi\|_4^4 - \|U^{n-1}\|_4^4\right) + \frac{\lambda\tau^2}{2}\left(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2\right) + \frac{\lambda\tau^2}{2}\left(\|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2\right) \\ &+ \frac{\lambda\tau^2}{2}\left(\|\delta_x\Psi\|_2^2 - \|\delta_x M^{n-1}\|_2^2\right) - \frac{\lambda\tau^2}{8}\left(\frac{1}{\epsilon_2} + 1\right)\|\Psi\|_2^2 - \frac{\lambda\tau^2}{8}\left(\epsilon_2 + \frac{1}{2}\right)\|\Phi\|_4^4 - \frac{\lambda}{3}\|\Phi\|_2^2 \\ &+ \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 + \left(1 - \frac{3\lambda+1}{20}\right)\|\Psi\|_2^2 - \frac{2\lambda\tau^2}{\epsilon_1 h}\|\Phi\|_4^4 - \frac{2\lambda\tau^2\epsilon_1}{h}\|\Psi\|_2^2 - C_6, \end{aligned} \tag{2.14}$$

where $C_6 = C_1 + C_2 + C_3 + C_4 + C_5$, and the constants $C_1, \dots, C_5 \in \mathbb{R}^+$ are those of Lemma 2.2.2. Rearranging terms and using the fact that $\lambda \in [0,1]$, we obtain that

$$K_1\|\Phi\|_2^2 + K_2\|\Phi\|_4^4 + K_3\|\Psi\|_2^2 \leq C, \tag{2.15}$$

with

$$K_1 = 1 + \frac{\lambda\tau^2}{2} - \frac{3\lambda+1}{6} \geq \frac{1}{3}, \tag{2.16}$$

$$K_2 = \frac{\lambda\tau^2}{4}\left[1 - \frac{8}{\epsilon_1 h} - \frac{1}{2}\left(\epsilon_2 + \frac{1}{2}\right)\right] > \frac{\lambda\tau^2}{4}\left(1 - \frac{1}{4} - \frac{1}{2}\right) = \frac{\lambda\tau^2}{16}, \tag{2.17}$$

$$K_3 = 1 - \lambda\tau^2\left(\frac{2\epsilon_1}{h} + \frac{1}{8\epsilon_2} + \frac{1}{8}\right) - \frac{3\lambda+1}{20} \geq 1 - \tau^2\left(\frac{1}{5\tau^2} + \frac{1}{5\tau^2} + \frac{1}{8}\right) - \frac{1}{5} > \frac{3}{10}, \tag{2.18}$$

$$C = C_6 + \frac{\tau^2}{2}\left(\frac{1}{2}\|U^{n-1}\|_4^4 + \|U^{n-1}\|_2^2 + \|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2 + \|\delta_x M^{n-1}\|_2^2\right). \tag{2.19}$$

The inequalities were obtained using the ranges of ϵ_1 and ϵ_2, and that $\lambda \in [0,1]$. It follows that K_1, K_2 and K_3 are positive and, moreover, that $\frac{1}{3}\|\Phi\|_2^2 + \frac{3}{10}\|\Psi\|_2^2 \leq C$. As a consequence, the set S is bounded, and the Leray–Schauder theorem guarantees that the system (2.2) has a solution (U^{n+1}, M^{n+1}). The result follows now by induction. □

2.3 Addendum

Throughout this stage, we will observe the discrete nomenclature of the previous section. An alternative finite-difference model to solve (2.1) is described by the following algebraic system:

$$\begin{array}{c}\delta_t^{(2)}U_j^n - \delta_x^{(\alpha)}U_j^n + \mu_t^{(1)}U_j^n + M_j^n\mu_t^{(1)}U_j^n + \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right) = 0, \qquad \forall (j,n) \in I \\ \delta_t^{(2)}M_j^n - \delta_x^{(2)}M_j^n - \delta_x^{(2)}|U_j^n|^2 = 0, \qquad \forall (j,n) \in I \\ \text{such that} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), \quad \forall j \in \overline{I}_J, \\ \delta_t^{(1)}U_j^0 = u_1(x_j) & \delta_t^{(1)}M_j^0 = m_1(x_j), \quad \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases}\end{array} \tag{2.20}$$

Note that the numerical model (2.20) is a three-step explicit technique. Indeed, notice that the first equation of that system yields an expression with complex parameters in which the only unknown is U_j^{n+1}. On the other hand, the second equation of (2.20) is a fully explicit difference equation which can be easily solved for M_j^{n+1}, for each $(j,n) \in I$. Obviously, this explicit character of the method has computational advantages over the implicit scheme in [35].

Observe that the initial conditions of the numerical model (2.20) require the knowledge of U^{-1} and M^{-1}. In order to eliminate them, we require for the difference equations of (2.20) to hold also for $n = 0$. Using then the initial data, we readily obtain that for each $j \in I_{J-1}$, the following identities hold:

$$\frac{2U_j^1 - 2u_0(x_j) - 2\tau u_1(x_j)}{\tau^2} = \delta_x^{(\alpha)}U_j^0 - (U_j^1 - \tau u_1(x_j))\left[1 + M_j^0 + \frac{1}{2}\left(|U_j^1|^2 + |U_j^1 - 2\tau u_1(x_j)|^2\right)\right], \tag{2.21}$$

$$M_j^1 = m_0(x_j) + \tau m_1(x_j) + \frac{\tau^2}{2}\delta_x^{(2)}\left(M_j^0 + |U_j^0|^2\right). \tag{2.22}$$

We will require the following lemma.

Lemma 2.3.1. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathring{\mathcal{V}}_h$, and assume that U is a sequence of complex functions while the functions of V are real. The following hold, for each $n \in I_{N-1}$ and $\Phi, \Psi \in \mathring{\mathcal{V}}_h$:*

(a) $2\,\mathrm{Re}\langle \mu_{t,\Phi}^{(1)}U^n, \Phi - U^{n-1}\rangle = \|\Phi\|_2^2 - \|U^{n-1}\|_2^2$.

(b) $4\,\mathrm{Re}\langle(\mu_{t,\Phi}^{(1)}|U^n|^2)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\rangle = \|\Phi\|_4^4 - \|U^{n-1}\|_4^4$.

In addition, the following inequalities hold for each $\lambda \in [0,1]$*:*

(c) $\lambda\tau^2\,\mathrm{Re}\langle M^n \mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1}\rangle \geq -\frac{1}{12}\lambda\tau^2\|\Phi\|_4^4 - C_1$.

(d) $\mathrm{Re}\langle \Phi - 2\lambda U^n + \lambda U^{n-1}, \Phi - U^{n-1}\rangle \geq \left[1 - \frac{1}{6}(3\lambda+1)\right]\|\Phi\|_2^2 - C_2$.

(e) $\mathrm{Re}\langle -\lambda\tau^2\delta_x^{(\alpha)}U^n, \Phi - U^{n-1}\rangle \geq -\frac{1}{6}\lambda\tau^2\|\Phi\|_2^2 - C_3$.

Here, the constants C_1 *and* C_2 *depend only on* U^{n-1} *and* U^n*.*

Proof. The proofs of these relations are similar to those in Lemma 2.2.2. We need only mention that, to reach the inequalities, we applied Young's theorem with $\epsilon = 3$, and the constants are $C_1 = \frac{3\tau^2}{4}\|M^n\|_2^2 + \frac{\tau^2}{2}|\langle M^n, |U^{n-1}|^2\rangle|$, $C_2 = 4\|U^{n-1}\|_2^2 + 3\|U^n\|_2^2 + 2|\langle U^n, U^{n-1}\rangle|$ and $C_3 = \frac{3}{2}\tau^2\|\delta_x^{(\alpha)}U^n\|_2^2 + \tau^2|\langle\delta_x^{(\alpha)}U^n, U^{n-1}\rangle|$. □

Theorem 2.3.2 (Solubility)**.** *The numerical model* (2.20) *is solvable for any set of initial conditions*

Proof. The proof is similar to that of Theorem 2.2.4, and we provide a shortened proof for that reason. Beforehand, note that M^{n+1} is defined explicitly in terms of M^n, M^{n-1}, U^n and U^{n-1}, for each $n \in I_{N-1}$, so we only need to establish the solubility of U. Let $n \in I_{N-1}$, and suppose that M^n, M^{n-1}, U^n and U^{n-1} have been calculated. Following the nomenclature of Theorem 2.2.4, we note in this case that

$$G_j(\Phi,\Psi) = 2U_j^n - U_j^{n-1} + \tau^2\delta_x^{(\alpha)}U_j^n - \tau^2\mu_{t,\Phi}^{(1)}U_j^n - \tau^2 M^n\mu_{t,\Phi}^{(1)}U_j^n - \tau^2\left(\mu_{t,\Phi}^{(1)}|U_j^n|^2\right)\left(\mu_{t,\Phi}^{(1)}U_j^n\right), \tag{2.23}$$

$$H_j(\Phi,\Psi) = 2M_j^n - M_j^{n-1} + \tau^2\delta_x^{(2)}M_j^n + \tau^2\delta_x^{(2)}|U_j^n|^2. \tag{2.24}$$

We take the real part of the inner product of both sides of the equation $0 = \Phi - \lambda G(\Phi,\Psi)$ with $\Phi - U^{n-1}$, use the results of Lemma 2.3.1, combine terms, simplify algebraically and bound from below using that $\lambda \in [0,1]$ to obtain

$$\frac{1}{6}\|\Phi\|_2^2 - C \leq \left(1 + \frac{\lambda\tau^2}{2} - \frac{4\lambda+1}{6}\right)\|\Phi\|_2^2 + \frac{\lambda\tau^2}{6}\|\Phi\|_4^4 - C_1 - C_2 - C_3 - \frac{\tau^2}{2}\|U^{n-1}\|_2^2 - \frac{\tau^2}{4}\|U^{n-1}\|_4^4 \leq 0, \tag{2.25}$$

where $C = C_1 + C_2 + C_3 + \frac{1}{2}\tau^2\|U^{n-1}\|_2^2 + \frac{1}{4}\tau^2\|U^{n-1}\|_4^4$. As a consequence, the set S of Lemma 2.2.3 is bounded, whence it follows that F has a fixed point. In this way, the existence of U^{n+1} is established. The existence of the approximation U^1 is carried out in similar fashion. The conclusion of the theorem follows now using induction. □

3. An implicit method for double-fractional system

3.1 Preliminaries

Let $B = (x_L, x_R)$ be a nonempty and bounded interval in $\mathbb{R}$, let $T > 0$ and define $\Omega = B \times (0, T)$. In this work, we will use the notation $\overline{S}$ to represent the closure of S with respect to the standard topology of $\mathbb{R}^2$, for each $S \subseteq \mathbb{R}^2$. Assume that u and m are a complex- and a real-valued functions, respectively, whose domains are both equal to $\overline{\Omega}$. Moreover, let $u_0, u_1 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions. In the following, all the relevant functions will be defined on $\overline{\Omega}$. However, for the sake of simplicity, we will extend their definitions to the set $\mathbb{R} \times [0, T]$, by letting them be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$.

Definition 3.1.1 (Podlubny [78])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 < \alpha \leq n$. The *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is defined (when it exists) as

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{|x - \xi|^{\alpha+1-n}}. \tag{3.1}$$

Here, Γ denotes the usual Gamma function. In the case that $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, and n and α are as above, then the *Riesz fractional partial derivative* of u of order α with respect to x at $(x, t) \in \mathbb{R} \times [0, T]$ is given (if it exists) by

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi, t) d\xi}{|x - \xi|^{\alpha+1-n}}. \tag{3.2}$$

The Riesz fractional partial derivative of u of order α with respect to x is also denoted by $\partial^\alpha_{|x|} u$ in this work.

Definition 3.1.2. If $z \in \mathbb{C}$ then we will represent its complex conjugate by $\overline{z}$. We will use $\mathbb{F}$ to represent the fields $\mathbb{R}$ or $\mathbb{C}$. Let us define the set $L_{x,p}(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{F} : f(\cdot, t) \in L_p(\overline{B}),$ for each $t \in [0, T]\}$, where $p \in [1, \infty]$. If $p \in [1, \infty)$ and $f \in L_{x,p}(\overline{\Omega})$ then we convey that

$$\|f\|_{x,p} = \left(\int_{\overline{B}} |f(x,t)|^p dx \right)^{1/p}, \quad \forall t \in [0, T]. \tag{3.3}$$

In the case when $p = \infty$, we set $\|f\|_{x,\infty} = \inf\{C \geq 0 : |f(x, t)| \leq C$ for almost all $x \in \overline{B}\}$. Obviously, $\|f\|_{x,p}$ is a function of $t \in [0, T]$ in any case. Moreover, for each pair $f, g \in L_{x,2}(\overline{\Omega})$, define the following function of t:

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x,t)\overline{g(x,t)} dx, \quad \forall t \in [0, T]. \tag{3.4}$$

For the remainder of this work and unless we mention otherwise, we will fix $\alpha, \beta \in (1,2]$. Under these circumstances, the fractional extension of the Klein–Gordon–Zakharov problem investigated in this work is given the coupled system of fractional differential equations with initial-boundary data

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \quad \forall (x,t) \in \Omega, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta} = 0, \quad \forall (x,t) \in \Omega, \\
\text{subject to} \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), & \forall x \in \overline{B}, \\ \dfrac{\partial u(x,0)}{\partial t} = u_1(x), & \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, & \forall t \in [0,T]. \end{cases}
\end{aligned}
\tag{3.5}
$$

Note that the system of partial differential equations of (3.5) reduces to the well-known Klein–Gordon–Zakharov system when $\alpha = \beta = 2$. It is worth recalling that the system of Klein–Gordon–Zakharov equations describes the propagation of strong turbulences of the Langmuir wave in a high-frequency plasma [91, 12], and that various other potential applications of this system have been proposed in the literature. Within the context of physics of high-frequency plasma, the function u denotes the fast time-scale component of an electric field raised by electrons, and m denotes the deviation of ion density from its equilibrium.

For the remainder of this work, we will let $v, w : \overline{\Omega} \to \mathbb{R}$ be functions satisfying

$$
w(x,t) = -\frac{\partial^{\beta/2} v(x,t)}{\partial |x|^{\beta/2}}, \qquad \forall (x,t) \in \Omega, \tag{3.6}
$$

$$
\frac{\partial^\beta v(x,t)}{\partial |x|^\beta} = \frac{\partial m(x,t)}{\partial t}, \qquad \forall (x,t) \in \Omega. \tag{3.7}
$$

Using these identities and the product rule, it is easy to check that

$$
m(x,t)\frac{\partial}{\partial t}\left(|u(x,t)|^2\right) = \frac{\partial}{\partial t}\left(m(x,t)|u(x,t)|^2\right) + |u(x,t)|^2 \frac{\partial^{\beta/2} w(x,t)}{\partial |x|^{\beta/2}}, \quad \forall (x,t) \in \Omega. \tag{3.8}
$$

Definition 3.1.3. Let u, m be a pair of functions satisfying the initial-boundary-value problem (3.5). We define the Hamiltonian of that fractional system as $\mathcal{H}(u(x,t), m(x,t)) = \mathcal{H}(x,t)$, where

$$
\begin{aligned}
\mathcal{H}(x,t) &= \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right|^2 + |u|^2 + m|u|^2 + \frac{1}{2}w^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \\
&= \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right|^2 + |u|^2 + m|u|^2 + \frac{1}{2}\left|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right|^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \quad \forall (x,t) \in \Omega.
\end{aligned}
\tag{3.9}
$$

Here, v and w satisfy (3.6) and (3.7). Notice that we obviated the dependence of all the functions on the right-hand side of this identity with respect to (x,t). The associated energy of the system at the time $t \in [0,T]$ is given then by

$$
\begin{aligned}
\mathcal{E}(t) &= \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\|w\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4 \\
&= \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4.
\end{aligned}
\tag{3.10}
$$

To establish our next result, recall that the additive inverse of the Riesz fractional derivative of order α has a unique square-root operators over the space of sufficiently integrable functions with compact support [46]. This unique operator is actually $\partial^{\alpha/2}_{|x|}$, and it satisfies the following, for any two functions u and v (see [27]):

$$\left\langle u, -\frac{\partial^\alpha v}{\partial |x|^\alpha} \right\rangle_x = \left\langle -\frac{\partial^\alpha u}{\partial |x|^\alpha}, v \right\rangle_x = \left\langle \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}, \frac{\partial^{\alpha/2} v}{\partial |x|^{\alpha/2}} \right\rangle_x, \quad \forall t \in [0,T], \tag{3.11}$$

Theorem 3.1.4 (Energy conservation)**.** *If u and m satisfy the problem* (3.5) *then the function $\mathcal{E}$ is constant.* □

Proof. Beforehand, note that the following identities are readily satisfied:

$$\frac{d}{dt}\left\| \frac{\partial u}{\partial t} \right\|^2_{x,2} = 2\,\mathrm{Re}\left\langle \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t} \right\rangle_x, \quad \forall t \in (0,T), \tag{3.12}$$

$$\frac{d}{dt}\left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|^2_{x,2} = 2\,\mathrm{Re}\left\langle \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}, \frac{\partial^{\alpha/2}}{\partial |x|^{\alpha/2}}\frac{\partial u}{\partial t} \right\rangle = 2\,\mathrm{Re}\left\langle -\frac{\partial^\alpha u}{\partial |x|^\alpha}, \frac{\partial u}{\partial t} \right\rangle, \quad \forall t \in (0,T), \tag{3.13}$$

$$\frac{d}{dt}\|u\|^2_{x,2} = 2\,\mathrm{Re}\left\langle u, \frac{\partial u}{\partial t} \right\rangle_x, \quad \forall t \in (0,T), \tag{3.14}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^4_{x,4} = 2\,\mathrm{Re}\left\langle |u|^2 u, \frac{\partial u}{\partial t} \right\rangle, \quad \forall t \in (0,T). \tag{3.15}$$

On the other hand, using the identity (3.8) and the second partial differential equation of (3.5), standard integration arguments, the formula for integration by parts and the boundary conditions of problem (3.5), we obtain , $\forall t \in (0,T)$,

$$\begin{aligned} 2\,\mathrm{Re}\left\langle mu, \frac{\partial u}{\partial t} \right\rangle_x &= \int_{-\infty}^{\infty} m \left[2\,\mathrm{Re}\left(u \cdot \overline{\frac{\partial u}{\partial t}} \right) \right] dx = \int_{-\infty}^{\infty} \left(m \cdot \frac{\partial\left(|u|^2\right)}{\partial t} \right) dx \\ &= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t}\left(m|u|^2 \right) - |u|^2 \frac{\partial m}{\partial t} \right] dx = \frac{d}{dt}\left\langle m, |u|^2 \right\rangle_x - \left\langle |u|^2, \frac{\partial^\beta v}{\partial |x|^\beta} \right\rangle_x . \end{aligned} \tag{3.16}$$

We estimate now the second term on the right-hand side of these identities using the second partial differential equation of (3.5). More precisely, notice that the following identities are satisfied, for each $t \in (0,T)$:

$$\begin{aligned} -\left\langle |u|^2, \frac{\partial^\beta v}{\partial |x|^\beta} \right\rangle_x &= -\left\langle \frac{\partial^2 m}{\partial t^2}, v \right\rangle_x + \left\langle \frac{\partial^\beta m}{\partial |x|^\beta}, v \right\rangle_x = -\left\langle \frac{\partial}{\partial t}\frac{\partial m}{\partial t}, v \right\rangle_x + \left\langle m, \frac{\partial^\beta v}{\partial |x|^\beta} \right\rangle_x \\ &= \left\langle -\frac{\partial^\beta}{\partial |x|^\beta}\frac{\partial v}{\partial t}, v \right\rangle_x + \left\langle m, \frac{\partial m}{\partial t} \right\rangle_x = \left\langle \frac{\partial}{\partial t}\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}, -w \right\rangle_x + \left\langle m, \frac{\partial m}{\partial t} \right\rangle_x \\ &= \left\langle w, \frac{\partial w}{\partial t} \right\rangle_x + \left\langle m, \frac{\partial m}{\partial t} \right\rangle_x . \end{aligned} \tag{3.17}$$

Using the last two sets of inequalities, we readily obtain that

$$2\,\mathrm{Re}\left\langle mu, \frac{\partial u}{\partial t} \right\rangle_x = \frac{d}{dt}\left[\left\langle m, |u|^2 \right\rangle_x + \frac{1}{2}\|w\|^2_{x,2} + \frac{1}{2}\|m\|^2_{x,2} \right], \quad \forall t \in (0,T). \tag{3.18}$$

Now, take the derivative of $\mathcal{E}$ with respect to t. Employing the first differential equation of (3.5), all the identities above and simplifying, we reach that $\mathcal{E}'(t) = 0$ for all $t \in (0,T)$, whence the conclusion of this result readily follows. □

Lemma 3.1.5 (Gagliardo-Nirenberg inequality [67]). *Let $m \in \mathbb{N}$, $q \geq 1$ and $0 < r \leq \infty$. Suppose that $f : \overline{B} \to \mathbb{R}$ satisfies $f \in L^q(\overline{B})$ and $f^{(m)} \in L^r(\overline{B})$. Let $j \in \mathbb{Z}$ satisfy $0 \leq j \leq m$, and suppose that $\frac{j}{m} \leq a \leq 1$ and $1 \leq p \leq \infty$. If*

$$\frac{1}{p} = \frac{j}{2} + a\left(\frac{1}{r} - \frac{m}{2}\right) + (1-a)\frac{1}{q}, \tag{3.19}$$

then there exists $C \in \mathbb{R}^+$ such that $\|f^{(j)}\|_p \leq C\|f^{(m)}\|_r^a \|f\|_q^{1-a}$.

Theorem 3.1.6 (Boundedness). *Let u and m satisfy the initial-boundary-value problem* (3.5), *and let $u, \partial u/\partial x \in L_{x,2}(\overline{\Omega})$. Then there exist a constant C such that*

$$\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \|w\|_{x,2}^2 + \|m\|_{x,2}^2 \leq C, \quad \forall t \in [0,T]. \tag{3.20}$$

Moreover, the constant function (3.10) *is nonnegative.*

Proof. From Theorem 3.1.4, there exists $C_0 \in \mathbb{R}$ such that $\mathcal{E}(t) = C_0$, for each $t \in [0,T]$. On the other hand, note that $|\langle m, |u|^2\rangle| \leq \frac{1}{2}(\|m\|_2^2 + \|u\|_4^4)$ is satisfied for all $\forall t \in [0,T]$. As a consequence, we obtain that, $\forall t \in [0,T]$,

$$\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \|w\|_{x,2}^2 \leq 2\left(\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \frac{1}{2}\|w\|_{x,2}^2\right) \leq 2C_0. \tag{3.21}$$

An application of Lemma 3.1.5 with $p = 4$, $j = 0$, $m = 1$, $r = q = 2$ and $a = \frac{1}{2}$, and the fact itself that $u, \partial u/\partial x \in L_{x,2}(\overline{\Omega})$ yield that there exist constants $C_1, C_2 \in \mathbb{R}^+$ with the property that

$$\|u\|_4^4 \leq C_1 \|u\|_{x,2}^2 \left\|\frac{\partial u}{\partial x}\right\|_{x,2}^2 \leq C_2, \quad \forall t \in [0,T]. \tag{3.22}$$

The conclusion of the theorem is reached now using then the bounds (3.21) and (3.22), and letting $C = 2C_0 + C_2$. □

The cornerstone in the design of our finite-difference scheme is the concept of fractional centered differences, which will be employed to discretize the Riesz fractional derivatives. We must point out that different approaches can be employed to provide such discretizations, like the weighted and shifted Grünwald differences [92]. We have opted to use fractional centered differences in view of the convenience of their computational implementations.

Definition 3.1.7 (Ortigueira [73]). Let $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ be the real sequence defined by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{N} \cup \{0\}, \tag{3.23}$$

and assume that $f : \mathbb{R} \to \mathbb{R}$ is any function. If $h > 0$ and $\alpha > -1$ then the *fractional-order centered difference* of order α of f at the point x is defined as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{3.24}$$

if the double series at the right-hand of (3.24) converges.

The following result provides some useful properties of the sequences $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$.

Lemma 3.1.8 (Wang *et al.* [97])**.** *If* $0 < \alpha \leq 2$ *and* $\alpha \neq 1$ *then*

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ *for all* $k \geq 1$, *and*

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$. *As a consequence, it follows that* $g_0^{(\alpha)} = -\displaystyle\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} g_k^{(\alpha)} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |g_k^{(\alpha)}|$.

For each nonnegative integer m, let $\mathcal{C}^m(\mathbb{R})$ denote the space of all the functions $f : \mathbb{R} \to \mathbb{R}$ which have continuous derivatives up to the mth order. As a consequence of Lemma 3.1.8, the series at the right-hand side of (3.24) converges absolutely for any bounded function $f \in L^1(\mathbb{R})$. This implies in particular that the function $\Delta_h^\alpha f : \mathbb{R} \to \mathbb{R}$ is well defined when $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\alpha \in (1, 2]$.

Theorem 3.1.9 (Wang *et al.* [97])**.** *Let* $\alpha \in (1, 2]$ *and* $h > 0$, *and suppose that* $f \in \mathcal{C}^5(\mathbb{R})$. *If all the derivatives of* f *up to order five belong to* $L^1(\mathbb{R})$ *then*

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{3.25}$$

3.2 Numerical approach

For the remainder of this work, we let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$, for each $q \in \mathbb{N}$. Throughout this manuscript, we let $J, N \in \mathbb{N}$ satisfy $J \geq 2$ and $N \geq 2$, and define the positive step-sizes $h = (x_R - x_L)/J$ and $\tau = T/N$. We consider uniform partitions of the intervals $[x_L, x_R]$ and $[0, T]$, respectively, of the forms

$$x_L = x_0 < x_1 < \ldots < x_j < \ldots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{3.26}$$

and

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T, \quad \forall n \in \overline{I}_N. \tag{3.27}$$

For each $(j, n) \in \overline{I}_J \times \overline{I}_N$, we let U_j^n and M_j^n represent numerical approximations to $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, respectively. In this manuscript, we let $\mathcal{R}_h = \{x_j : j \in \overline{I}_J\}$, and represent by $\mathcal{V}_h$ the vector space over $\mathbb{F}$ of all $\mathbb{F}$-valued functions on the grid space $\mathcal{R}_h$ which vanish at x_0 and x_J. If $V \in \mathcal{V}_h$ then we set $V_j = V(x_j)$, for each $j \in \overline{I}_J$. Moreover, in this work we will let $U^n = (U_j^n)_{j\in\overline{I}_J} \in \mathcal{V}_h$ and $M^n = (M_j^n)_{j\in\overline{I}_J} \in \mathcal{V}_h$, and set $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$.

Definition 3.2.1. Let p be any number satisfying $1 \leq p < \infty$. The inner product $\langle\cdot,\cdot\rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{C}$ and the norms $\|\cdot\|_p, \|\cdot\|_\infty : \mathcal{V}_h \to \mathbb{R}$ are defined, respectively, by

$$\langle U, V\rangle = h \sum_{j\in\overline{I}_J} U_j \overline{V_j}, \quad \forall U, V \in \mathcal{V}_h, \tag{3.28}$$

$$\|U\|_p^p = h \sum_{j\in\overline{I}_J} |U_j|^p, \quad \forall U \in \mathcal{V}_h, \tag{3.29}$$

$$\|U\|_\infty = \max\left\{|U_j| : j \in \overline{I}_J\right\}, \quad U \in \mathcal{V}_h. \tag{3.30}$$

Additionally, we let $|||V|||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}$, for each $V = (V^n)_{n\in\overline{I}_N} \subseteq \mathcal{V}_h$.

Definition 3.2.2. Let V represent any of the functions U or M, and suppose that $\alpha \in (1,2]$. In the present manuscript, we will employ the linear difference operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j,n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{3.31}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{3.32}$$

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{3.33}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j,n) \in \overline{I}_J \times I_{N-1}. \tag{3.34}$$

We agree that $\delta_x^{(2)} V_j^n = (\delta_x \circ \delta_x) V_{j-1}^n$, for each $(j,n) \in I_{J-1} \times \overline{I}_N$. Here, the symbol $\circ$ represents composition of operators, and it will be obviated in the future for the sake of simplicity. Using this convention, let $\delta_t^{(1)} V_j^n = \mu_t \delta_t V_j^{n-1}$, $\delta_t^{(2)} V_j^n = \delta_t \delta_t V_j^{n-1}$ and $\mu_t^{(2)} V_j^n = \mu_t \mu_t V_j^{n-1}$, for each $(j,n) \in \overline{I}_J \times I_{N-1}$. Moreover, using the notation in Definition 3.1.7, we define the discrete linear operator

$$\delta_x^{(\alpha)} V_j^n = -\frac{1}{h^\alpha} \sum_{k \in \overline{I}_J} g_{j-k}^{(\alpha)} V_k^n, \qquad \forall (j,n) \in I_{J-1} \times \overline{I}_N. \tag{3.35}$$

Lemma 3.2.3 (Macías-Díaz [49]). *If $\alpha \in (1,2]$ and $U, V \in \mathcal{V}_h$ then $\langle -\delta_x^{(\alpha)} U, V \rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V \rangle$.*

Using the nomenclature of this section, the finite-difference model employed in this work to approximate the solutions of (3.5) is described by the algebraic system of difference equations

$$\begin{gathered}
\delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n + M_j^n \mu_t^{(1)} U_j^n + \left(\mu_t^{(2)} |U_j^n|^2 \right) \left(\mu_t^{(1)} U_j^n \right) = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\delta_t^{(2)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2 = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\text{subject to} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), \quad \forall j \in I_{J-1}, \\ \delta_t^{(1)} U_j^0 = u_1(x_j) & \delta_t^{(1)} M_j^0 = m_1(x_j), \quad \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases}
\end{gathered} \tag{3.36}$$

It is worth pointing out that the discrete system (3.36) requires to consider fictitious approximations at the time t_{-1}. Notice also that the initial conditions on the discrete velocities yield $U_j^{-1} = U_j^1 - 2\tau u_1(x_j)$ and $M_j^{-1} = M_j^1 - 2\tau m_1(x_j)$, for each $j \in I_{J-1}$. Substituting these identities into the difference equations of (3.36) when $n = 0$, we readily obtain

$$\begin{aligned}
2[U_j^1 - \tau u_1(x_j) - u_0(x_j)] = \tau^2 \left(\tau u_1(x_j) - U_j^1 \right) \left[1 + \frac{|U_j^1|^2 + 2|u_0(x_j)|^2 + |U_j^1 - 2\tau u_1(x_j)|^2}{4} \right] \\
\tau^2 \left(\tau u_1(x_j) - U_j^1 \right) m_0(x_j) + \tau^2 \delta_x^{(\alpha)} (U_j^1 - \tau u_1(x_j)), \quad \forall j \in I_{J-1},
\end{aligned} \tag{3.37}$$

and

$$2[M_j^1 - \tau m_1(x_j) - m_0(x_j)] = 2\tau^2 \delta_x^{(\beta)} |u_0(x_j)|^2 + \tau^2 \delta_x^{(\beta)} (M_j^1 - \tau m_1(x_j)), \quad \forall j \in I_{J-1}. \tag{3.38}$$

Note that the numerical model (3.36) is a three-step implicit nonlinear technique. Indeed, if the approximations at the times t_{n-1} and t_n are known, then the difference equations of (3.36) have the vectors U^{n+1} and M^{n+1} as unknowns. Moreover, it is easy to check that the first difference equation of (3.36) contains U^{n+1} as the only unknown vector, while the only unknown of the second equation is M^{n+1}. In that sense, the discrete model (3.36) is a decoupled nonlinear system. In the following and for the sake of convenience, we will let $\{V_j^n : (j,n) \in \overline{I}_J \times \overline{I}_N\}$ be such that

$$\delta_x^{(\beta)} V_j^n = \delta_t M_j^n, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.39}$$

$$V_0^n = V_J^n = 0, \quad \forall n \in \overline{I}_N. \tag{3.40}$$

Under these circumstances, (U, M) will denote a solution of (3.36), and $V = (V^n)_{n\in\overline{I}_N}$ will satisfy (3.39) and (3.40).

Definition 3.2.4. Let (U, M) be a solution of (3.36). We define the associated discrete energy density of the system at the point x_j and time t_n as $H(U_j^n, M_j^n) = H_j^n$, where

$$\begin{aligned} H_j^n = {} & |\delta_t U_j^n|^2 + \mu_t|\delta_x^{(\alpha/2)} U_j^n|^2 + \mu_t|U_j^n|^2 + \frac{1}{4}\mu_t|U_j^n|^4 + \frac{1}{2}\mu_t|M_j^n|^2 + \frac{1}{2}|\delta_x^{(\beta/2)} V_j^n|^2 + \frac{1}{4}|U_j^{n+1}|^2|U_j^n|^2 \\ & + \frac{1}{2}\left[M_j^n|U_j^{n+1}|^2 + M_j^{n+1}|U_j^n|^2\right], \quad (j,n) \in I_{J-1} \times \overline{I}_{N-1}. \end{aligned} \tag{3.41}$$

Meanwhile, the discrete energy of the system (3.36) at the time t_n is defined by

$$\begin{aligned} E^n = {} & \|\delta_t U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{4}\mu_t\|U^n\|_4^4 + \frac{1}{2}\mu_t\|M^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 \\ & + \frac{1}{4}\langle |U^{n+1}|^2, |U^n|^2\rangle + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2\rangle\right], \quad \forall n \in \overline{I}_{N-1}. \end{aligned} \tag{3.42}$$

Here, we employ the notation $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$. Also, we will set the computational parameter $R = \tau^2 h^{-\beta}$.

Lemma 3.2.5 (Young's inequality)**.** *Let $a, b \in \mathbb{R}^+ \cup \{0\}$, and let $p, q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For each $\epsilon > 0$, the following inequality holds:*

$$ab \le \frac{|a|^p}{p\epsilon} + \frac{\epsilon |b|^q}{q}. \tag{3.43}$$

Definition 3.2.6. Let $(U^n)_{n\in\overline{I}_N}$ be any sequence in $\mathcal{V}_h$, let $\Phi \in \mathcal{V}_h$ and assume that $g : \mathbb{R} \to \mathbb{R}$ is a function. We define

$$\mu_{t,\Phi}^{(1)}[g(U_j^n)] = \tfrac{1}{2}\left[g(\Phi_j) + g(U_j^{n-1})\right], \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{3.44}$$

$$\mu_{t,\Phi}^{(2)}[g(U_j^n)] = \tfrac{1}{4}\left[g(\Phi_j) + 2g(U_j^n) + g(U_j^{n-1})\right], \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_N. \tag{3.45}$$

Lemma 3.2.7. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathcal{V}_h$. More precisely, assume that U is a sequence of complex functions while the functions of M are real. The following identities hold, for each $n \in \overline{I}_{N-1}$ and $\Phi \in \mathcal{V}_h$:*

(a) $8\,\mathrm{Re}\left\langle (\mu_{t,\Phi}^{(2)}|U^n|^2)(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1}\right\rangle = \|\Phi\|_4^4 - \|U^{n-1}\|_4^4 + 2\langle |U^n|^2, |\Phi|^2 - |U^{n-1}|^2\rangle$,

(b) $2\,\mathrm{Re}\left\langle M^n(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1}\right\rangle = \langle M^n, |\Phi|^2 - |U^{n-1}|^2\rangle$,

(c) $2\,\mathrm{Re}\left\langle \mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1}\right\rangle = \|\Phi\|_2^2 - \|U^{n-1}\|_2^2$, *and*

(d) $2\,\mathrm{Re}\left\langle -\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1}\right\rangle = \|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2$.

Additionally, the following inequalities are satisfied for each $\lambda \in [0,1]$ and $\epsilon > 0$:

(e) $\mathrm{Re}\langle -2\lambda U^n, \Phi - U^{n-1}\rangle \geq -\frac{\lambda}{3}\|\Phi\|_2^2 - C_1$,

(f) $\mathrm{Re}\langle \Phi + \lambda U^{n-1}, \Phi - U^{n-1}\rangle \geq \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 - C_2$, *and*

(g) $\langle 2M^n, |\Phi|^2 - |U^{n-1}|^2\rangle + \langle |U^n|^2, |\Phi|^2 - |U^{n-1}|^2\rangle \geq -\frac{3\epsilon}{2}\|\Phi\|_4^4 - C_3$.

Here, the constants $C_1, C_2 \in \mathbb{R}^+$ depend only on U^n, U^{n-1}, M^n and M^{n-1}. Additionally, C_3 depends also on ϵ.

Proof. The proofs of the identities (a)–(d) are straightforward. To prove (e), notice that

$$-2\lambda\,\mathrm{Re}\langle U^n, \Phi\rangle + 2\lambda\,\mathrm{Re}\langle U^n, U^{n-1}\rangle \geq -2\lambda\,|\langle \Phi, U^n\rangle| - 2\lambda\left|\langle U^n, U^{n-1}\rangle\right|. \tag{3.46}$$

Applying Young's inequality to $|\langle \Phi, U^n\rangle|$ with $p = q = 2$ and $\epsilon = 3$, and using the fact that $\lambda \in [0,1]$, we obtain that

$$\begin{aligned}\mathrm{Re}\left\langle -2\lambda U^n, \Phi - U^{n-1}\right\rangle &\geq -\frac{\lambda}{3}\|\Phi\|_2^2 - 3\lambda\|U^n\|_2^2 - 2\lambda\left|\langle U^n, U^{n-1}\rangle\right| \\ &\geq -\frac{\lambda}{3}\|\Phi\|_2^2 - 3\|U^n\|_2^2 - 2\left|\langle U^n, U^{n-1}\rangle\right|.\end{aligned} \tag{3.47}$$

The inequality (e) follows now with $C_1 = 3\|U^n\|_2^2 + 2|\langle U^n, U^{n-1}\rangle|$. It is worth pointing out that the inequality (f) is obtained similarly using Lemma 3.2.5 with $\epsilon = 3$ and $C_2 = 4\|U^{n-1}\|_2^2$. Indeed, notice that

$$\begin{aligned}\mathrm{Re}\langle \Phi + \lambda U^{n-1}, \Phi - U^{n-1}\rangle &\geq \|\Phi\|_2^2 - (\lambda+1)\left|\langle \Phi, U^{n-1}\rangle\right| - \lambda\|U^{n-1}\|_2^2 \\ &\geq \|\Phi\|_2^2 - \frac{\lambda+1}{6}\|\Phi\|_2^2 - 3\|U^{n-1}\|_2^2 - \|U^{n-1}\|_2^2 \\ &= \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 - C_2.\end{aligned} \tag{3.48}$$

Finally, expanding the inner product of (g) and applying Young's inequality, we have

$$\begin{aligned}\langle 2M^n + |U^n|^2, |\Phi|^2 - |U^{n-1}|^2\rangle &\geq -2|\langle M^n, |\Phi|^2\rangle| - 2|\langle M^n, |U^{n-1}|^2\rangle| \\ &\quad - |\langle |U^n|^2, |\Phi|^2\rangle| - |\langle |U^n|^2, |U^{n-1}|^2\rangle| \\ &\geq -\frac{1}{\epsilon}\|M^n\|_2^2 - \epsilon\|\Phi\|_4^4 - \frac{1}{2\epsilon}\|U^n\|_4^4 - \frac{\epsilon}{2}\|\Phi\|_4^4 \\ &\quad - 2|\langle M^n, |U^{n-1}|^2\rangle| - |\langle |U^n|^2, |U^{n-1}|^2\rangle| \\ &\geq -\frac{3\epsilon}{2}\|\Phi\|_4^4 - C_3,\end{aligned} \tag{3.49}$$

where the constant C_3 obviously depends on M^n, U^n, U^{n-1} and ϵ. The conclusion of this lemma readily follows. □

For the remainder of this work, we will employ the real matrix $A = C + D^{(\beta)}$ of size $(J+1) \times (J+1)$, where

$$C = \frac{1}{\tau^2}\begin{pmatrix} \tau^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tau^2 \end{pmatrix} \tag{3.50}$$

and

$$D^{(\beta)} = \frac{1}{2h^\beta} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & g_2^{(\beta)} & \cdots & g_{J-4}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-2}^{(\beta)} & g_{J-1}^{(\beta)} \\ g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & \cdots & g_{J-5}^{(\beta)} & g_{J-4}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-2}^{(\beta)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ g_{J-2}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-4}^{(\beta)} & g_{J-5}^{(\beta)} & \cdots & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & g_2^{(\beta)} \\ g_{J-1}^{(\beta)} & g_{J-2}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-4}^{(\beta)} & \cdots & g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.51}$$

Lemma 3.2.8. *The real matrix A is strictly diagonally dominant.*

Proof. Using Lemma 3.1.8, we readily check that the following inequalities and identity hold, for each $i \in \{2, \dots, J\}$:

$$\sum_{\substack{j=1 \\ j\neq i}}^{J} |a_{ij}| \leq \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{|g_l^{(\beta)}|}{2h^\beta} < \frac{1}{\tau^2} + \frac{g_0^{(\beta)}}{2h^\beta} = |a_{ii}|. \tag{3.52}$$

Obviously, the first and the last rows of the matrix A also satisfy this condition, whence we readily conclude that A is strictly diagonally dominant, as desired. □

Being strictly diagonally dominant, the matrix A is nonsingular. In addition, the off-diagonal entries of A are non-positive, while the diagonal components are all positive numbers. This means that A is an M-matrix, whence it follows that the entries of A^{-1} are positive real numbers [28]. We will not exploit this feature of A in this work.

3.3 Physical properties

The purpose of the present section is to establish the main physical properties of the discrete model (3.36). More precisely, we establish the existence of solutions of the discrete model and prove that the quantities (3.42) are invariant. Moreover, we prove rigorously the boundedness of the solutions of (3.36). Obviously, these results will be in perfect qualitative agreement with the properties established in the continuous-case scenario (see Section 3.1).

The following result will be the cornerstone to establish the existence of solutions of (3.36).

Lemma 3.3.1 (Leray–Schauder fixed-point theorem)**.** *Let X be a Banach space, and let $F : X \to X$ be continuous and compact. If the set $S = \{x \in X : \lambda F(x) = x$ for some $\lambda \in [0, 1]\}$ is bounded then F has a fixed point.*

Theorem 3.3.2 (Solubility)**.** *The numerical model* (3.36) *is solvable for any set of initial conditions.*

Proof. Observe that the approximation (U^0, M^0) is defined by the initial conditions of (3.36). Let us assume firstly that (U^{n-1}, M^{n-1}) and (U^n, M^n) have been already obtained, for some $n \in I_{N-1}$. Consider the function $F : \mathcal{V}_h \to \mathcal{V}_h$ given by $F = (F_j)_{j\in\overline{I}_J}$, where each $F_j : \mathcal{V}_h \to \mathbb{C}$ is a function. More precisely, let $\Phi \in \mathcal{V}_h$ and set $F_j(\Phi) = 0$ if $j \in \{0, J\}$. In the case when $j \in I_{J-1}$, we define $\forall \Phi \in \mathcal{V}_h$,

$$F_j(\Phi) = 2U_j^n - U_j^{n-1} + \tau^2 \delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U_j^n - \tau^2 \mu_{t,\Phi}^{(1)} U_j^n - \tau^2 M_j^n \left(\mu_{t,\Phi}^{(1)} U_j^n\right) - \tau^2 \left(\mu_{t,\Phi}^{(2)} |U_j^n|^2\right) \left(\mu_{t,\Phi}^{(1)} U_j^n\right). \tag{3.53}$$

It is obvious that F is a continuous and compact map from the Banach space $\mathcal{V}_h$ into itself. We will prove next that the set S of Lemma 3.3.1 is a bounded subset of $\mathcal{V}_h$. To that end, let $\Phi \in \mathcal{V}_h$ and $\lambda \in [0,1]$ satisfy $\lambda F(\Phi) = \Phi$. Equivalently, the following identity holds, for each $j \in I_{J-1}$:

$$0 = \Phi_j - 2\lambda U_j^n + \lambda U_j^{n-1} - \lambda\tau^2 \delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U_j^n + \lambda\tau^2 \mu_{t,\Phi}^{(1)} U_j^n + \lambda\tau^2 \left(\mu_{t,\Phi}^{(1)} U_j^n\right) \left[M_j^n + \mu_{t,\Phi}^{(2)} |U_j^n|^2\right]. \tag{3.54}$$

Take the real part of the inner product of both sides of the equation $0 = \Phi - \lambda F(\Phi)$ with $\Phi - U^{n-1}$ and use the identities and inequalities of Lemma 3.2.7 with $\epsilon < \frac{1}{12}$. Then rearrange terms and simplify to obtain

$$\begin{aligned}0 &\geq \frac{\lambda\tau^2}{8}\left(\|\Phi\|_4^4 - \|U^{n-1}\|_4^4\right) + \frac{\lambda\tau^2}{2}\left(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2 + \|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2\right) - \frac{\lambda}{3}\|\Phi\|_2^2 \\ &\quad + \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 + \frac{\lambda\tau^2}{4}\left[\langle 2M^n, |\Phi|^2 - |U^{n-1}|^2\rangle + \langle |U^n|^2, |\Phi|^2 - |U^{n-1}|^2\rangle\right] - C_1 - C_2 \\ &\geq \left(1 + \frac{\lambda\tau^2}{2} - \frac{3\lambda+1}{6}\right)\|\Phi\|_2^2 + \lambda\tau^2\left(\frac{1}{8} - \frac{3\epsilon}{2}\right)\|\Phi\|_4^4 - C.\end{aligned} \tag{3.55}$$

Here, the constants $C_1, C_2, C_3 \in \mathbb{R}^+$ are those of Lemma 3.2.7, and C is given by

$$C = C_1 + C_2 + C_3 + \frac{\lambda\tau^2}{8}\left(4\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2 + 4\|U^{n-1}\|_2^2 + \|U^{n-1}\|_4^4\right). \tag{3.56}$$

Using the fact that $\lambda \in [0,1]$ and letting $\epsilon < \frac{1}{12}$, we obtain that

$$K_1 = 1 + \frac{\lambda\tau^2}{2} - \frac{3\lambda+1}{6} \geq \frac{1}{3}, \quad \text{and} \quad K_2 = \frac{1}{8} - \frac{3\epsilon}{2} > 0. \tag{3.57}$$

It follows that $\frac{1}{3}\|\Phi\|_2^2 \leq C$. As a consequence, the set S is bounded, and the Leray–Schauder theorem guarantees that the system consisting of all the first difference equations of (3.36) at time t_n has a solution U^{n+1}. It only remains to prove that the system consisting of the second equations is likewise solvable. To that end, observe that those identities can be expressed in vector form as $A\Psi = b$, where Ψ is the unknown vector of approximations at the time t_{n+1}. Additionally, the vector $b \in \mathbb{R}^{J+1}$ is given by

$$b = \begin{pmatrix} 0 \\ \delta_x^{(\beta)}|U_1^n|^2 + \frac{1}{2}\delta_x^{(\beta)}M_1^{n-1} + \frac{1}{\tau^2}\left(2M_1^n - M_1^{n-1}\right) \\ \delta_x^{(\beta)}|U_2^n|^2 + \frac{1}{2}\delta_x^{(\beta)}M_2^{n-1} + \frac{1}{\tau^2}\left(2M_2^n - M_2^{n-1}\right) \\ \vdots \\ \delta_x^{(\beta)}|U_{J-1}^n|^2 + \frac{1}{2}\delta_x^{(\beta)}M_{J-1}^{n-1} + \frac{1}{\tau^2}\left(2M_{J-1}^n - M_{J-1}^{n-1}\right) \\ 0 \end{pmatrix}. \tag{3.58}$$

Observe now that A is strictly diagonally dominant by Lemma 3.2.8, so nonsingular. As a consequence, there exists a vector M^{n+1} which satisfies the system consisting of all the second difference equations of (3.36) at time t_n. Finally, if $n = 0$ then the solubility of the identities (3.37)-(3.38) can be established in a similar fashion. The conclusion of the theorem readily follows now by induction. □

Definition 3.3.3. Given any arbitrary $U, V \in \mathcal{V}_h$, we define their product point-wisely, that is, $UV = (U_j V_j)_{j \in \bar{I}_J}$.

Lemma 3.3.4. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathcal{V}_h$, and assume that U is a sequence of complex functions while the functions of V are real. Then the following are satisfied for each $n \in I_{N-1}$:*

(a) $2\,\mathrm{Re}\langle\delta_t^{(2)}U^n, \delta_t^{(1)}U^n\rangle = \delta_t\|\delta_t U^{n-1}\|_2^2$,

(b) $2\,\mathrm{Re}\langle-\delta_x^{(\alpha)}\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \delta_t\mu_t\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2$,

(c) $2\,\mathrm{Re}\langle\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \delta_t\mu_t\|U^{n-1}\|_2^2$,

(d) $2\,\mathrm{Re}\langle M^n\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \langle M^n, \delta_t^{(1)}|U^n|^2\rangle$, *and*

(e) $8\,\mathrm{Re}\left\langle(\mu_t^{(2)}|U^n|^2)(\mu_t^{(1)}U^n), \delta_t^{(1)}U^n\right\rangle = \delta_t\left[\mu_t\|U^{n-1}\|_4^4 + \langle|U^n|^2, |U^{n-1}|^2\rangle\right]$.

Suppose additionally that there exists $(V^n)_{n\in\overline{I}_N} \subseteq \mathcal{V}_h$ such that (3.39) holds. Then for each $n \in I_{N-1}$,

(f) $2\langle\delta_x^{(\beta)}\mu_t^{(1)}M^n, \mu_t V^{n-1}\rangle = \delta_t\mu_t\|M^{n-1}\|_2^2$,

(g) $\langle\delta_x^{(\beta)}|U^n|^2, \mu_t V^{n-1}\rangle = \langle|U^n|^2, \delta_t^{(1)}M^n\rangle$, *and*

(h) $-2\langle\delta_t^{(2)}M^n, \mu_t V^{n-1}\rangle = \delta_t\|\delta_x^{(\beta/2)}V^{n-1}\|_2^2$.

Proof. The identities (a)–(c) can be established following arguments similar to those in [49]. On the other hand, the formulas (d) and (e) make use of the product introduced in Definition 3.3.3. Indeed, notice that $\forall n \in I_{N-1}$,

$$2\,\mathrm{Re}\langle M^n\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \frac{1}{2\tau}\,\mathrm{Re}\left\langle M^n(U^{n+1}+U^{n-1}), (U^{n+1}-U^{n-1})\right\rangle = \langle M^n, \delta_t^{(1)}\,|U^n|^2\rangle, \tag{3.59}$$

and computing $8\,\mathrm{Re}\left\langle\left(\mu_t^{(2)}\,|U^n|^2\right)\left(\mu_t^{(1)}U^n\right), \delta_t^{(1)}U^n\right\rangle$, we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\,\mathrm{Re}\left\langle\left(\left|U^{n+1}\right|^2 + 2|U^n|^2 + \left|U^{n-1}\right|^2\right)\left(U^{n+1}+U^{n-1}\right), \left(U^{n+1}-U^{n-1}\right)\right\rangle \\
&\quad = \frac{1}{2\tau}\left[h\sum_j\left(\left|U_j^{n+1}\right|^4 + 2|U_j^{n+1}|^2|U_j^n|^2 - 2|U_j^n|^2|U_j^{n-1}|^2 - \left|U_j^{n-1}\right|^4\right)\right] \\
&\quad = \delta_t\left[\mu_t\|U^{n-1}\|_4^4 + \langle|U^n|^2, |U^{n-1}|^2\rangle\right], \quad \forall n \in I_{N-1}.
\end{aligned} \tag{3.60}$$

To establish now (f) and (g), we assume that $(V^n)_{n\in\overline{I}_N}$ satisfies condition (3.39). Observe then that $\forall n \in I_{N-1}$,

$$2\langle\delta_t^{(\beta)}\mu_t^{(1)}M^n, \mu_t V^{n-1}\rangle = 2\langle\mu_t^{(1)}M^n, \mu_t\delta_x^{(\beta)}V^{n-1}\rangle = 2\langle\mu_t^{(1)}M^n, \delta_t^{(1)}M^n\rangle = \delta_t\mu_t\|M^{n-1}\|_2^2, \tag{3.61}$$

and

$$\langle\delta_x^{(\beta)}|U^n|^2, \mu_t V^{n-1}\rangle = \langle|U^n|^2, \mu_t\delta_x^{(\beta)}V^{n-1}\rangle = \langle|U^n|^2, \mu_t\delta_t M^{n-1}\rangle = \langle|U^n|^2, \delta_t^{(1)}M^n\rangle. \tag{3.62}$$

In similar fashion, it is easy to establish the validity of identity (h) using Lemma 3.2.3 and the formula (3.39). □

Let $(U^n)_{n\in I_{N-1}}, (M^n)_{n\in I_{N-1}} \subseteq \mathcal{V}_h$. We define $L = L_U \times L_M : \mathcal{V}_h \times \mathcal{V}_h \to \mathcal{V}_h \times \mathcal{V}_h$ by

$$L_U(U_j^n, M_j^n) = \delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n \left[1 + M_j^n + \mu_t^{(2)} |U_j^n|^2\right], \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_N, \tag{3.63}$$

and

$$L_M(U_j^n, M_j^n) = \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2, \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_N. \tag{3.64}$$

Then, $L(U^n, M^n) = (L(U_j^n, M_j^n))_{j\in\overline{I}_J}$ for each $n \in \overline{I}_{N-1}$, and let $L(U,M) = (L(U^n, M^n))_{n\in\overline{I}_{N-1}}$. On the other hand, we will let $u^n = (u_j^n)_{j\in\overline{I}_J}$ and $m^n = (m_j^n)_{j\in\overline{I}_J}$. Moreover, we agree that $u = (u^n)_{n\in\overline{I}_N}$ and $m = (m^n)_{n\in\overline{I}_N}$. Let us introduce also the continuous operator $\mathcal{L} = \mathcal{L}_u \times \mathcal{L}_m$, defined for each (u,m) and $\forall (x,t) \in \Omega$, by

$$\mathcal{L}_u(u(x,t), m(x,t)) = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t), \tag{3.65}$$

$$\mathcal{L}_m(u(x,t), m(x,t)) = \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta}. \tag{3.66}$$

Also, for each $x \in \{x_L, x_R\}$ and $t \in [0,T]$, we let $\mathcal{L}(u(x,t), m(x,t)) = 0$. Let $\mathcal{L}(u^n, m^n) = (\mathcal{L}(u_j^n, m_j^n))_{j\in\overline{I}_J}$ for each $n \in I_{N-1}$, and define $\mathcal{L}(u,v) = (\mathcal{L}(u^n, m^n))_{n\in I_{N-1}}$. Similarly, let $L(u^n, m^n) = (L(u_j^n, m_j^n))_{j\in\overline{I}_J} \in \mathcal{V}_h$ for each $n \in I_{N-1}$, and introduce $L(u,m) = (L(u^n, m^n))_{n\in I_{N-1}}$.

Theorem 3.3.5 (Energy conservation)**.** *If (U,M) is a solution of* (3.5) *then the quantities* (3.42) *are constant.*

Proof. Suppose that (U,M) is a solution of the finite-difference scheme (3.36), and let $n \in I_{N-1}$. Notice beforehand that $\operatorname{Re}\langle L_U(U^n, M^n), \delta_t^{(1)} U^n\rangle = \langle L_M(U^n, M^n), \mu_t V^{n-1}\rangle = 0$. As a consequence of this and the identities in Lemma 3.3.4, we obtain the system of algebraic equations

$$\begin{aligned} 0 = {} & \frac{1}{2}\delta_t \left[\|\delta_t U^{n-1}\|_2^2 + \mu_t \left(\|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2 + \|U^{n-1}\|_2^2 + \frac{1}{4}\|U^{n-1}\|_4^4\right) + \frac{1}{4}\langle |U^n|^2, |U^{n-1}|^2\rangle\right] \\ & + \frac{1}{2}\langle M^n, \delta_t^{(1)} |U^n|^2\rangle, \end{aligned} \tag{3.67}$$

$$0 = \frac{1}{2}\delta_t \left[\|\delta_x^{(\beta/2)} V^{n-1}\|_2^2 + \mu_t \|M^{n-1}\|_2^2\right] + \langle |U^n|^2, \delta_t^{(1)} M^n\rangle. \tag{3.68}$$

Multiply both ends of (3.67) by 2, and notice that

$$\langle M^n, \delta_t^{(1)} |U^n|^2\rangle + \langle |U^n|^2, \delta_t^{(1)} M^n\rangle = \frac{1}{2}\delta_t \left[\langle M^{n-1}, |U^n|^2\rangle + \langle M^n, |U^{n-1}|^2\rangle\right]. \tag{3.69}$$

Finally, add the result to (3.68) to obtain that $\delta_t E^{n-1} = 0$. As a consequence, $E^n = E^{n-1}$ for all $n \in I_{N-1}$. The fact that the quantities (3.42) are constant follows now by induction. □

Definition 3.3.6. Let $\sigma \in [0,1]$. Define the *fractional Sobolev norm* and *semi-norm* $\|\cdot\|_{H^\sigma}, |\cdot|_{H^\sigma} : \mathcal{V}_h \to \mathbb{R}$ by

$$\|U\|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} (1 + |k|^{2\sigma}) |\widehat{U}(k)|^2 dk, \quad \forall U \in \mathcal{V}_h, \tag{3.70}$$

$$|U|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} |k|^{2\sigma} |\widehat{U}(k)|^2 dk, \quad \forall U \in \mathcal{V}_h, \tag{3.71}$$

respectively. Alternatively, $\|U\|_{H^\sigma}^2 = \|U\|_2^2 + |U|_{H^\sigma}^2$ and $|U|_{H^0}^2 = \|U\|_2^2$, for each $U \in \mathcal{V}_h$. The *fractional Sobolev space* H^σ is the set of all $U \in \mathcal{V}_h$ such that $\|U\|_{H^\sigma} < \infty$.

Lemma 3.3.7 (Wang *et al.* [94]).

(a) *For each* $\frac{1}{2} < \sigma \leq 1$, *there exists a constant* $C_\sigma > 0$ *independent of* h *such that if* $U \in H^\sigma$ *then* $\|U\|_\infty \leq C_\sigma \|U\|_{H^\sigma}$.

(b) *For each* $\frac{1}{4} < \sigma_0 \leq 1$, *there exists a constant* $C_{\sigma_0} > 0$ *independent of* h *such that if* $\sigma_0 \leq \sigma \leq 1$ *and* $U \in H^\sigma$ *then* $\|U\|_\infty \leq C_{\sigma_0} \|U\|_{H^\sigma}^{\sigma_0/\sigma} \|U\|_2^{1-\sigma_0/\sigma}$.

(c) *For each* $\alpha \in (1,2)$ *there is a constant* $C > 0$ *such that* $C|U|^2_{H^{\alpha/2}} \leq |\langle \delta_x^{(\alpha)} U, U\rangle| \leq |U|^2_{H^{\alpha/2}}$, *for each* $U \in H^\alpha$.

(d) *For each* $\alpha \in (1,2)$ *and* $U, V \in H^\alpha$, *there exists* $C > 0$ *such that* $C|U|_{H^{\alpha/2}}|V|_{H^{\alpha/2}} \leq |\langle \delta_x^{(\alpha)} U, V\rangle| \leq |U|_{H^{\alpha/2}}|V|_{H^{\alpha/2}}$.

Theorem 3.3.8 (Boundedness). *Let* $u_0, m_0 \in H^1$ *and* $u_1, m_1 \in L_2$, *and suppose that* (U, M) *is the corresponding solution of* (3.36). *Then there exists* $C \geq 0$ *such that*

$$\max\left\{\|\delta_t U^{n-1}\|_2^2, \|\delta_x^{(\alpha/2)} U^n\|_2^2, \|U^n\|_2^2, \|U^n\|_\infty, \|\delta_x^{(\beta/2)} V^{n-1}\|_2^2, \|M^n\|_2^2, \|U^n\|_4^4\right\} \leq C, \quad \forall n \in \overline{I}_N. \tag{3.72}$$

Proof. The conclusion of the theorem will be reached using mathematical induction. Note beforehand that Theorem 3.3.2 assures that the numerical model (3.36) has a solution. Moreover, in light of Theorem 3.3.5, the quantities E^n are all equal to a constant $C_0 \geq 0$. On the other hand, some applications of Young's inequality readily show that the following relations are satisfied:

$$\frac{1}{2}\left|\langle M^n, |U^{n+1}|^2\rangle\right| \leq \|M^n\|_2^2 + \frac{1}{16}\|U^{n+1}\|_4^4, \quad \forall n \in \overline{I}_{N-1}, \tag{3.73}$$

$$\frac{1}{2}\left|\langle M^{n+1}, |U^n|^2\rangle\right| \leq \frac{1}{8}\|M^{n+1}\|_2^2 + \frac{1}{2}\|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{3.74}$$

In addition, note that $\langle |U^{n+1}|^2, |U^n|^2\rangle$ is nonnegative, for each $n \in \overline{I}_{N-1}$. Using all these inequalities and the expression of the discrete energy quantities (3.42), rearranging terms and simplifying, we obtain that, for each $n \in \overline{I}_{N-1}$,

$$\begin{aligned} C_0 + \frac{3}{4}\|M^n\|_2^2 + \frac{3}{8}\|U^n\|_4^4 &\geq \|\delta_t U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 \\ &\quad + \frac{1}{16}\|U^{n+1}\|_4^4 + \frac{1}{8}\|M^{n+1}\|_2^2 \\ &\geq \|\delta_t U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\alpha/2)} U^{n+1}\|_2^2 + \frac{1}{2}\|U^{n+1}\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 \\ &\quad + \frac{1}{16}\|U^{n+1}\|_4^4 + \frac{1}{8}\|M^{n+1}\|_2^2. \end{aligned} \tag{3.75}$$

Note that there exists $C^n \geq 0$ such that

$$\max\left\{\|\delta_t U^{n-1}\|_2^2, \|\delta_x^{(\alpha/2)} U^n\|_2^2, \|U^n\|_2^2, \|U^n\|_\infty, \|\delta_x^{(\beta/2)} V^{n-1}\|_2^2, \|M^n\|_2^2, \|U^n\|_4^4\right\} \leq C^n$$

when $n = 0$. Proceeding by induction, suppose now that this assertion is satisfied for some $n \in \overline{I}_{N-1}$. This assumption and the inequality (3.75) show that

$$\|\delta_t U^n\|_2^2 + \|\delta_x^{(\alpha/2)} U^{n+1}\|_2^2 + \|U^{n+1}\|_2^2 + \|\delta_x^{(\beta/2)} V^n\|_2^2 + \|U^{n+1}\|_4^4 + \|M^{n+1}\|_2^2 \leq C_1^{n+1}, \tag{3.76}$$

where $C_1^{n+1} = 16(C_0 + \frac{9}{8}C^n) \geq 0$. On the other hand, an application of Lemma 3.3.7(c) shows that there exists a constant $C_2 \in \mathbb{R}^+$ independent of U^n, such that $C_2|U^n|^2_{H^{\alpha/2}} \leq \|\delta_x^{(\alpha/2)} U^n\|_2^2$.

This together with Lemma 3.3.7(a) and the properties of fractional Sobolev norms show then that there exists a constant $C_3 \in \mathbb{R}^+$ independent of U^n, such that

$$\|U^n\|_\infty^2 \leq C_3^2 \|U^n\|_{H^{\alpha/2}}^2 = C_3^2 \left(\|U^n\|_2^2 + |U^n|_{H^{\alpha/2}}^2 \right) \leq (C_2^{n+1})^2, \quad \forall n \in I_{N-1}, \tag{3.77}$$

where $C_2^{n+1} = C_1 C_3 \sqrt{1 + C_2} / \sqrt{C_2}$. Let $C^{n+1} = C_1^{n+1} \vee C_2^{n+1}$. The conclusion follows now by induction when we define the nonnegative constant $C = \max\{C^n : n \in \overline{I}_N\}$. □

3.4 Numerical properties

The purpose of the present section is to provide the most important numerical properties of the finite-difference scheme (3.5), namely, its consistency, stability and convergence. As a consequence of the stability and the existence of solutions of the discrete model, the uniqueness of solutions will be readily derived as a corollary. In a first stage, we will establish the consistency properties of the numerical model (3.36) and its associated discrete energy density.

Theorem 3.4.1 (Consistency). *If $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$ then there exist constants $C, C' \in \overline{\mathbb{R}^+}$ which are independent of h and τ, such that $|||\mathcal{L}(u,m) - L(u,m)|||_\infty \leq C(\tau^2 + h^2)$ and $|||\mathcal{H}(u,m) - H(u,m)|||_\infty \leq C(\tau + h)$*

Proof. Using the regularity of the functions u and m along with Theorem 3.1.9 and Taylor's theorem, there exist constants $C_{1,i} \geq 0$ which are independent of h and τ for each $i \in I_5$, such that the following inequalities are satisfied:

$$\left| \frac{\partial^2 u(x_j, t_n)}{\partial t^2} - \delta_t^{(2)} u_j^n \right| \leq C_1^u \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.78}$$

$$\left| \frac{\partial^\alpha u(x_j, t_n)}{\partial |x|^\alpha} - \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n \right| \leq C_2^u (h^2 + \tau^2), \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.79}$$

$$\left| u(x_j, t_n) - \mu_t^{(1)} u_j^n \right| \leq C_3^u \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.80}$$

$$\left| m(x_j, t_n) u(x_j, t_n) - m_j^n \mu_t^{(1)} u_j^n \right| \leq C_4^u \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.81}$$

$$\left| |u(x_j, t_n)|^2 u(x_j, t_n) - \left(\mu_t^{(2)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right) \right| \leq C_5^u \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}. \tag{3.82}$$

It is obvious that $|||\mathcal{L}_u(u,m) - L_u(u,m)|||_\infty \leq C^u(\tau^2 + h^2)$ if we let $C^u = \max\{C_i^u : i \in I_5\}$. Similarly, it is easy to see that there exist constants $C_i^m \geq 0$ which are independent of h and τ for each $i \in I_3$, with the property that

$$\left| \frac{\partial^2 m(x_j, t_n)}{\partial t^2} - \delta_t^{(2)} m_j^n \right| \leq C_1^m \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.83}$$

$$\left| \frac{\partial^\beta m(x_j, t_n)}{\partial |x|^\beta} - \delta_x^{(\beta)} \mu_t^{(1)} m_j^n \right| \leq C_2^m (h^2 + \tau^2), \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{3.84}$$

$$\left| \frac{\partial^\beta \left(|u(x_j, t_n)|^2 \right)}{\partial |x|^\beta} - \delta_x^{(\beta)} |u_j^n|^2 \right| \leq C_3^m \tau^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}. \tag{3.85}$$

Defining $C^m = \max\{C_i^m : i \in I_3\}$ and using the triangle inequality, it is possible to establish that the inequality $|||\mathcal{L}_m(u,m) - L_m(u,m)|||_\infty \leq C^m(\tau^2 + h^2)$ holds. As a consequence, the first inequality of the conclusions is reached when $C = C^u \vee C^m$. The second inequality of this theorem may be readily reached using similar arguments. □

We turn our attention to the properties of stability and convergence of the discrete model (3.36). The following technical results will be important tools to establish those properties.

Lemma 3.4.2 (Macías-Díaz [49]). *If $V \in \mathcal{V}_h$ and $\alpha \in (1,2]$ then*

(a) $\|\delta_x^{(\alpha/2)} V\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|V\|_2^2$,

(b) $\|\delta_x^{(\alpha)} V\|_2^2 = \|\delta_x^{(\alpha/2)} \delta_x^{(\alpha/2)} V\|_2^2$, *and*

(c) $\|\delta_x^{(\alpha)} V\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|\delta_x^{(\alpha/2)} V\|_2^2 \leq 4 \left(g_0^{(\alpha)} h^{1-\alpha}\right)^2 \|V\|_2^2$.

Lemma 3.4.3 (Pen-Yu [75]). *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative real numbers, assume that $\tau > 0$ and suppose that there exists $C \geq 0$ such that*

$$\omega^k \leq \rho^k + C\tau \sum_{n=0}^{k} \omega^n, \quad \forall k \in \overline{I}_N. \tag{3.86}$$

If τ is sufficiently small then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in \overline{I}_N$.

In the following theorem, we will consider two sets of initial conditions (u_0, u_1, m_0, m_1) and $(\tilde{u}_0, \tilde{u}_1, \tilde{m}_0, \tilde{m}_1)$ for the discrete problem (3.36). The functions will satisfy the conditions of Theorem 3.3.2, whence the solutions corresponding to these two sets will exist, and will be denoted by (U, M) and $(\tilde{U}, \tilde{M})$, respectively. More precisely, the pair (U, M) satisfies (3.36), while $(\tilde{U}, \tilde{M})$ satisfies the discrete initial-boundary-value problem

$$\begin{aligned}
\delta_t^{(2)} \tilde{U}_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} \tilde{U}_j^n + \mu_t^{(1)} \tilde{U}_j^n + \tilde{M}_j^n \mu_t^{(1)} \tilde{U}_j^n + \left(\mu_t^{(2)} |\tilde{U}_j^n|^2\right) \left(\mu_t^{(1)} \tilde{U}_j^n\right) = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\delta_t^{(2)} \tilde{M}_j^n - \delta_x^{(\beta)} \mu_t^{(1)} \tilde{M}_j^n - \delta_x^{(\beta)} |\tilde{U}_j^n|^2 = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\text{subject to} \begin{cases} \tilde{U}_j^0 = \tilde{u}_0(x_j), & \tilde{M}_j^0 = \tilde{m}_0(x_j), \quad \forall j \in I_{J-1}, \\ \delta_t^{(1)} \tilde{U}_j^0 = \tilde{u}_1(x_j) & \delta_t^{(1)} \tilde{M}_j^0 = \tilde{m}_1(x_j), \quad \forall j \in I_{J-1}, \\ \tilde{U}_0^n = \tilde{U}_J^n = 0, & \tilde{M}_0^n = \tilde{M}_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases}
\end{aligned} \tag{3.87}$$

Definition 3.4.4. Suppose that (U, M) and $\tilde{U}, \tilde{M})$ are two solutions of (3.36) corresponding to the initial conditions (u_0, u_1, m_0, m_1) and $(\tilde{u}_0, \tilde{u}_1, \tilde{m}_0, \tilde{m}_1)$, respectively. If $g : \mathbb{R}^2 \to \mathbb{R}$ is any function then we define $\tilde{\delta} : \mathcal{V}_h \times \mathcal{V}_h \to \mathcal{V}_h$ for each $(j,n) \in \overline{I}_J \times \overline{I}_N$, through $\tilde{\delta}[g(U_j^n, M_j^n)] = g(U_j^n, M_j^n) - g(\tilde{U}_j^n, \tilde{M}_j^n)$.

Lemma 3.4.5. *Let $u_0, \tilde{u}_0, m_0, \tilde{m}_0 \in H^1$ and $u_1, \tilde{u}_1, m_1, \tilde{m}_1 \in L_2$, and suppose that (u_0, u_1, m_0, m_1) and $(\tilde{u}_0, \tilde{u}_1, \tilde{m}_0, \tilde{m}_1)$ are sets of initial conditions, and let (U, M) and $\tilde{U}, \tilde{M})$ be the respective solutions obtained by (3.36). Let $\epsilon^n = U^n - \tilde{U}^n$ and $\zeta^n = M^n - \tilde{M}^n$, for each $n \in \overline{I}_N$, and define the constants*

$$\rho = 2 \left(\|\delta_t \epsilon^{-1}\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} \epsilon^{-1}\|_2^2 + \mu_t \|\epsilon^{-1}\|_2^2 + \|\delta_x^{(\beta/2)} \upsilon^{-1}\|_2^2 + \mu_t \|\zeta^{-1}\|_2^2 \right), \tag{3.88}$$

$$\omega^k = \|\delta_t \epsilon^{k-1}\|_2^2 + \|\delta_x^{(\alpha/2)} \epsilon^k\|_2^2 + \|\epsilon^k\|_2^2 + \|\delta_t \zeta^{k-1}\|_2^2 + \|\zeta^k\|_2^2, \quad \forall k \in \overline{I}_{N-1}. \tag{3.89}$$

There exists a constant $C \geq 0$ such that, if τ is sufficiently small, then $\omega^k \leq C\rho$ for each $k \in \overline{I}_{N-1}$.

Proof. Beforehand, notice that Theorems 3.3.2 and 3.3.8 guarantee that the solutions (U, M) and $(\tilde{U}, \tilde{M})$ exist and are bounded. This implies in particular that the set $\{\|U^n\|_p, \|\tilde{U}^n\|_p, \|M^n\|_p, \|\tilde{M}^n\|_p :$

$n \in \overline{I}_N, p = 2, \infty\}$ is bounded. On the other hand, note that the pair (ϵ, ζ) satisfies the algebraic system

$$
\begin{aligned}
\delta_t^{(2)}\epsilon_j^n - \delta_x^{(\alpha)}\mu_t^{(1)}\epsilon_j^n + \mu_t^{(1)}\epsilon_j^n + \tilde{\delta}\left[M_j^n\mu_t^{(1)}U_j^n + \left(\mu_t^{(2)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\delta_t^{(2)}\zeta_j^n - \delta_x^{(\beta)}\mu_t^{(1)}\zeta_j^n - \tilde{\delta}\left(\delta_x^{(\beta)}|U_j^n|^2\right) = 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\text{subject to}
\begin{cases}
\epsilon_j^0 = u_0(x_j) - \tilde{u}_0(x_j), & \zeta_j^0 = m_0(x_j) - \tilde{m}_0(x_j), \quad \forall j \in I_{J-1}, \\
\delta_t^{(1)}\epsilon_j^0 = u_1(x_j) - \tilde{u}_1(x_j) & \delta_t^{(1)}\zeta_j^0 = m_1(x_j) - \tilde{m}_1(x_j), \quad \forall j \in I_{J-1}, \\
\epsilon_0^n = \epsilon_J^n = 0, & \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N.
\end{cases}
\end{aligned}
\tag{3.90}
$$

For the sake of convenience, we agree that $\upsilon^n = V^n - \tilde{V}^n$, for each $n \in \overline{I}_N$. In light of (3.39), it is clear that $\delta_x^{(\beta)}\upsilon^n = \delta_t\zeta^n$, for each $n \in \overline{I}_{N-1}$. Observe that Lemma 3.3.4 guarantees that the identities (a)–(c) are satisfied when U is replaced by ϵ. Additionally, the identities (f) and (h) of that lemma are also satisfied with ζ and υ instead of M and V, respectively. More precisely, the following relations hold for each $n \in \overline{I}_{N-1}$:

(i) $2\,\mathrm{Re}\langle\delta_t^{(2)}\epsilon^n, \delta_t^{(1)}\epsilon^n\rangle = \delta_t\|\delta_t\epsilon^{n-1}\|_2^2$,

(ii) $2\,\mathrm{Re}\langle-\delta_x^{(\alpha)}\mu_t^{(1)}\epsilon^n, \delta_t^{(1)}\epsilon^n\rangle = \delta_t\mu_t\|\delta_x^{(\alpha/2)}\epsilon^{n-1}\|_2^2$,

(iii) $2\,\mathrm{Re}\langle\mu_t^{(1)}\epsilon^n, \delta_t^{(1)}\epsilon^n\rangle = \delta_t\mu_t\|\epsilon^{n-1}\|_2^2$,

(iv) $-2\langle\delta_t^{(2)}\zeta^n, \mu_t\upsilon^{n-1}\rangle = \delta_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2$, and

(v) $2\langle\delta_x^{(\beta)}\mu_t^{(1)}\zeta^n, \mu_t\upsilon^{n-1}\rangle = \delta_t\mu_t\|\zeta^{n-1}\|_2^2$.

In addition, some applications of Young's inequality and the boundedness of the numerical solutions show that there exists a constant $C_1 \geq 0$ such that

$$
\begin{aligned}
\mathrm{Re}\left\langle\tilde{\delta}\left[M^n\mu_t^{(1)}U^n\right], \delta_t^{(1)}\epsilon^n\right\rangle &= \left|\left\langle\zeta^n\mu_t^{(1)}U^n, \mu_t\delta_t\epsilon^{n-1}\right\rangle\right| + \left|\left\langle\tilde{M}^n\mu_t^{(1)}\epsilon^n, \mu_t\delta_t\epsilon^{n-1}\right\rangle\right| \\
&\leq C_1\left(\|\zeta^n\|_2^2 + \mu_t^{(1)}\|\epsilon^n\|_2^2 + \mu_t\|\delta_t\epsilon^{n-1}\|_2^2\right), \quad \forall n \in \overline{I}_{N-1}.
\end{aligned}
\tag{3.91}
$$

Similarly, using again Young's inequality and some straightforward algebraic arguments, it is easy to check that there exist nonnegative constants C_2 and C_3, such that

$$
\mathrm{Re}\left\langle\tilde{\delta}\left[\left(\mu_t^{(2)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right], \delta_t^{(1)}\epsilon^n\right\rangle \leq C_2\left(\mu_t^{(1)}\|\epsilon^n\|_2^2 + \mu_t\|\delta_t\epsilon^{n-1}\|_2^2\right), \quad \forall n \in \overline{I}_{N-1}, \tag{3.92}
$$

$$
\left\langle\tilde{\delta}\left(\delta_x^{(\beta)}|U^n|^2\right), \mu_t\upsilon^{n-1}\right\rangle \leq C_3\left(\|\epsilon^n\|_2^2 + \mu_t\|\delta_t\zeta^{n-1}\|_2^2\right), \quad \forall n \in \overline{I}_{N-1}. \tag{3.93}
$$

Now, take the inner product of $\delta_t^{(1)}\epsilon^n$ with the vector system consisting of the first equations of (3.90) at the time t_n, take then the real part and rearrange terms. At the same time, take the inner product of $\mu_t\upsilon^{n-1}$ with the vector of the second difference equations of (3.90) at the time t_n. Next, use the identities of Lemma 3.3.4 and the inequalities above to show that there exist nonnegative constants C_4 and C_5, such that the following inequalities hold for each $n \in \overline{I}_{N-1}$:

$$
\delta_t\|\delta_t\epsilon^{n-1}\|_2^2 + \delta_t\mu_t\|\delta_x^{(\alpha/2)}\epsilon^{n-1}\|_2^2 + \delta_t\mu_t\|\epsilon^{n-1}\|_2^2 \leq C_4\left(\|\zeta^n\|_2^2 + \mu_t^{(1)}\|\epsilon^n\|_2^2 + \mu_t\|\delta_t\epsilon^{n-1}\|_2^2\right), \tag{3.94}
$$

$$
\delta_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \delta_t\mu_t\|\zeta^{n-1}\|_2^2 \leq C_5\left(\|\epsilon^n\|_2^2 + \mu_t\|\delta_t\zeta^{n-1}\|_2^2\right). \tag{3.95}
$$

Fix $k \in \overline{I}_{N-1}$, add the last two inequalities, multiply the result by τ and calculate the sum over all indexes $n \in \overline{I}_k$ on both sides of the resulting expression. After using the formula for telescoping sums, using the properties of Lemma 3.4.2, rearranging terms and simplifying algebraically, it is possible to check that there exist nonnegative constants C_6, C_7 and C_8, such that the following inequalities are satisfied for each $k \in \overline{I}_{N-1}$:

$$\begin{aligned}\omega^{k+1} &\leq C_6 \left(\|\delta_t \epsilon^k\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} \epsilon^k\|_2^2 + \mu_t \|\epsilon^k\|_2^2 + \|\delta_x^{(\beta/2)} \upsilon^k\|_2^2 + \mu_t \|\zeta^k\|_2^2 \right) \\ &\leq C_6 \rho + C_7 \tau \sum_{n=0}^{k} \left(\|\zeta^n\|_2^2 + \mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 + \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \zeta^{n-1}\|_2^2 \right) \\ &\leq C_6 \rho + C_8 \tau \sum_{n=-1}^{k+1} \omega^n.\end{aligned} \tag{3.96}$$

It is worth pointing out here that the first inequality was obtained using the fact that

$$\|\delta_t \zeta^n\|_2^2 \leq 2 g_0^{(\beta)} h^{1-\beta} \|\delta_x^{(\beta/2)} \upsilon^n\|_2^2.$$

It is obvious that the conclusion of this theorem readily follows from Lemma 3.4.3, letting $C = C_6 e^{C_8 T}$. ☐

The following results are straightforward consequences of Lemma 3.4.5.

Theorem 3.4.6 (Stability)**.** *Let (u_0, u_1, m_0, m_1) and $(\tilde{u}_0, \tilde{u}_1, \tilde{m}_0, \tilde{m}_1)$ be sets of initial conditions with $u_0, \tilde{u}_0, m_0, \tilde{m}_0 \in H^1$ and $u_1, \tilde{u}_1, m_1, \tilde{m}_1 \in L_2$, and let (U, M) and $(\tilde{U}, \tilde{M})$ be the respective numerical solutions. If ρ, ϵ^n and ζ^n are as in Lemma 3.4.5 for each $n \in \overline{I}_N$, then there exists a constant $C \geq 0$ such that $\max\{\|\epsilon^k\|_2^2, \|\zeta^k\|_2^2\} \leq C\rho$, for each $k \in \overline{I}_N$.* ☐

Theorem 3.4.7 (Uniqueness)**.** *Let (u_0, u_1, m_0, m_1) be a set of initial conditions satisfying $u_1, \tilde{u}_1 \in L_2$ and $u_0, \tilde{u}_0, m_0 \in H^1$. For sufficiently small values of τ, the finite-difference scheme (3.36) is uniquely solvable.* ☐

Theorem 3.4.8 (Convergence)**.** *Let $u_0, m_0 \in H^1$ and $u_1, m_1 \in L_2$, and assume that there exists a unique solution (u, m) of (3.5) which satisfies $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. If τ is sufficiently small then the solutions of (3.36) converge to the exact solution with order of convergence $\mathcal{O}(h^2 + \tau^2)$ in the norm $\|\cdot\|_2$.*

Finally, we tackle the problem of establishing the convergence of the finite-difference scheme (3.36). More precisely, we will show that the numerical model converges to the exact solution of the continuous system (3.5) with quadratic order in both space and time. To that end, we will consider a fixed (though arbitrary) set of initial conditions (u_0, u_1, m_0, m_1) which is common to the continuous problem (3.5) and the discrete model (3.36). Recall that the solution of the former problem is denoted by (u, m), while the solution of the latter is represented by (U, M). Moreover, the pair (u, m) satisfies the discrete system

$$\begin{aligned}\delta_t^{(2)} u_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n + \mu_t^{(1)} u_j^n + m_j^n \mu_t^{(1)} u_j^n + \left(\mu_t^{(2)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right) = \rho_j^n, \quad \forall (j, n) \in I_{J-1} \times \overline{I}_N, \\ \delta_t^{(2)} m_j^n - \delta_x^{(\beta)} \mu_t^{(1)} m_j^n - \delta_x^{(\beta)} |u_j^n|^2 = \sigma_j^n, \quad \forall (j, n) \in I_{J-1} \times \overline{I}_N, \\ \text{subject to} \begin{cases} u_j^0 = u_0(x_j), & m_j^0 = m_0(x_j), & \forall j \in I_{J-1}, \\ \delta_t^{(1)} u_j^0 = u_1(x_j) & \delta_t^{(1)} m_j^0 = m_1(x_j), & \forall j \in I_{J-1}, \\ u_0^n = u_J^n = 0, & m_0^n = m_J^n = 0, & \forall n \in \overline{I}_N. \end{cases}\end{aligned} \tag{3.97}$$

Here, ρ_j^n and σ_j^n represent the respective local truncation errors. Notice that, under the assumptions of Theorem 3.4.1, there is a constant C independent of h and t, such that $|||\rho|||_\infty \leq C(\tau^2 + h^2)$ and $|||\sigma|||_\infty \leq C(\tau^2 + h^2)$. In the following, if $g : \mathbb{R}^2 \to \mathbb{R}$ then we let $\hat{\delta} : \mathcal{V}_h \times \mathcal{V}_h \to \mathcal{V}_h$ be given by $\hat{\delta}[g(u_j^n, m_j^n)] = g(u_j^n, m_j^n) - g(U_j^n, M_j^n)$, for each $(j, n) \in \overline{I}_J \times \overline{I}_N$.

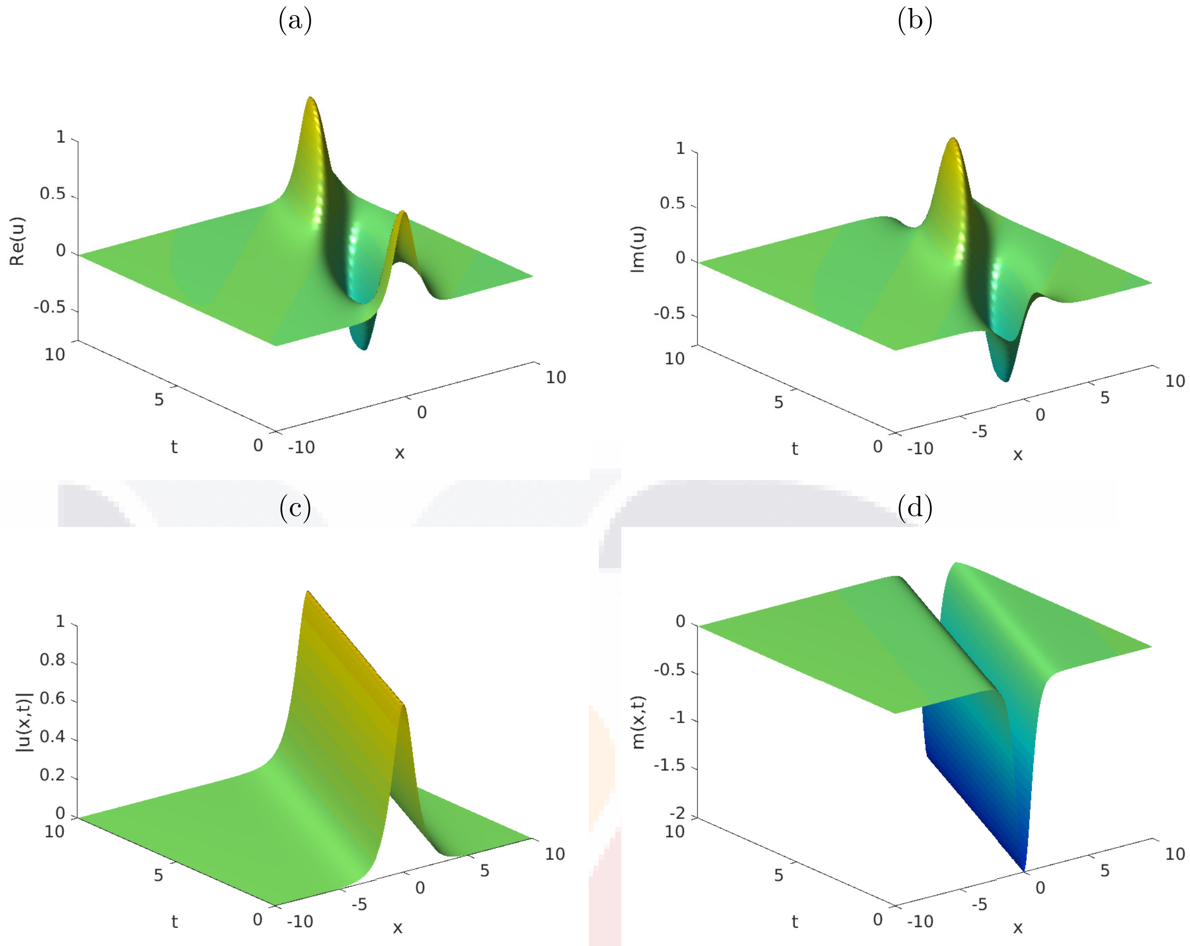

Figure 3.1: Approximate solution of the problem (3.5) versus x and t, using $\Omega = (-20, 20) \times (0, 10)$ and $\alpha = 2$. The graphs correspond to (a) $\operatorname{Re} u$, (a) $\operatorname{Im} u$, (c) $|u(x,t)|$ and (d) $m(x,t)$, and they were obtained using the initial data (3.105)–(3.108). Computationally, we used the finite-difference method (3.36) with $h = 0.05$ and $\tau = 0.1$.

Proof. To start with, let $\epsilon^n = u^n - U^n$ and $\zeta^n = m^n - M^n$, for each $n \in \overline{I}_N$. Again, notice that Theorems 3.1.6 and 3.3.8 guarantee that the boundedness of $\{\|u^n\|_{x,p}, \|m^n\|_{x,p}, \|U^n\|_p, \|M^n\|_p : n \in \overline{I}_N, p = 2, \infty\}$. Moreover, it is easy to check that the following discrete problem is satisfied by (ϵ, ζ):

$$\begin{aligned} \delta_t^{(2)} \epsilon_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} \epsilon_j^n + \mu_t^{(1)} \epsilon_j^n + \hat{\delta} \left[m_j^n \mu_t^{(1)} u_j^n + \left(\mu_t^{(2)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right) \right] = \rho_j^n, \quad \forall (j,n) \in I, \\ \delta_t^{(2)} \zeta_j^n - \delta_x^{(\beta)} \mu_t^{(1)} \zeta_j^n - \hat{\delta} \left(\delta_x^{(\beta)} |u_j^n|^2 \right) = \sigma_j^n, \quad \forall (j,n) \in I, \\ \text{subject to} \begin{cases} \epsilon_j^0 = 0, & \zeta_j^0 = 0, & \forall j \in \overline{I}_J, \\ \delta_t^{(1)} \epsilon_j^0 = 0 & \delta_t^{(1)} \zeta_j^0 = 0, & \forall j \in I_{J-1}, \\ \epsilon_0^n = \epsilon_J^n = 0, & \zeta_0^n = \zeta_J^n = 0, & \forall n \in \overline{I}_N. \end{cases} \end{aligned} \tag{3.98}$$

Proceeding now as in the proof of Lemma 3.4.5, we let $(v^n)_{n \in \overline{I}_N}$ be a sequence in $\mathcal{V}_h$ which satisfies $\delta_x^{(\beta/2)} v^n = \delta_t \zeta^n$, for each $n \in \overline{I}_{N-1}$. It is easy to check then that the identities (i)–(v) of the proof of that lemma are also satisfied in the present case, along with the inequalities (3.91)–(3.93). Now, take the inner product of $\delta_t^{(1)} \epsilon^n$ with the vector the first equations of (3.98) at the time t_n, take then the real part and rearrange terms. Also, take the inner product of $\mu_t v^{n-1}$ with

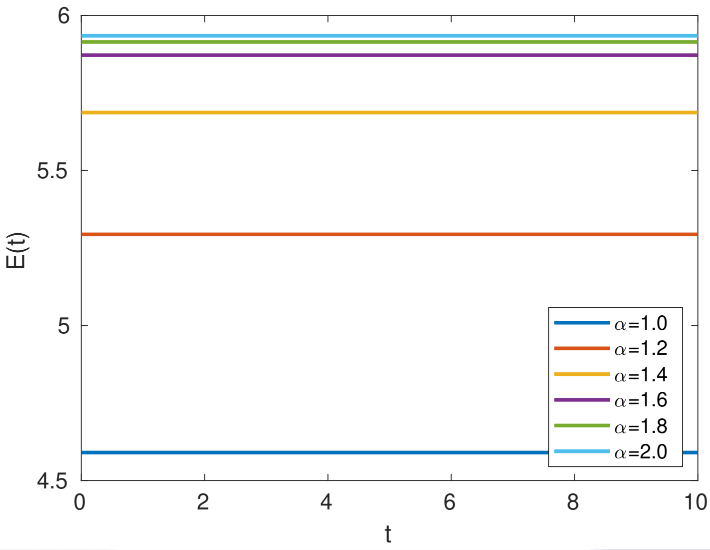

Figure 3.2: Graphs of the total energy of the system (3.5) versus t, using $\Omega = (-20, 20) \times (0, 10)$ and various values of α (see the values in the legend for the correspondence with the colors of the graphs). The graphs were obtained using the initial data (3.105)–(3.108). Computationally, we used the finite-difference method (3.36) with $h = 0.05$ and $\tau = 0.1$.

the vector of the second difference equations of (3.98) at the time t_n. It is possible to heck then that there exist nonnegative constants C_4 and C_5, such that the following inequalities hold, for each $n \in I_{N-1}$:

$$\delta_t \|\delta_t \epsilon^{n-1}\|_2^2 + \delta_t \mu_t \|\delta_x^{(\alpha/2)} \epsilon^{n-1}\|_2^2 + \delta_t \mu_t \|\epsilon^{n-1}\|_2^2 \leq C_4 \left(\|\rho^n\|_2^2 + \|\zeta^n\|_2^2 + \mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \right), \tag{3.99}$$

$$\delta_t \|\delta_x^{(\beta/2)} \upsilon^{n-1}\|_2^2 + \delta_t \mu_t \|\zeta^{n-1}\|_2^2 \leq C_5 \left(\|\sigma^n\|_2^2 + \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \zeta^{n-1}\|_2^2 \right). \tag{3.100}$$

Let $C_6 = C_4 + C_5$, define the constants ω^k as in Lemma 3.4.5, for each $k \in \overline{I}_N$, and let $\forall k \in \overline{I}_N$,

$$\frac{\rho^k}{2} = \|\delta_t \epsilon^{-1}\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} \epsilon^{-1}\|_2^2 + \mu_t \|\epsilon^{-1}\|_2^2 + \|\delta_x^{(\beta/2)} \upsilon^{-1}\|_2^2 + \mu_t \|\zeta^{-1}\|_2^2 + C_6 \tau \sum_{n=0}^{k} \left(\|\rho^n\|_2^2 + \|\sigma^n\|_2^2 \right). \tag{3.101}$$

Using arguments similar to those in Lemma 3.4.5 and employing Lemma 3.4.3, one may readily check that, for sufficiently small values of τ, there exists constant $C \geq 0$ such that $\omega^k \leq C\rho^k$, for each $k \in \overline{I}_{N-1}$. As a consequence of this inequality, the initial conditions of the problem (3.98) and the local truncation errors at the initial times, there exists $C_0 \geq 0$ such that

$$\frac{1}{\tau} \left(\|\epsilon^k\|_2 - \|\epsilon^{k-1}\|_2 \right) \leq \|\delta_t \epsilon^{k-1}\|_2 \leq \sqrt{\omega^k} \leq C_0 \sqrt{T} (h^2 + \tau^2), \quad \forall k \in \overline{I}_N. \tag{3.102}$$

Observe that this inequality readily implies that $\|\epsilon^k\|_2 - \|\epsilon^{k-1}\|_2 \leq C_0 \sqrt{T} \tau (h^2 + \tau^2)$ is satisfied, for each $k \in I_N$. If we let $n \in I_N$, take the sum over all indexes $k \in I_n$ on both sides of the last inequality and use the initial conditions, we obtain $\|\epsilon^n\|_2 \leq C_0 \sqrt{T} \tau n (h^2 + \tau^2) \leq C_0 T (h^2 + \tau^2)$. We conclude that $\|\epsilon^n\|_2 \leq C_0 T (h^2 + \tau^2)$, for each $n \in I_N$, and the fact that this inequality is also satisfied when we replace ϵ by ζ is proved in a similar fashion. We conclude that the solutions of (3.36) converge quadratically to those of the continuous problem (3.5) for sufficiently small τ. □

τ	h	$T=5$		$T=10$	
		$\epsilon_{\tau,h}$	$\rho^x_{\tau,h}$	$\epsilon_{\tau,h}$	$\rho^x_{\tau,h}$
0.04	1×2^{-1}	1.0736×10^{-2}	–	1.1948×10^{-2}	–
	1×2^{-2}	2.8810×10^{-3}	1.8978	3.2964×10^{-3}	1.8578
	1×2^{-3}	7.6005×10^{-4}	1.9224	8.8160×10^{-4}	1.9027
	1×2^{-4}	1.9709×10^{-4}	1.9472	2.3164×10^{-4}	1.9282
	1×2^{-5}	5.0803×10^{-5}	1.9559	6.0467×10^{-5}	1.9377
0.02	1×2^{-1}	2.7124×10^{-3}	–	3.0867×10^{-3}	–
	1×2^{-2}	7.2136×10^{-4}	1.9108	8.2296×10^{-4}	1.9072
	1×2^{-3}	1.8815×10^{-4}	1.9388	2.1795×10^{-4}	1.9168
	1×2^{-4}	4.7891×10^{-5}	1.9741	5.6496×10^{-5}	1.9478
	1×2^{-5}	1.2118×10^{-5}	1.9826	1.4496×10^{-5}	1.9624
0.01	1×2^{-1}	6.4674×10^{-4}	–	7.7247×10^{-4}	–
	1×2^{-2}	1.6895×10^{-4}	1.9366	2.0413×10^{-4}	1.9200
	1×2^{-3}	4.2952×10^{-5}	1.9758	5.2912×10^{-5}	1.9478
	1×2^{-4}	1.0793×10^{-5}	1.9926	1.3522×10^{-5}	1.9683
	1×2^{-5}	2.6296×10^{-6}	2.0372	3.4087×10^{-6}	1.9880

Table 3.1: Table of absolute errors and standard convergence rates in space when approximating the solution m of (3.5) with $\alpha=2$, using the method (3.36). We employed the spatial domain $B=(-20,20)$ and two periods of time, namely, $T=5$ and $T=10$. The initial conditions were prescribed by the functions (3.105)–(3.108). Various sets of computational parameters were employed.

We have implemented computationally the discrete model (3.36) and performed some experiments. Before closing this section, we will provide some examples on the performance of our computer implementation of the finite-difference scheme. In particular, our computational experiments will assess the capability of the numerical method to conserve the total energy of the system at each discrete temporal step. To that end, we will consider the absolute error at the time T between the exact solution u of (3.5) and the corresponding approximations U, which is given by

$$\epsilon_{\tau,h} = |||u-U|||_\infty, \tag{3.103}$$

and consider the standard rates

$$\rho^t_{\tau,h} = \log_2\left(\frac{\epsilon_{2\tau,h}}{\epsilon_{\tau,h}}\right), \qquad \rho^x_{\tau,h} = \log_2\left(\frac{\epsilon_{\tau,2h}}{\epsilon_{\tau,h}}\right). \tag{3.104}$$

Example 3.4.9. Consider (3.5) with $\alpha=2$, let $B=(-20,20)$ and define the functions u_0, m_0, u_1 and m_1 by

$$u_0(x) = \frac{\sqrt{10}-\sqrt{2}}{2}\operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x\in B, \tag{3.105}$$

$$m_0(x) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x\in B, \tag{3.106}$$

h	τ	$T=5$		$T=10$	
		$\epsilon_{\tau,h}$	$\rho^t_{\tau,h}$	$\epsilon_{\tau,h}$	$\rho^t_{\tau,h}$
0.04	0.02×2^{-1}	3.4982×10^{-6}	–	3.5705×10^{-6}	–
	0.02×2^{-2}	9.0677×10^{-7}	1.9478	9.3887×10^{-7}	1.9271
	0.02×2^{-3}	2.2950×10^{-7}	1.9822	2.4178×10^{-7}	1.9572
	0.02×2^{-4}	5.7756×10^{-8}	1.9905	6.1700×10^{-8}	1.9704
	0.02×2^{-5}	1.3973×10^{-8}	2.0473	1.5496×10^{-8}	1.9933
0.02	0.02×2^{-1}	8.8095×10^{-7}	–	9.2985×10^{-7}	–
	0.02×2^{-2}	2.2297×10^{-7}	1.9822	2.3772×10^{-7}	1.9677
	0.02×2^{-3}	5.4637×10^{-8}	2.0289	5.9873×10^{-8}	1.9893
	0.02×2^{-4}	1.2880×10^{-8}	2.0847	1.4658×10^{-8}	2.0302
	0.02×2^{-5}	3.0678×10^{-9}	2.0699	3.6476×10^{-9}	2.0067
0.01	0.02×2^{-1}	2.2696×10^{-7}	–	2.3860×10^{-7}	–
	0.02×2^{-2}	5.5263×10^{-8}	2.0381	5.9940×10^{-8}	1.9930
	0.02×2^{-3}	1.3172×10^{-8}	2.0688	1.4790×10^{-8}	2.0189
	0.02×2^{-4}	3.1629×10^{-9}	2.0582	3.7237×10^{-9}	1.9898
	0.02×2^{-5}	7.9589×10^{-10}	1.9906	9.5684×10^{-10}	1.9604

Table 3.2: Table of absolute errors and standard convergence rates in time when approximating the solution m of (3.5) with $\alpha = 2$, using the method (3.36). We employed the spatial domain $B = (-20, 20)$ and two periods of time, namely, $T = 5$ and $T = 10$. The initial conditions were prescribed by the functions (3.105)–(3.108). Various sets of computational parameters were employed.

$$u_1(x) = \frac{\sqrt{10}-\sqrt{2}}{2}(\tanh x - 1)\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x \in B, \tag{3.107}$$

$$m_1(x) = -4\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x \in B, \tag{3.108}$$

where $i^2 = -1$. With these data, the corresponding continuous initial-value problem (3.5) has the following exact traveling-wave solution on $\mathbb{R}\times\overline{\mathbb{R}^+}$ (see [45, 44]):

$$u(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right)\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \quad \forall (x,t)\in\mathbb{R}\times\overline{\mathbb{R}^+}, \tag{3.109}$$

$$m(x,t) = -2\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \quad \forall (x,t)\in\mathbb{R}\times\overline{\mathbb{R}^+}. \tag{3.110}$$

For comparison purposes, consider the discrete model (3.36) with initial data (3.105)–(3.108). For illustration purposes, Figure 3.1 shows the numerical solutions using $h = 0.05$ and $\tau = 0.1$. It is easy to check that the computational results are in qualitative agreement with the exact solutions (3.109) and (3.110). Moreover, Tables 3.1 and 3.2 provide a numerical study of the convergence of the method. The results confirm the quadratic order of convergence of the scheme (3.36), in agreement with Theorem 3.4.8.Finally, we considered various orders of differentiation and checked the dynamics of the total energy of the problem under investigation. The differentiation orders considered here are $\alpha = 1.0$, 1.2, 1.4, 1.6, 1.8 and 2.0. The results of our simulations are presented in Figure 3.2. It is obvious that the total energy is a constant function of time for each of the values of α used in this example. This is in agreement with Theorem 3.1.4. □

It is worth pointing out that, in addition to the illustrative example provided in this section, we have carried out more computational experiments with different initial data and considering various values of the computational parameters. The results are not included in this work in order to avoid redundancy, but they invariably confirm both the capability of the numerical model (3.36) to preserve the total energy of the system and the quadratic order of convergence of the scheme. Obviously, these remarks are in perfect qualitative agreement with the theoretical results derived in this work. In particular, they confirm the validity of Theorems 3.1.4 and 3.4.8.

4. An implicit semi-linear method

4.1 Preliminaries

Throughout, we fix a nonempty, open and bounded interval $B = (x_L, x_R)$ of $\mathbb{R}$. Let $T > 0$, and define the set $\Omega = B \times (0, T)$. In general, for each $S \subseteq \mathbb{R}^2$, we let $\overline{S}$ be the closure of S with respect to the standard topology of $\mathbb{R}^2$. In particular, this means that $\overline{\Omega} = \overline{B} \times [0, T]$. Let $u : \overline{\Omega} \to \mathbb{C}$, $m : \overline{\Omega} \to \mathbb{R}$, $u_0, u_1 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions. In this report, functions defined on $\overline{\Omega}$ will be extended to all $\mathbb{R} \times [0, T]$ by letting them be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$.

Definition 4.1.1 (Podlubny [77])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 < \alpha \leq n$. The *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is defined (when it exists) as

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{4.1}$$

Here, Γ is the usual Gamma function. In the case that $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, and n and α are as above, then the *Riesz fractional partial derivative* of u of order α with respect to x at $(x, t) \in \mathbb{R} \times [0, T]$ is given (if it exists) by

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi,t)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{4.2}$$

Definition 4.1.2. If $z \in \mathbb{C}$, then $\overline{z}$ will represent its complex conjugate. We will use $\mathbb{F}$ to denote the fields $\mathbb{R}$ or $\mathbb{C}$. Let us introduce the set $L_{x,p}(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{F} : f(\cdot, t) \in L_p(\overline{B}), \text{ for each } t \in [0, T]\}$ and $p \in [1, \infty]$. If $p \in [1, \infty)$ and $f \in L_{x,p}(\overline{\Omega})$, then we convey that

$$\|f\|_{x,p} = \left(\int_B |f(x,t)|^p dx \right)^{1/p}, \quad \forall t \in [0, T]. \tag{4.3}$$

In the case when $p = \infty$, we set $\|f\|_{x,\infty} = \inf\{C \geq 0 : |f(x,t)| \leq C \text{ for almost all } x \in \overline{B}\}$. Obviously, $\|f\|_{x,p}$ is a function of $t \in [0, T]$ in any case. Moreover, for each pair $f, g \in L_{x,2}(\overline{\Omega})$, define the following function of t:

$$\langle f, g \rangle_x = \int_B f(x,t)\overline{g(x,t)}dx, \quad \forall t \in [0, T]. \tag{4.4}$$

In this work, we will fix $\alpha, \beta \in (1, 2]$, and consider the bi-fractional extension of the Klein–Gordon–Zakharov model given by the following coupled system of fractional-order differential equations with initial-boundary data:

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \quad \forall (x,t) \in \Omega, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta} = 0, \quad \forall (x,t) \in \Omega, \\
\text{subject to} \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), & \forall x \in \overline{B}, \\ \dfrac{\partial u(x,0)}{\partial t} = u_1(x), & \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, & \forall t \in [0,T]. \end{cases}
\end{aligned}
\tag{4.5}
$$

It is important to point out here that the case $\alpha = \beta = 2$ in (4.5) is precisely the well-known Klein–Gordon–Zakharov system. Moreover, as we mentioned before, the Klein–Gordon–Zakharov system describes physical phenomena, specifically the interaction between Langmuir waves in a high-frequency plasma. Under this context, the function u represents the fast time-scale component of an electric field raised by electrons, and the function m is the deviation of ion density from its equilibrium.

In the sequel and for convenience, we will suppose that the function $v : \overline{\Omega} \to \mathbb{R}$ satisfies the following property:

$$
\frac{\partial^\beta v(x,t)}{\partial |x|^\beta} = \frac{\partial m(x,t)}{\partial t}, \quad \forall (x,t) \in \Omega. \tag{4.6}
$$

It is worth pointing out that the existence of the function v is guaranteed, for example, in the case that the right-hand side of (4.6) is a Lebesgue-integrable function or, more particularly, when it is a continuous function [81, Ch. 6].

Definition 4.1.3. Let u and m satisfy the problem (4.5). For the sake of convenience, let us agree that $u = u(x,t)$ and $m = m(x,t)$. Define the Hamiltonian or energy density functional $\mathcal{H}(x,t) = \mathcal{H}(u(x,t), m(x,t))$ as

$$
\mathcal{H}(x,t) = \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right|^2 + |u|^2 + m|u|^2 + \frac{1}{2}\left|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right|^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \quad \forall (x,t) \in \Omega. \tag{4.7}
$$

In turn, the associated total energy of the system at the time $t \in [0,T]$ is given by

$$
\mathcal{E}(t) = \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4. \tag{4.8}
$$

Using these conventions, the following results were proved in [62].

Theorem 4.1.4 (Energy conservation)**.** *If u and m satisfy the problem* (4.5)*, then the function $\mathcal{E}$ is a constant.*

Theorem 4.1.5 (Boundedness)**.** *Let u and m satisfy the initial-boundary-value problem* (4.5)*, and let $u, \partial u/\partial x \in L_{x,2}(\overline{\Omega})$. Then there exist a constant C such that*

$$
\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \|m\|_{x,2}^2 \leq C, \quad \forall t \in [0,T]. \tag{4.9}
$$

Moreover, the constant function (4.8) *is non-negative.*

4.2 Discrete model

The first aim of this section is to provide the necessary discrete nomenclature which will be required later on in this work. To give a fresh start, we will focus firstly on the concept of fractional-order centered differences, which will allow us to obtain a discretization of Riesz space-fractional derivatives. We must point out that we opted to use fractional centered differences for computational implementation reasons. However, there are various other alternative approaches to provide such discretizations, like the well-known weighted-and-shifted Grünwald differences [92].

Definition 4.2.1 (Ortigueira [73])**.** Define the real sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and assume that $h, \alpha \in \mathbb{R}$ satisfy $h > 0$ and $\alpha > -1$. The *fractional centered difference* of f of order α and spatial step h at the point x is given as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x-kh), \quad \forall x \in \mathbb{R}, \tag{4.11}$$

whenever the double series at the right-hand side of (4.11) converges.

The following results provide important analytical properties on fractional-order centered differences.

Lemma 4.2.2 (Wang *et al.* [97])**.** *If* $1 < \alpha \leq 2$*, then*

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$*, for all* $k \geq 1$*, and*

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$.

Lemma 4.2.3 (Wang *et al.* [97])**.** *Suppose that* $1 < \alpha \leq 2$*, let* $h > 0$*, and assume that* $f \in \mathcal{C}^5(\mathbb{R})$ *has all its derivatives up to order five in* $L^1(\mathbb{R})$*. Then*

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{4.12}$$

In the sequel, we agree that $I_n = \{1, \ldots, n\}$ and $\overline{I}_n = I_n \cup \{0\}$, for each $n \in \mathbb{N}$. We will let $J, N \in \mathbb{N}$ be such that $J \geq 2$ and $N \geq 2$, and introduce the positive constants $h = (x_R - x_L)/J$ and $\tau = T/N$. We employ uniform partitions of $[x_L, x_R]$ and $[0, T]$, respectively, of the forms

$$x_L = x_0 < x_1 < \ldots < x_j < \ldots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{4.13}$$

and

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T, \quad \forall n \in \overline{I}_N. \tag{4.14}$$

For each $(j, n) \in \overline{I}_J \times \overline{I}_N$, agree that U_j^n and M_j^n denote numerical estimates of $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, respectively. Set $\mathcal{R}_h = \{x_j : j \in \overline{I}_J\}$, and use the nomenclature $\mathcal{V}_h$ to represent the vector space over $\mathbb{F}$ of all $\mathbb{F}$-valued functions on the grid space $\mathcal{R}_h$, which vanish at the endpoints x_0 and x_J. In general, if $V \in \mathcal{V}_h$, then let $V_j = V(x_j)$, for each $j \in \overline{I}_J$. Additionally, we agree that $U^n = (U_j^n)_{j \in \overline{I}_J} \in \mathcal{V}_h$ and $M^n = (M_j^n)_{j \in \overline{I}_J} \in \mathcal{V}_h$. Moreover, for the sake of convenience, we will let $U = (U^n)_{n \in \overline{I}_N}$ and $M = (M^n)_{n \in \overline{I}_N}$.

Definition 4.2.4. Let $p \in \mathbb{R}$ be such that $1 \leq p < \infty$. Then the inner product $\langle \cdot, \cdot \rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{C}$ and the norms $\| \cdot \|_p, \| \cdot \|_\infty : \mathcal{V}_h \to \mathbb{R}$ will be respectively defined as

$$\langle U, V \rangle = h \sum_{j \in \overline{I}_J} U_j \overline{V_j}, \quad \forall U, V \in \mathcal{V}_h, \tag{4.15}$$

$$\|U\|_p^p = h \sum_{j \in \overline{I}_J} |U_j|^p, \quad \forall U \in \mathcal{V}_h, \tag{4.16}$$

$$\|U\|_\infty = \max \left\{ |U_j| : j \in \overline{I}_J \right\}, \quad U \in \mathcal{V}_h. \tag{4.17}$$

Moreover, let us set $|||V|||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}$, for each $V = (V^n)_{n \in \overline{I}_N} \subseteq \mathcal{V}_h$.

Definition 4.2.5. Let V represent any of the functions U or M. We will employ the linear difference operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j,n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{4.18}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{4.19}$$

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{4.20}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j,n) \in \overline{I}_J \times I_{N-1}. \tag{4.21}$$

If we omit the composition symbol for simplicity, we have the operators $\delta_x^{(2)} V_j^n = \delta_x \delta_x V_{j-1}^n$, $\delta_t^{(1)} V_j^n = \mu_t \delta_t V_j^{n-1}$, $\delta_t^{(2)} V_j^n = \delta_t \delta_t V_j^{n-1}$, $\mu_t^{(2)} V_j^n = \mu_t \mu_t V_j^{n-1}$ and $\delta_{xt} V_j^n = \delta_x \delta_t V_j^n$, for each $(j,n) \in I_{J-1} \times \overline{I}_N$. Moreover, if $f : \mathbb{R} \to \mathbb{R}$, we define the difference operator $\mathbb{A}_x$ of f at the point $x \in \mathbb{R}$ as

$$\mathbb{A}_x f(x) = \frac{1}{12} f(x-h) + \frac{10}{12} f(x) + \frac{1}{12} f(x+h), \quad \forall x \in \mathbb{R}. \tag{4.22}$$

Definition 4.2.6. Let $V = U$ (respectively, $V = M$) and $v = u$ (respectively, $v = m$). Using the nomenclature of Definition 4.2.1 and motivated by the lemmas afterwards, we introduce the following consistent estimate operator of the fractional partial derivative of order α of v with respect to x at the point (x_j, t_n):

$$\delta_x^{(\alpha)} V_j^n = -\frac{1}{h^\alpha} \sum_{k \in \overline{I}_J} g_{j-k}^{(\alpha)} V_k^n, \qquad \forall (j,n) \in \overline{I}_J \times \overline{I}_N. \tag{4.23}$$

Obviously, this discrete operator is a second-order consistent approximation for the Riesz space-fractional partial derivative of v of order α at the point (x_j, t_n). Moreover, we define the discrete operator

$$\mathbb{A}_x V_j^n = \frac{1}{12} V_{j+1}^n + \frac{10}{12} V_j^n + \frac{1}{12} V_{j-1}^n \qquad \forall (j,n) \in \overline{I}_J \times I_{N-1}. \tag{4.24}$$

The following lemma will be frequently required in the sequel. This result summarizes some important properties of fractional-order centered differences.

Lemma 4.2.7 (Macías-Díaz [49])**.** *If* $\alpha \in (1,2]$ *and* $U, V \in \mathcal{V}_h$*, then* $\langle -\delta_x^{(\alpha)} U, V \rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V \rangle$. *Moreover,*

(a) $\|\delta_x^{(\alpha/2)} V\|_2^2 \leq 2 g_0^{(\alpha)} h^{1-\alpha} \|V\|_2^2$, *for each* $V \in \mathcal{V}_h$,

(b) $\|\delta_x^{(\alpha)} V\|_2^2 = \|\delta_x^{(\alpha/2)} \delta_x^{(\alpha/2)} V\|_2^2$, *for each* $V \in \mathcal{V}_h$, *and*

(c) $\|\delta_x^{(\alpha)} V\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|\delta_x^{(\alpha/2)} V\|_2^2 \leq 4\left(g_0^{(\alpha)} h^{1-\alpha}\right)^2 \|V\|_2^2$, *for each* $V \in \mathcal{V}_h$. □

Using the nomenclature introduced above, we will employ the following fully discrete initial-boundary-value problem to approximate the solutions of the continuous fractional system (4.5):

$$\begin{aligned} \delta_t^{(2)} \mathbb{A}_x U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n \left[1 + M_j^n + \mu_t^{(1)} |U_j^n|^2\right] &= 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\ \delta_t^{(2)} \mathbb{A}_x M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2 &= 0, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \\ \text{subject to} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), \quad \forall j \in I_{J-1}, \\ \delta_t^{(1)} U_j^0 = u_1(x_j) & \delta_t^{(1)} M_j^0 = m_1(x_j), \quad \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases} \end{aligned} \tag{4.25}$$

Notice that the scheme (4.25) requires to employ approximations at the time t_{-1}. However, it is worth noting that the initial conditions yield the identities $U_j^{-1} = U_j^1 - 2\tau u_1(x_j)$ and $M_j^{-1} = M_j^1 - 2\tau m_1(x_j)$, for each $j \in I_{J-1}$. Moreover, letting $n = 0$ in the recursive equation of (4.25) and substituting the initial conditions, we readily obtain that

$$\begin{aligned} \frac{2}{\tau^2} \mathbb{A}_x [U_j^1 - u_0(x_j) - \tau u_1(x_j)] = -\left(U_j^1 - \tau u_1(x_j)\right) \left[1 + m_0(x_j) - \tfrac{1}{2}\left(|U_j^1|^2 + |U_j^1 - 2\tau u_1(x_j)|^2\right)\right] \\ + \delta_x^{(\alpha)} \left(U_j^1 - \tau u_1(x_j)\right), \quad \forall j \in I_{J-1}, \end{aligned} \tag{4.26}$$

and

$$\frac{2}{\tau^2} \mathbb{A}_x [M_j^1 - m_0(x_j) - \tau m_1(x_j)] = \delta_x^{(\beta)} \left(M_j^1 - \tau m_1(x_j) + |u_0(x_j)|^2\right), \quad \forall j \in I_{J-1}. \tag{4.27}$$

Observe that the discrete model (4.25) is a three-step implicit nonlinear technique. This means in particular that, if the estimations at the times t_{n-1} and t_n are known in that model, then the approximations U^{n+1} and M^{n+1} will be the only unknowns. In fact, the scheme (4.25) is a decoupled nonlinear system since the only unknown in the first equation of (4.25) is the vector U^{n+1}. Meanwhile, the only unknown of the second equation is the vector M^{n+1}. In the following and for the sake of convenience, we will let $\{V_j^n : (j,n) \in \overline{I}_J \times \overline{I}_N\}$ be such that

$$\delta_x^{(\beta)} V_j^n = \delta_t M_j^n, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{4.28}$$

$$V_0^n = V_J^n = 0, \quad \forall n \in \overline{I}_N. \tag{4.29}$$

Under these circumstances, (U, M) will denote a solution of (4.25), and $V = (V^n)_{n \in \overline{I}_N}$ will satisfy (4.28) and (4.29).

Definition 4.2.8. Suppose that (U, M) solves the discrete system (4.25). The discrete Hamiltonian or discrete energy density of the system at the point x_j and time t_n is the quantity $H_j^n = H(U_j^n, M_j^n)$, which is defined through

$$\begin{aligned} H_j^n = |\delta_t U_j^n|^2 - \frac{h^2}{12} |\delta_{xt} U_j^n|^2 + \mu_t |\delta_x^{(\alpha/2)} U_j^n|^2 + \mu_t |U_j^n|^2 + \frac{1}{2}\left[M_j^n |U_j^{n+1}|^2 + M_j^{n+1} |U_j^n|^2\right] \\ + \frac{1}{2} |\delta_x^{(\beta/2)} V_j^n|^2 + \frac{1}{2} \mu_t |U_j^n|^4 - \frac{h^2}{24} |\delta_x \delta_x^{(\beta/2)} V_j^n|^2 + \frac{1}{2} \mu_t |M_j^n|^2, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}. \end{aligned} \tag{4.30}$$

Agree that $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$, for each $n \in \overline{I}_N$. The total discrete energy of the model (4.25) at the time t_n is given by

$$\begin{aligned} E^n = \|\delta_t U^n\|_2^2 - \frac{h^2}{12}\|\delta_{xt}U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{2}\langle M^n, |U^{n+1}|^2\rangle + \frac{1}{2}\mu_t\|M^n\|_2^2 \\ + \frac{1}{2}\langle M^{n+1}, |U^n|^2\rangle + \frac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \frac{1}{2}\mu_t\|U^n\|_4^4 - \frac{h^2}{24}\|\delta_x\delta_x^{(\beta/2)}V^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \end{aligned} \tag{4.31}$$

4.3 Structural properties

The present section is devoted to establishing the main structural properties of the discrete model (4.25). More precisely, we will prove herein that the discrete model is solvable, and that the quantities (4.31) are non-negative temporal invariants of the scheme (4.25). Firstly, we prove and state some crucial results in our analysis.

Definition 4.3.1. If $U, V \in \mathcal{V}_h$, then we define the product of U and V point-wisely. More precisely, $UV = (U_jV_j)_{j\in\overline{I}_J}$.

Lemma 4.3.2. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathcal{V}_h$. Assume that U is a sequence of complex functions while the functions of M are real. Suppose additionally that there exists $(V^n)_{n\in\overline{I}_N} \subseteq \mathcal{V}_h$ such that (4.28) holds. Then the following are satisfied, for each $n \in I_{N-1}$:*

(a) $2\,\mathrm{Re}\langle\delta_t^{(2)}\mathbb{A}_xU^n, \delta_t^{(1)}U^n\rangle = \delta_t\left[\|\delta_tU^{n-1}\|_2^2 - \frac{1}{12}h^2\|\delta_{xt}U^{n-1}\|_2^2\right]$,

(b) $2\,\mathrm{Re}\langle-\delta_x^{(\alpha)}\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \delta_t\mu_t\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2$,

(c) $2\,\mathrm{Re}\langle\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \delta_t\mu_t\|U^{n-1}\|_2^2$,

(d) $2\,\mathrm{Re}\langle M^n\mu_t^{(1)}U^n, \delta_t^{(1)}U^n\rangle = \langle M^n, \delta_t^{(1)}|U^n|^2\rangle$,

(e) $4\,\mathrm{Re}\langle(\mu_t^{(1)}|U^n|^2)(\mu_t^{(1)}U^n), \delta_t^{(1)}U^n\rangle = \delta_t\mu_t\|U^{n-1}\|_4^4$,

(f) $-2\langle\delta_t^{(2)}\mathbb{A}_xM^n, \mu_tV^{n-1}\rangle = \delta_t\left[\|\delta_x^{(\beta/2)}V^{n-1}\|_2^2 - \frac{1}{12}h^2\|\delta_x\delta_x^{(\beta/2)}V^{n-1}\|_2^2\right]$,

(g) $2\langle\delta_x^{(\beta)}\mu_t^{(1)}M^n, \mu_tV^{n-1}\rangle = \delta_t\mu_t\|M^{n-1}\|_2^2$, *and*

(h) $\langle\delta_x^{(\beta)}|U^n|^2, \mu_tV^{n-1}\rangle = \langle|U^n|^2, \delta_t^{(1)}M^n\rangle$.

Proof. Beforehand, we must point out that most of the identities have been proved already in [62]. We only give here the proofs of the new identities (a) and (f). To that end, notice that Lemma 4.2.7 implies that

$$\begin{aligned} -2\langle\delta_t^{(2)}\mathbb{A}_xM^n, \mu_tV^{n-1}\rangle &= -2\langle\delta_t^{(2)}M^n, \mu_tV^{n-1}\rangle - \frac{2}{12}\left\langle\delta_t^{(2)}\left(M_{j-1}^n - 2M_j^n + M_{j+1}^n\right), \mu_tV^{n-1}\right\rangle \\ &= -2\langle\delta_t\delta_x^{(\beta)}V^{n-1}, \mu_tV^{n-1}\rangle - \frac{2}{12}h^2\langle\delta_t^{(2)}\delta_x^{(2)}M^n, \mu_tV^{n-1}\rangle \\ &= 2\langle\delta_t\delta_x^{(\beta/2)}V^{n-1}, \mu_t\delta_x^{(\beta/2)}V^{n-1}\rangle - \frac{2}{12}h^2\langle\delta_t\delta_x\delta_x^{(\beta/2)}V^{n-1}, \mu_t\delta_x\delta_x^{(\beta/2)}V^{n-1}\rangle \\ &= \delta_t\left[\|\delta_x^{(\beta/2)}V^{n-1}\|_2^2 - \tfrac{1}{12}h^2\|\delta_x\delta_x^{(\beta/2)}V^{n-1}\|_2^2\right], \end{aligned} \tag{4.32}$$

which establishes the validity of (f). Finally, we just mention that the proof of (a) is similar to (f). □

The following result is the well-known Young's inequality, which will be crucial in the present report.

Lemma 4.3.3. *Let $a, b \in \mathbb{R}^+ \cup \{0\}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For each $\epsilon > 0$, the following holds:*

$$ab \leq \frac{|a|^p}{p\epsilon} + \frac{\epsilon |b|^q}{q}. \tag{4.33}$$

Next, we prove that the quantities (4.31) are non-negative, and that they are conserved throughout time.

Theorem 4.3.4 (Energy conservation)**.** *Suppose that (U, M) is a solution of* (4.5)*. Then the quantities* (4.31) *are non-negative and constant with respect to time.*

Proof. Take the real part of the inner product of the first equation of the scheme (4.25) with $\delta_t^{(1)} U^n$, and compute the inner product between the second equation in (4.25) and $\mu_t V^{n-1}$, for each $n \in I_{N-1}$. Then add both equations and apply Lemma 4.3.2 to confirm that $\delta_t E^{n-1} = 0$. This identity and induction establish the temporal invariance of the quantities (4.31). To prove now the non-negativity of these quantities, notice that

$$\begin{aligned} h^2 \|\delta_x \delta_x^{(\beta/2)} V^n\|_2^2 &= h \sum_{j=1}^{J-1} \left(\delta_x^{(\beta/2)} V_j^n - \delta_x^{(\beta/2)} V_{j-1}^n \right)^2 \\ &\leq 2h \sum_{j=1}^{J-1} \left(|\delta_x^{(\beta/2)} V_j^n|^2 + |\delta_x^{(\beta/2)} V_{j-1}^n|^2 \right) = 4 \|\delta_x^{(\beta/2)} V_j^n\|_2^2, \end{aligned} \tag{4.34}$$

holds for each $n \in \overline{I}_{N-1}$. In similar fashion, we can obtain that $h^2 \|\delta_{xt} U^n\|_2^2 \leq 4 \|\delta_t U^n\|_2^2$, for each $\forall n \in \overline{I}_{N-1}$. On the other hand, using Cauchy–Schwarz inequality, it follows that

$$|\langle M^n, |U^{n+1}|^2 \rangle + \langle M^{n+1}, |U^n|^2 \rangle| \leq \mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{4.35}$$

Finally, applying all of these estimates to (4.31) yields

$$E^n \geq \frac{2}{3} \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{6} \|\delta_x^{(\beta/2)} V^n\|_2^2 \geq 0, \quad \forall n \in \overline{I}_{N-1}, \tag{4.36}$$

whence the non-negativity of the quantities (4.31) readily follows. □

The next result proves the boundedness of the solutions of (4.25). To establish this property, the non-negativity and temporal invariance of the quantities (4.31) will be of utmost importance.

Theorem 4.3.5 (Boundedness)**.** *Let $u_0, m_0 \in H^1$ and $u_1, m_1 \in L_2$, and suppose that (U, M) is a solution corresponding to the problem* (4.25)*. Then there exists a constant $C^* \in \mathbb{R}^+$ with the property that*

$$\max \left\{ \|\delta_t U^n\|_2, \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2, \mu_t \|U^n\|_2, \mu_t \|M^n\|_2, \mu_t \|U^n\|_4, \|\delta_x^{(\beta/2)} V^n\|_2 \right\} \leq C^*, \quad \forall n \in \overline{I}_N. \tag{4.37}$$

Proof. Note that Theorem 4.3.4 guarantees that the quantities E^n are equal to a constant $C_0 \in \mathbb{R}$, for each $n \in I_{N-1}$. Proceeding as in Theorem and simplifying, we obtain that

$$\begin{aligned} C_0 \geq{} & \frac{2}{3} \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2} \mu_t \|U^n\|_4^4 + \frac{1}{6} \|\delta_x^{(\beta/2)} V^n\|_2^2 \\ & + \frac{1}{2} \mu_t \|M^n\|_2^2 - \frac{1}{2} \left| \langle M^n, |U^{n+1}|^2 \rangle \right| - \frac{1}{2} \left| \langle M^{n+1}, |U^n|^2 \rangle \right|, \quad \forall n \in \overline{I}_{N-1}. \end{aligned} \tag{4.38}$$

On the other hand, applying Young's inequality twice, we observe that

$$\frac{1}{2}\left|\langle M^n,|U^{n+1}|^2\rangle\right| + \frac{1}{2}\left|\langle M^{n+1},|U^n|^2\rangle\right| \leq \frac{1}{2}\mu_t\|M^n\|_2^2 + \frac{1}{2}\mu_t\|U^{n+1}\|_4^4, \quad \forall n \in \overline{I}_{N-1}, \tag{4.39}$$

$$\frac{1}{2}\left|\langle M^n,|U^{n+1}|^2\rangle\right| + \frac{1}{2}\left|\langle M^{n+1},|U^n|^2\rangle\right| \leq \frac{1}{4}\mu_t\|M^n\|_2^2 + \mu_t\|U^{n+1}\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{4.40}$$

Using now the inequality (4.39) to bound (5.51) from below, we obtain that

$$C_0 \geq \frac{2}{3}\|\delta_t U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{6}\|\delta_x^{(\beta/2)}V^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{4.41}$$

Since $\mu_t\|U^n\|_2$ is bounded, it follows that $\mu_t\|U^n\|_4$ is also bounded. In fact, the last inequality and the properties of the discrete norms show that $\|U^n\|_4^4 \leq \|U^n\|_2^4 \leq 4C_0^2$. Similarly, $\|U^{n+1}\|_4^4 \leq 4C_0^2$ holds. Using these facts, keeping the last three quantities of inequality (5.51) and employing the bound (4.40), we have that

$$\frac{1}{4}\mu_t\|M^n\|_2^2 \leq C_0 + \mu_t\|U^n\|_4^4 \leq C_0 + 2C_0^2, \quad \forall n \in \overline{I}_{N-1}. \tag{4.42}$$

This means that the norms $\|M^n\|_2$ (and, thus, also the norms $\|M^n\|_\infty$) are uniformly bounded by the non-negative constant $2\sqrt{2}(C_0 + 2C_0^2)^{1/2} \geq C_0$. It is easy to see now that the quantities $\|\delta_t U^n\|_2$, $\mu_t\|\delta_x^{(\alpha/2)}U^n\|_2$, $\mu_t\|U^n\|_2$, $\mu_t\|M^n\|_2$, $\mu_t\|U^n\|_4$ and $\|\delta_x^{(\beta/2)}V^n\|_2$ can be uniformly bounded by a single constant $C^* \geq 0$, valid for all $n \in \overline{I}_N$. □

For the remainder of this manuscript, we will employ the real matrix $A = C - D^{(\beta)}$ of size $(J+1)\times(J+1)$, where

$$C = \frac{1}{12}\begin{pmatrix} 12 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 12 \end{pmatrix} \tag{4.43}$$

and

$$D^{(\beta)} = -\frac{\tau^2}{2h^\beta}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & g_2^{(\beta)} & \cdots & g_{J-4}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-2}^{(\beta)} & g_{J-1}^{(\beta)} \\ g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & \cdots & g_{J-5}^{(\beta)} & g_{J-4}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-2}^{(\beta)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ g_{J-2}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-4}^{(\beta)} & g_{J-5}^{(\beta)} & \cdots & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} & g_2^{(\beta)} \\ g_{J-1}^{(\beta)} & g_{J-2}^{(\beta)} & g_{J-3}^{(\beta)} & g_{J-4}^{(\beta)} & \cdots & g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & g_1^{(\beta)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.44}$$

Lemma 4.3.6. *The real matrix A is strictly diagonally dominant.*

Proof. Using Lemma 4.2.2, we readily check that the following inequalities and identity hold, for each $i \in \{2,\ldots,J\}$:

$$\sum_{\substack{j=1 \\ j\neq i}}^{J} |a_{ij}| \leq \frac{1}{6} + \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{\tau^2|g_l^{(\beta)}|}{2h^\beta} < \frac{5}{6} + \frac{\tau^2 g_0^{(\beta)}}{2h^\beta} = |a_{ii}|. \tag{4.45}$$

Obviously, the first and the last rows of the matrix A also satisfy this condition, whence we readily conclude that A is strictly diagonally dominant, as desired. □

Definition 4.3.7. Let $(U^n)_{n\in\overline{I}_N}$ be any sequence in $\mathcal{V}_h$, let $\Phi \in \mathcal{V}_h$ and assume that $g : \mathbb{R} \to \mathbb{R}$ is a function. We define

$$\mu_{t,\Phi}^{(1)}[g(U_j^n)] = \frac{1}{2}\left[g(\Phi_j) + g(U_j^{n-1})\right], \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}. \tag{4.46}$$

Notice in particular that $\mu_t^{(1)} U_j^n = \mu_{t,U^{n+1}} U_j^n$ and $\mu_t^{(1)} |U_j^n|^2 = \mu_{t,U^{n+1}} |U_j^n|^2$, for each $(j,n) \in \overline{I}_J \times I_{N-1}$. On the other hand, if $\Phi \in \mathcal{V}_h$, then we agree that

$$\Phi^+ = (\Phi_1, \Phi_2, \Phi_3, \dots, \Phi_{J-1}, \Phi_J, 0), \tag{4.47}$$

$$\Phi^- = (0, \Phi_0, \Phi_1, \dots, \Phi_{J-2}, \Phi_{J-1}). \tag{4.48}$$

Lemma 4.3.8. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathcal{V}_h$, and assume that U is a sequence of complex functions, while the functions of M are real. The following identities are satisfied, for each $n \in I_{N-1}$ and $\Phi \in \mathcal{V}_h$:*

(a) $\operatorname{Re}\left\langle (\mu_{t,\Phi}^{(1)}|U^n|^2)(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1}\right\rangle = \frac{1}{4}(\|\Phi\|_4^4 - \|U^{n-1}\|_4^4)$.

(b) $\operatorname{Re}\left\langle M^n(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1}\right\rangle = \frac{1}{4}\langle M^n, |\Phi|^2 - |U^{n-1}|^2\rangle$.

(c) $\operatorname{Re}\left\langle \mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1}\right\rangle = \frac{1}{2}(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2)$.

(d) $\operatorname{Re}\left\langle -\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1}\right\rangle = \frac{1}{2}(\|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2)$.

Additionally, the following inequalities are satisfied for each $\lambda \in [0,1]$:

(e) $|\operatorname{Re}\langle \mathbb{A}_x U^n, \Phi - U^{n-1}\rangle| \leq \frac{1}{24}\|\Phi\|_2^2 + C_1$.

(f) $|\operatorname{Re}\langle \mathbb{A}_x U^{n-1}, \Phi - U^{n-1}\rangle| \leq \frac{1}{12}\|\Phi\|_2^2 + C_2$.

(g) *If $C^* \geq 0$ satisfies $\|M^n\|_\infty \leq C*$, then* $|\langle M^n, |\Phi|^2 - |U^{n-1}|\rangle| \leq C^*\|\Phi\|_2^2 + C_3$.

(h) $\operatorname{Re}\langle \Phi, \Phi - U^{n-1}\rangle \geq \frac{1}{2}(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2)$.

(i) $|\operatorname{Re}\langle \Phi^+ + \Phi^-, \Phi - U^{n-1}\rangle| \leq \frac{5}{2}\|\Phi\|_2^2 + 2\|U^{n-1}\|_2^2$.

Here, the constants $C_1, C_2, C_3 \in \mathbb{R}$ are non-negative and depend only on U^n, U^{n-1} and M^n.

Proof. The proofs of the identities (a)–(d) can be found in [61]. To establish (e), we employ the triangle inequality and Lemma 4.3.3 (with a suitable $\epsilon > 0$) to obtain that

$$\begin{aligned}\left|\langle \mathbb{A}_x U^n, \Phi - U^{n-1}\rangle\right| &\leq |\langle \mathbb{A}_x U^n, \Phi\rangle| + \left|\langle \mathbb{A}_x U^n, U^{n-1}\rangle\right| \\ &\leq \frac{1}{24}\|\Phi\|_2^2 + 6\|\mathbb{A}_x U^n\|_2^2 + \frac{1}{2}\left(\|\mathbb{A}_x U^n\|_2^2 + \|U^{n-1}\|_2^2\right).\end{aligned} \tag{4.49}$$

The inequality (e) follows now letting $C_1 = \frac{13}{2}\|\mathbb{A}_x U^n\|_2^2 + \frac{1}{2}\|U^{n-1}\|_2^2$, which obviously is a constant that depends only on the vectors U^n and U^{n-1}. Part (f) is proved in similar fashion. On the other hand, to prove the inequality (g) we use the triangle inequality and Hölder's inequality to obtain that

$$\left|\langle M^n, |\Phi|^2 - |U^{n-1}|\rangle\right| \leq \left|\langle M^n, |\Phi|^2\rangle\right| + \left|\langle M^n, |U^{n-1}|\rangle\right| \leq C^*\|\Phi\|_2^2 + C_3, \tag{4.50}$$

where $C_3 = |\langle M^n, |U^{n-1}|\rangle|$. The inequality (h) is straightforward, so it only remains to prove (i). To that end, we need to point out firstly that the identities $\|\Phi^+\|_2 = \|\Phi^-\|_2 = \|\Phi\|_2$ are satisfied. Observe now that the triangle inequality, Cauchy's inequality and Young's inequality yield

$$\begin{aligned} |\operatorname{Re}\langle \Phi^+ + \Phi^-, \Phi - U^{n-1}\rangle| &\leq |\langle \Phi^+, \Phi\rangle| + |\langle \Phi^-, \Phi\rangle| + |\langle \Phi^+, U^{n-1}\rangle| + |\langle \Phi^-, U^{n-1}\rangle| \\ &\leq \frac{5\|\Phi\|_2^2}{2} + 2\|U^{n-1}\|_2^2, \end{aligned} \tag{4.51}$$

which is what we wanted to proof. □

Lemma 4.3.9 (Leray–Schauder fixed-point theorem)**.** *Let X be a Banach space, and let $F : X \to X$ be continuous and compact. If the set $S = \{x \in X : \lambda F(x) = x \text{ for some } \lambda \in [0,1]\}$ is bounded then F has a fixed point.*

We establish next the existence of solutions for the finite-difference model (4.25). In the proof, we will require to use the non-negative constant $C^{**} = 2C^*$, where C^* is the same constant as in Theorem 4.3.5.

Theorem 4.3.10 (Solubility)**.** *The numerical model* (4.25) *is solvable for any set of initial conditions if* $12\tau^2 C^{**} < 1$.

Proof. The proof will make use of mathematical induction.

- Notice firstly that the approximations (U^0, M^0) are explicitly defined through the initial conditions in the discrete initial-boundary-value problem (4.25).

- Suppose now that $n \in I_{J-1}$, and that the approximations (U^m, M^m) have been obtained already, for each $m \in I_n$. Following an argument similar to that in Theorem 4.3.5 yields that $\forall m \in \overline{I}_{n-1}$,

  $$\max\left\{\|\delta_t U^m\|_2, \mu_t\|\delta_x^{(\alpha/2)} U^m\|_2, \mu_t\|U^m\|_2, \mu_t\|M^m\|_2, \mu_t\|U^m\|_4, \|\delta_x^{(\beta/2)} V^m\|_2\right\} \leq C^*. \tag{4.52}$$

  Using the properties of discrete norms, it follows that $\|M^m\|_\infty \leq C^{**}$, for each $m \in \overline{I}_n$. On the one hand, notice that the second equation of (4.25) can be rewritten in matrix form as $A\Psi = b$, where Ψ is the unknown vector of approximations of the functions M at time t_{n+1}, and $b \in \mathbb{R}^{J+1}$ is given by

  $$b = \begin{pmatrix} 0 \\ \frac{1}{2}\tau^2\delta_x^{(\beta)} M_1^{n-1} + \tau^2\delta_x^{(\beta)}|U_1^n|^2 + 2\mathbb{A}_x M_1^n - \mathbb{A}_x M_1^{n-1} \\ \frac{1}{2}\tau^2\delta_x^{(\beta)} M_2^{n-1} + \tau^2\delta_x^{(\beta)}|U_2^n|^2 + 2\mathbb{A}_x M_2^n - \mathbb{A}_x M_2^{n-1} \\ \vdots \\ \frac{1}{2}\tau^2\delta_x^{(\beta)} M_{J-1}^{n-1} + \tau^2\delta_x^{(\beta)}|U_{J-1}^n|^2 + 2\mathbb{A}_x M_{J-1}^n - \mathbb{A}_x M_{J-1}^{n-1} \\ 0 \end{pmatrix} \tag{4.53}$$

  From Lemma 4.3.6, the matrix A is strictly diagonally dominant, so non-singular. It follows that there exists a (unique) vector M^{n+1} which satisfies the second difference equation of (4.25). To establish now the existence of U^{n+1}, we will employ the Leray–Schauder fixed-point theorem. Let $F : \mathcal{V}_h \to \mathcal{V}_h$ be the function whose jth component is represented by $F_j : \mathcal{V}_h \to \mathbb{C}$, and defined, for each $\Phi \in \mathcal{V}_h$ and $j \in I_{J-1}$, as

  $$\begin{aligned} F_j(\Phi) = -\frac{12}{10}\Bigg[&\frac{1}{12}(\Phi_{j+1} + \Phi_{j-1}) - 2\mathbb{A}_x U_j^n + \mathbb{A}_x U_j^{n-1} - \tau^2\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)} U_j^n \\ &+\tau^2\left(1 + M_j^n + \mu_{t,\Phi}^{(1)}|U_j^n|^2\right)\left(\mu_{t,\Phi}^{(1)} U_j^n\right)\Bigg]. \end{aligned} \tag{4.54}$$

Let F_0 and F_J be identically equal to zero. It is obvious that $\mathcal{V}_h$ is a Banach space with the Euclidean norm induced by the inner product in $\mathcal{V}_h$, and that F is a continuous and compact map. Take now a function $\Phi \in \mathcal{V}_j$ such that $\Phi = \lambda F(\Phi)$, for some $\lambda \in [0, 1]$. From this and the identity (4.54), it follows that

$$\begin{aligned} 0 = \frac{10}{12}\Phi + \lambda \left[\frac{1}{12}(\Phi^+ + \Phi^-) - 2\mathbb{A}_x U^n + \mathbb{A}_x U^{n-1} - \tau^2 \delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U^n \right] \\ + \lambda \left[\tau^2 \left(1 + M^n + \mu_{t,\Phi}^{(1)} |U^n|^2 \right) \left(\mu_{t,\Phi}^{(1)} U^n \right) \right]. \end{aligned} \tag{4.55}$$

We take now the inner product of both sides of (4.55) with $\Phi - U^{n-1}$, and calculate then the real parts on both sides. Using then the identities and inequalities in Lemma 4.3.8, it follows that

$$\begin{aligned} 0 \geq \frac{5}{12} \left(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2 \right) - \frac{5\lambda}{24}\|\Phi\|_2^2 - \frac{\lambda \|U^{n-1}\|_2^2}{6} - \frac{\lambda}{12}\|\Phi\|_2^2 - 2\lambda C_1 - \frac{\lambda}{12}\|\Phi\|_2^2 - \lambda C_2 \\ + \frac{\lambda\tau^2}{2} \left(\|\delta_x^{(\alpha/2)}\Phi\|_2^2 - \|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2 + \|\Phi\|_2^2 - \|U^{n-1}\|_2^2 \right) \\ + \frac{\lambda\tau^2}{4} \left(\|\Phi\|_4^4 - C^* \|\Phi\|_2^2 - C_3 - \|U^{n-1}\|_4^4 \right) \geq \frac{1}{24}(1 - 12\tau^2 C^{**})\|\Phi\|_2^2 - C_4, \end{aligned} \tag{4.56}$$

where

$$C_4 = 2C_1 + C_2 + \frac{7}{12}\|U^{n-1}\|_2^2 + \frac{\tau^2}{4} \left(C_3 + 2\|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2 + 2\|U^{n-1}\|_2^2 + \|U^{n-1}\|_4^4 \right). \tag{4.57}$$

Notice firstly that the hypothesis guarantees that $1 - 12\tau^2 C^{**} > 0$. Moreover, observe that

$$\|\Phi\|_2^2 \leq \frac{24 C_4}{1 - 12\tau^2 C^{**}}, \tag{4.58}$$

and that the constant C_4 only depends on the approximations U^n, U^{n-1} and M^n. The Leray–Schauder theorem and the uniform boundedness of the elements of the set S in that theorem imply now that there exists a $\Phi \in \mathcal{V}_h$ with the property that $\Phi = F(\Phi)$. In other words, the solution U^{n+1} of the finite-difference method (4.25) exists. Following now an argument similar to the proof of Theorem 4.3.5, it is easy to show that $\|M^{n+1}\|_\infty \leq C^*$.

- Finally, the existence of the approximations (U^1, M^1) can be established similarly, by applying complex matrix properties and the Leray–Schauder theorem to the system (4.26)–(4.27). In fact, M^1 is the solution of the system $A\Psi = \widetilde{b}$, where A is as before, and the $(J+1)$-dimensional vector b is defined now as

$$b = \begin{pmatrix} 0 \\ \mathbb{A}_x m_0(x_1) + \tau \mathbb{A}_x m_1(x_1) - \frac{1}{2}\tau^3 \delta_x^{(\beta)} m_1(x_1) + \frac{1}{2}\tau^2 |u_0(x_1)|^2 \\ \mathbb{A}_x m_0(x_2) + \tau \mathbb{A}_x m_1(x_2) - \frac{1}{2}\tau^3 \delta_x^{(\beta)} m_1(x_2) + \frac{1}{2}\tau^2 |u_0(x_2)|^2 \\ \vdots \\ \mathbb{A}_x m_0(x_{J-1}) + \tau \mathbb{A}_x m_1(x_{J-1}) - \frac{1}{2}\tau^3 \delta_x^{(\beta)} m_1(x_{J-1}) + \frac{1}{2}\tau^2 |u_0(x_{J-1})|^2 \\ 0 \end{pmatrix}. \tag{4.59}$$

The non-singularity of A guarantees that the approximation M^n exists and satisfies the second recursive equation of (4.25). In similar fashion, the existence of U^1 will follow using the same arguments as in the general case. We just need to point out in this case that the function $F_j : \mathcal{V}_h \to \mathbb{C}$ is defined for each $\Phi \in \mathcal{V}_h$ and $\forall j \in I_{J-1}$, by

$$\begin{aligned} F_j(\Phi) = -\frac{12}{10}\Bigg[&\frac{1}{2}(\Phi_{j+1}+\Phi_{j-1}) - \mathbb{A}_x u_0(x_j) - \tau A_x u_1(x_j) - \frac{1}{2}\tau^2\delta_x^{(\alpha)}\left(\Phi_j - \tau u_1(x_j)\right) \\ &+\frac{1}{2}\tau^2\left(\Phi_j - \tau u_1(x_j)\right)\left(1 + m_0(x_j) - \frac{1}{2}\left(|\Phi_j|^2 + |\Phi_j - 2\tau u_1(x_j)|^2\right)\right)\Bigg]. \end{aligned} \tag{4.60}$$

The conclusion follows from mathematical induction. □

4.4 Numerical properties

The present section will be devoted to prove rigorously the main numerical properties of the scheme (4.25). More precisely, we will show that the numerical approximations are second-order consistent estimates of the solutions of the continuous model (4.5). Moreover, we will establish the stability and the second-order convergence of the scheme using a suitable discrete Gronwall inequality. As a corollary of the stability property of our scheme, we will prove that the solutions of the finite-difference method are unique for sufficiently small values of τ.

Our first result will establish the quadratic consistency property of the discrete model (4.25). To that end, consider sequences $(U^n)_{n\in I_{N-1}}, (M^n)_{n\in I_{N-1}} \subseteq \mathcal{V}_h$. Under these circumstances, define $L = L_U \times L_M : \mathcal{V}_h \times \mathcal{V}_h \to \mathcal{V}_h \times \mathcal{V}_h$ by

$$L_U(U_j^n, M_j^n) = \delta_t^{(2)}\mathbb{A}_x U_j^n - \delta_x^{(\alpha)}\mu_t^{(1)}U_j^n + \mu_t^{(1)}U_j^n\left[1 + M_j^n + \mu_t^{(1)}|U_j^n|^2\right], \quad \forall (j,n) \in I, \tag{4.61}$$

$$L_M(U_j^n, M_j^n) = \delta_t^{(2)}\mathbb{A}_x M_j^n - \delta_x^{(\beta)}\mu_t^{(1)}M_j^n - \delta_x^{(\beta)}|U_j^n|^2, \quad \forall (j,n) \in I. \tag{4.62}$$

in turn, let us convey that $L(U^n, M^n) = (L(U_j^n, M_j^n))_{j\in\bar{I}_J}$ for each $n \in I_{N-1}$, and agree that $L(U,M) = (L(U^n, M^n))_{n\in I_{N-1}}$. We also consider the differential operator $\mathcal{L} = \mathcal{L}_u \times \mathcal{L}_m$, defined for each pair (u, m) of functions and $\forall (x,t) \in \Omega$, by

$$\mathcal{L}_u(u(x,t), m(x,t)) = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t), \tag{4.63}$$

$$\mathcal{L}_m(u(x,t), m(x,t)) = \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta\left(|u(x,t)|^2\right)}{\partial |x|^\beta}. \tag{4.64}$$

If $x \in \{x_L, x_R\}$ and $t \in [0,T]$, then we let $\mathcal{L}(u(x,t), m(x,t)) = 0$. Let $\mathcal{L}(u^n, m^n) = (\mathcal{L}(u_j^n, m_j^n))_{j\in\bar{I}_J}$ for each $n \in I_{N-1}$, and define $\mathcal{L}(u,m) = (\mathcal{L}(u^n, m^n))_{n\in I_{N-1}}$. In similar fashion, define $L(u^n, m^n) = (L(u_j^n, m_j^n))_{j\in\bar{I}_J} \in \mathcal{V}_h$ for each $n \in I_{N-1}$, and introduce $L(u,m) = (L(u^n, m^n))_{n\in I_{N-1}}$.

Theorem 4.4.1 (Consistency)**.** *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$, and let $h < 1$. Then there exist constants C and C' which are independent of τ and h, such that $|||(\mathcal{L} - L)(u,m)|||_\infty \leq C(\tau^2 + h^2)$ and $|||(\mathcal{H} - H)(u,m)|||_\infty \leq C'(\tau + h^2)$.*

Proof. To establish this result, we use the traditional approach using Taylor's theorem, the mean value theorem and the regularity of the functions u and m. Under those circumstances, it is possible to prove that there exists constants $C_i \in \mathbb{R}^+$ independent of τ and h, for each $i \in I_5$, with the property that

$$\left|\frac{\partial^2 u(x_j,t_n)}{\partial t^2} - \delta_t^{(2)}\mathbb{A}_x u_j^n\right| \leq C_1(\tau^2+h^2), \quad \forall (j,n)\in I, \tag{4.65}$$

$$\left|\frac{\partial^\alpha u(x_j,t_n)}{\partial |x|^\alpha} - \delta_x^{(\alpha)}\mu_t^{(1)} u_j^n\right| \leq C_2(\tau^2+h^2), \quad \forall (j,n)\in I, \tag{4.66}$$

$$\left|u(x_j,t_n) - \mu_t^{(1)} u_j^n\right| \leq C_3\tau^2, \quad \forall (j,n)\in I, \tag{4.67}$$

$$\left|m(x_j,t_n)u(x_j,t_n) - m_j^n\mu_t^{(1)} u_j^n\right| \leq C_4\tau^2, \quad \forall (j,n)\in I, \tag{4.68}$$

$$\left||u(x_j,t_n)|^2 u(x_j,t_n) - \left(\mu_t^{(1)}|u_j^n|^2\right)\left(\mu_t^{(1)} u_j^n\right)\right| \leq C_5\tau^2, \quad \forall (j,n)\in I. \tag{4.69}$$

Using the triangle inequality, we verify that there exists a constant $C^* \in \mathbb{R}^+$ which is independent of τ and h, such that $|||(L_U - L_U)(u,m)|||_\infty < C^*(\tau^2+h^2)$. Also, there exist constants $C_6, C_7, C_8 \in \mathbb{R}^+$ independent of τ and h, for which

$$\left|\frac{\partial^2 m(x_j,t_n)}{\partial t^2} - \delta_t^{(2)}\mathbb{A}_x m_j^n\right| \leq C_6(\tau^2+h^2), \quad \forall (j,n)\in I, \tag{4.70}$$

$$\left|\frac{\partial^\beta m(x_j,t_n)}{\partial |x|^\beta} - \delta_x^{(\beta)}\mu_t^{(1)} m_j^n\right| \leq C_7(\tau^2+h^2), \quad \forall (j,n)\in I, \tag{4.71}$$

$$\left|\frac{\partial^\beta\left(|u(x_j,t_n)|^2\right)}{\partial |x|^\beta} - \delta_x^{(\beta)}|u_j^n|^2\right| \leq C_8 h^2, \quad \forall (j,n)\in I, \tag{4.72}$$

are satisfied. Using the triangle inequality, we can readily check that there exists a constant $C^{**} \in \mathbb{R}^+$ which is independent of τ and h, with the property that $|||(\mathcal{L}_M - L_M)(u,m)|||_\infty < C^{**}(\tau^2+h^2)$. Then the constant C of the conclusion is the maximum of C^* and C^{**}. The second inequality of this result can be obtained in similar fashion. □

It is important to point out that the existence and uniqueness of solutions satisfying the regularity conditions in the hypotheses of Theorem 4.4.1 have been established in the completion of $\mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$. This fact was thoroughly proved in reference [32] for the non-fractional scenario $\alpha = \beta = 2$. However, to the best of our knowledge, the existence and uniqueness of sufficiently smooth solutions for the fully fractional case is still an open problem of investigation.

Definition 4.4.2. If $f : \mathbb{F} \to \mathbb{F}$ and $V \in \mathcal{V}_h$ then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

The next objective is to prove the stability and convergence properties of (4.5). For that reason and in the following, (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ will represent two sets of initial conditions of (4.5). We will assume also that the initial data for (4.25) are provided exactly.

Lemma 4.4.3 (Gronwall's inequality [105])**.** *Assume that $N \in \mathbb{N}$ with $N > 1$. Let $(\omega^n)_{n\in\overline{I}_N}$ and $(C_n)_{n\in I_N}$ be sequences of real numbers, and let A, B and C_n be non-negative numbers, for each $n \in I_N$. Suppose that $\tau \in \mathbb{R}^+$ is such that*

$$\omega^n - \omega^{n-1} \leq A\tau\omega^n + B\tau\omega^{n-1} + C_n\tau, \quad \forall n \in I_N. \tag{4.73}$$

If $(A+B)\tau \leq (N-1)/(2N)$, then

$$\max_{n\in I_N}|\omega^n| \leq \left(\omega^0 + \tau\sum_{k\in I_N} C_k\right) e^{2(A+B)N\tau}. \tag{4.74}$$

Lemma 4.4.4. *Let $u_0, m_0, \widetilde{u}_0, \widetilde{m}_0 \in H^1(\overline{B})$ and $u_1, m_1, \widetilde{u}_1, \widetilde{m}_1 \in L_2(\overline{B})$. Suppose that (U, M) and $(\widetilde{U}, \widetilde{M})$ are the solutions of (4.25) corresponding to (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$, respectively. Let $\varepsilon^n = \widetilde{U}^n - U^n$, $\zeta^n = \widetilde{M}^n - M^n$ and $\upsilon^n = \widetilde{V}^n - V^n$, for each $n \in \overline{I}_N$, and define $\forall n \in I_{N-1}$,*

$$\omega^n = \|\delta_t \varepsilon^n\|_2^2 - \tfrac{h^2}{12}\|\delta_{xt}\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 + \mu_t\|\varepsilon^n\|_2^2 + \|\delta_x^{(\beta/2)}\upsilon^n\|_2^2 - \tfrac{h^2}{12}\|\delta_x\delta_x^{(\beta/2)}\upsilon^n\|_2^2 + \mu_t\|\zeta^n\|_2^2. \tag{4.75}$$

For τ sufficiently small, there exists $C \in \mathbb{R}^+$ independent of h and τ, such that $\omega^n \leq \omega^0 \exp(CT)$, for each $n \in \overline{I}_{N-1}$.

Proof. Notice that the sequence (ε, ζ) satisfies the system

$$\begin{aligned} \delta_t^{(2)}\mathbb{A}_x\varepsilon_j^n - \delta_x^{(\alpha)}\mu_t^{(1)}\varepsilon_j^n + \mu_t^{(1)}\varepsilon_j^n + \widetilde{\delta}\left[M_j^n\mu_t^{(1)}U_j^n\right] + \widetilde{\delta}\left[\left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] &= 0, \quad \forall (j,n) \in I, \\ \delta_t^{(2)}\mathbb{A}_x\zeta_j^n - \delta_x^{(\beta)}\mu_t^{(1)}\zeta_j^n - \widetilde{\delta}\left(\delta_x^{(\beta)}|U_j^n|^2\right) &= 0, \quad \forall (j,n) \in I, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N. \end{aligned} \tag{4.76}$$

Proceeding as in Lemma 4.3.2, we can readily obtain the following identities:

(i) $2\,\mathrm{Re}\langle\delta_t^{(2)}\mathbb{A}_x\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle = \delta_t\left[\|\delta_t\varepsilon^{n-1}\|_2^2 - \frac{1}{12}h^2\|\delta_{xt}\varepsilon^{n-1}\|_2^2\right]$,

(ii) $2\,\mathrm{Re}\langle-\delta_x^{(\alpha)}\mu_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle = \delta_t\mu_t\|\delta_x^{(\alpha/2)}\varepsilon^{n-1}\|_2^2$,

(iii) $2\,\mathrm{Re}\langle\mu_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle = \delta_t\mu_t\|\varepsilon^{n-1}\|_2^2$,

(iv) $-2\langle\delta_t^{(2)}\mathbb{A}_x\zeta^n, \mu_t\upsilon^{n-1}\rangle = \delta_t\left[\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 - \frac{1}{12}h^2\|\delta_x\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right]$, and

(v) $2\langle\delta_x^{(\beta)}\mu_t^{(1)}\zeta^n, \mu_t\upsilon^{n-1}\rangle = \delta_t\mu_t\|\zeta^{n-1}\|_2^2$.

Using the fact $\delta_x^{(\beta)}\upsilon^n = \delta_t\zeta^n$ and Theorem 4.3.5, it is easy to show that there exist $C_1, C_2, C_3 \in \mathbb{R}^+$, such that

$$\mathrm{Re}\left\langle\widetilde{\delta}\left[M^n\mu_t^{(1)}U^n\right], \delta_t^{(1)}\varepsilon^n\right\rangle \leq C_1\left(\mu_t\|\delta_t\varepsilon^{n-1}\|_2^2 + \mu_t^{(1)}\left[\|\zeta^n\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2\right]\right) \tag{4.77}$$

$$\mathrm{Re}\left\langle\widetilde{\delta}\left[\left(\mu_t^{(1)}|U^n|^2\right)\left(\mu_t^{(1)}U^n\right)\right], \delta_t^{(1)}\varepsilon^n\right\rangle \leq C_2\left(\mu_t^{(1)}\|\varepsilon^n\|_2^2 + \mu_t\|\delta_t\varepsilon^{n-1}\|_2^2\right), \tag{4.78}$$

$$\left|\left\langle\widetilde{\delta}\left(\delta_x^{(\beta)}|U^n|^2\right), \mu_t\upsilon^{n-1}\right\rangle\right| \leq C_3\left(\|\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2\right). \tag{4.79}$$

Take the real part of the inner product of first equation in (4.76) with $2\delta_t^{(1)}\varepsilon^n$, and calculate the product between the second equation and $2\delta_t\zeta^{n-1}$. Use then the identities (i)–(v) and the inequalities (4.77)–(4.79) to show that there exists $C_4 \in \mathbb{R}^+$, such that the following inequalities are satisfied for each $n \in I_{N-1}$:

$$\begin{aligned} \delta_t\left[\|\delta_t\varepsilon^{n-1}\|_2^2 - \tfrac{h^2}{12}\|\delta_{xt}\varepsilon^{n-1}\|_2^2 + \mu_t\left(\|\delta_x^{(\alpha/2)}\varepsilon^{n-1}\|_2^2 + \|\varepsilon^{n-1}\|_2^2\right)\right] \leq C_4\Big[\mu_t^{(1)}\|\zeta^n\|_2^2 + \mu_t^{(1)}\|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 & \\ + \mu_t^{(1)}\|\varepsilon^n\|_2^2 + \mu_t\|\delta_t\varepsilon^{n-1}\|_2^2\Big], & \end{aligned} \tag{4.80}$$

$$\begin{aligned} \delta_t\left[\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 - \tfrac{h^2}{12}\|\delta_x\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \mu_t\|\zeta^{n-1}\|_2^2\right] \leq C_4\Big[\|\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 & \\ + \|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2\Big]. & \end{aligned} \tag{4.81}$$

Adding these last inequalities, we obtain that for each $n \in I_N$,

$$\begin{aligned}\omega^n - \omega^{n-1} &\leq C_5\tau\left[\tfrac{4}{3}\mu_t\left(\|\delta_t\varepsilon^{n-1}\|_2^2 + \|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right) + \mu_t^{(1)}\left(\|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2 + \|\zeta^n\|_2^2\right)\right.\\ &\qquad \left. + \|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2 + \|\zeta^n\|_2^2\right]\\ &\leq C_5\tau\left(\omega^n + \omega^{n-1}\right).\end{aligned} \tag{4.82}$$

If τ is sufficiently small (namely, if $C_5\tau \leq (N-1)/(4N)$), then Gronwall's inequality readily establishes the conclusion of this lemma with $C = 4C_5$. □

The following results are immediate consequences of Lemma 4.4.4.

Theorem 4.4.5 (Stability)**.** *If the initial data satisfy $u_0, m_0 \in H^1(\overline{B})$ and $u_1, m_1 \in L_2(\overline{B})$ then the solutions of the numerical model* (4.25) *are stable for sufficiently small values of τ.* □

Corollary 4.4.6 (Uniqueness)**.** *Assume that the hypotheses of Theorem 4.4.5 are satisfied. If τ is sufficiently small then the numerical model* (4.25) *is uniquely solvable.* □

Before closing this section, we would like to prove that the discrete model proposed in this manuscript has quadratic order of convergence in both, space and time. To that end, we consider the local truncation errors of the finite-difference system (4.25) at the node (x_j, t_n). More precisely, we will let

$$\rho_j^n = \delta_t^{(2)}\mathbb{A}_x u_j^n - \delta_x^{(\alpha)}\mu_t^{(1)}u_j^n + \mu_t^{(1)}u_j^n + m_j^n\mu_t^{(1)}u_j^n + \left(\mu_t^{(1)}|u_j^n|^2\right)\left(\mu_t^{(1)}u_j^n\right), \quad \forall(j,n) \in I, \tag{4.83}$$

$$\sigma_j^n = \delta_t^{(2)}\mathbb{A}_x m_j^n - \delta_x^{(\beta)}\mu_t^{(1)}m_j^n - \delta_x^{(\beta)}|u_j^n|^2, \quad \forall(j,n) \in I. \tag{4.84}$$

According to the theorem on the consistency of the finite-difference scheme (4.25), $|\rho_j^n| + |\sigma_j^n| = \mathcal{O}(\tau^2 + h^2)$. In the sequel, we will use (u, m) to represent a solution of the differential model (4.5), while (U, M) denotes a solution of the discrete system (4.25) corresponding to the same set of initial data. Under these circumstances, let $\epsilon_j^n = u_j^n - U_j^n$, $\eta_j^n = m_j^n - M_j^n$ and $\delta_x^{(\beta)}\theta_j^n = \delta_t\eta_j^n$, for each $(j, n) \in I$.

Definition 4.4.7. If $f : \mathbb{F} \to \mathbb{F}$ is a function and $V \in \mathcal{V}_h$, then $\widehat{\delta}(f(v_j)) = f(v_j) - f(V_j)$, for each $j \in I_{J-1}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Theorem 4.4.8 (Convergence)**.** *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. If τ is sufficiently small, then the solution of the problem* (4.25) *converges to that of* (4.5) *with order $\mathcal{O}(\tau^2 + h^2)$ in the L_2-norm.*

Proof. Throughout this proof, we will follow the notation and conventions mentioned in the paragraph preceding Definition 4.4.7. As a consequence, observe that the ordered pair (ϵ, η) satisfies the discrete system

$$\begin{aligned}\delta_t^{(2)}\mathbb{A}_x\epsilon_j^n - \delta_x^{(\alpha)}\mu_t^{(1)}\epsilon_j^n + \mu_t^{(1)}\epsilon_j^n + \widehat{\delta}\left[m_j^n\left(\mu_t^{(1)}u_j^n\right)\right] + \widehat{\delta}\left[\left(\mu_t^{(1)}|u_j^n|^2\right)\left(\mu_t^{(1)}u_j^n\right)\right] &= \rho_j^n, \ \forall(j,n) \in I,\\ \delta_t^{(2)}\mathbb{A}_x\eta_j^n - \delta_x^{(\beta)}\mu_t^{(1)}\eta_j^n - \widehat{\delta}\left(\delta_x^{(\beta)}|u_j^n|^2\right) &= \sigma_j^n, \ \forall(j,n) \in I,\\ \text{subject to}\begin{cases}\epsilon^0 = \eta^0 = \epsilon^1 = \eta^1 = 0,\\ \epsilon_0^n = \epsilon_J^n = \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N.\end{cases}&\end{aligned} \tag{4.85}$$

Proceeding as in the proof of Lemma 4.4.4, it is easy to show that there exists a common $C_1 \in \mathbb{R}^+$ such that

$$\operatorname{Re}\langle \rho^n, \delta_t^{(1)}\epsilon^n \rangle \leq C_1 \left(\|\rho^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \right), \tag{4.86}$$

$$\operatorname{Re}\left\langle \widehat{\delta}\left[m^n \left(\mu_t^{(1)} u^n \right) \right], \delta_t^{(1)}\epsilon^n \right\rangle \leq C_1 \left(\mu_t \|\delta_t \epsilon^{n-1}\|_2^2 + \mu_t^{(1)} \left[\|\eta^n\|_2^2 + \|\epsilon^n\|_2^2 + \|\delta_x^{(\alpha/2)}\epsilon^n\|_2^2 \right] \right), \tag{4.87}$$

$$\operatorname{Re}\left\langle \widehat{\delta}\left[\left(\mu_t^{(1)} |u^n|^2 \right) \left(\mu_t^{(1)} u^n \right) \right], \delta_t^{(1)}\epsilon^n \right\rangle \leq C_1 \left(\mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \right), \tag{4.88}$$

$$\langle \sigma^n, \mu_t \theta^{n-1} \rangle \leq C_1 \left(\|\sigma^n\|_2^2 + \mu_t \|\theta^{n-1}\|_2^2 \right), \tag{4.89}$$

$$\langle \widehat{\delta}\left(\delta_x^{(\beta)} |u_j^n|^2 \right), \mu_t \theta^{n-1} \rangle \leq C_1 \left(\|\epsilon^n\|_2^2 + \mu_t \|\delta_x^{(\beta/2)}\theta^{n-1}\|_2^2 + \|\delta_x^{(\alpha/2)}\epsilon^n\|_2^2 \right). \tag{4.90}$$

Firstly, take the inner product between the first vector equation of (4.85) and $2\delta_t^{(1)}\epsilon^n$, and employ next the inequalities (4.86)–(4.88). At the same time, we take the inner product of the second vector equation with $2\mu_t\theta^{n-1}$, and use (4.89) and (4.90). It follows that there exists a constant $C_2 \in \mathbb{R}^+$, with the property that

$$\begin{aligned}\delta_t \left[\|\delta_t \epsilon^{n-1}\|_2^2 - \tfrac{h^2}{12} \|\delta_{xt}\epsilon^{n-1}\|_2^2 + \mu_t \left(\|\delta_x^{(\alpha/2)}\epsilon^{n-1}\|_2^2 + \|\epsilon^{n-1}\|_2^2 \right) \right] \leq C_2 \Big[& \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 + \mu_t^{(1)} \|\eta^n\|_2^2 \\ & + \mu_t^{(1)} \left(\|\delta_x^{(\alpha/2)}\epsilon^n\|_2^2 + \|\epsilon^n\|_2^2 \right) \Big] \\ & + \|\rho^n\|_2^2 \Big], \end{aligned} \tag{4.91}$$

and

$$\begin{aligned}\delta_t \left[\|\delta_x^{(\beta/2)}\theta^{n-1}\|_2^2 - \frac{h^2}{12} \|\delta_x \delta_x^{(\beta/2)}\theta^{n-1}\|_2^2 + \mu_t \|\eta^{n-1}\|_2^2 \right] \leq C_2 \Big[& \|\sigma^n\|_2^2 + \|\epsilon^n\|_2^2 + \mu_t \|\delta_x^{(\beta/2)}\theta^{n-1}\|_2^2 \\ & + \|\delta_x^{(\alpha/2)}\epsilon^n\|_2^2 + \mu_t \|\theta^{n-1}\|_2^2 \Big]. \end{aligned} \tag{4.92}$$

Adding the inequalities (4.91) and (4.92), we can show that there exists $C_3 \in \mathbb{R}^+$ such that, for each $n \in I_{N-1}$, the following inequality holds: $\xi^n - \xi^{n-1} \leq C_3\tau(\|\rho^n\|_2^2 + \|\sigma^n\|_2^2) + C_3\tau(\xi^n + \xi^{n-1})$. Here, $\forall n \in \overline{I}_{N-1}$, we have

$$\xi^n = \|\delta_t \epsilon^n\|_2^2 - \tfrac{h^2}{12} \left(\|\delta_{xt}\epsilon^n\|_2^2 + \|\delta_x \delta_x^{(\beta/2)}\theta^n\|_2^2 \right) + \mu_t \left(\|\delta_x^{(\alpha/2)}\epsilon^n\|_2^2 + \|\epsilon^n\|_2^2 + \|\eta^n\|_2^2 \right) + \|\delta_x^{(\beta/2)}\theta^n\|_2^2. \tag{4.93}$$

Finally, by Lemma 4.4.3, we know that there exists $C \in \mathbb{R}^+$ such that $\xi^n \leq C(\tau^2 + h^2)$. In particular, it is easy to establish now that $\|\epsilon^n\|_2, \|\eta^n\|_2 \leq C(\tau^2 + h^2)$, for each $n \in \overline{I}_{N-1}$, as desired. □

4.5 Computer implementation

We describe now the computational implementation of the scheme (4.25) and provide some illustrative simulations. About the computational implementation, it is important to point out that the left-hand side of the first recursive equation of (4.25) is a function of the unknown complex vector U^{n+1}. That function considers the presence of $|U^{n+1}|$ which, unfortunately, is not analytic in the variable U^{n+1}. As a consequence, the use of numerical methods to approximate the roots of complex functions which consider derivatives of the functions of interest (including Newton–Raphson techniques) is discarded. To avoid using such methods, we will employ an iterative approach. More precisely, in the following discussion, we will assume that τ is sufficiently small

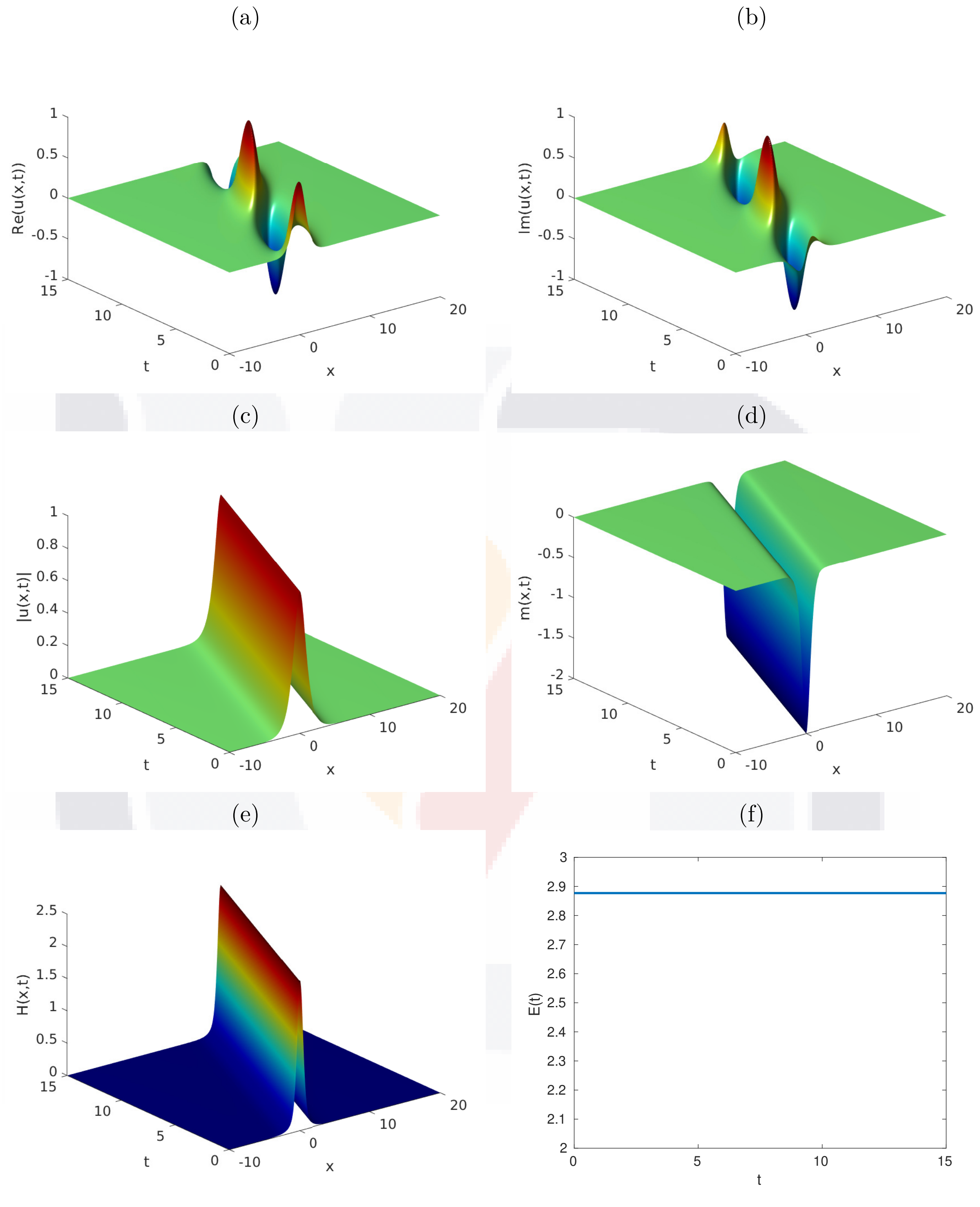

Figure 4.1: Results of approximating numerically the solution of the model (4.5) on the set $\Omega = (-10, 20) \times (0, 15)$, with initial data (4.99)–(4.102). In our numerical implementation, we used $h = 0.1$, $\tau = 0.05$, a tolerance of 1×10^{-8}, and a maximum number of iterations equal to 20. In this simulations, we employed $\alpha = \beta = 2$. The graphs provide the approximate behavior of (a) $\mathrm{Re}(u(x,t))$, (b) $\mathrm{Im}(u(x,t))$, (c) $|u(x,t)|$, (d) $m(x,t)$ and (e) $\mathcal{H}(x,t)$ as functions of $(x,t) \in \overline{\Omega}$. Meanwhile, (f) is the approximation to the total energy $\mathcal{E}(t)$ with respect to time $t \in [0, 15]$.

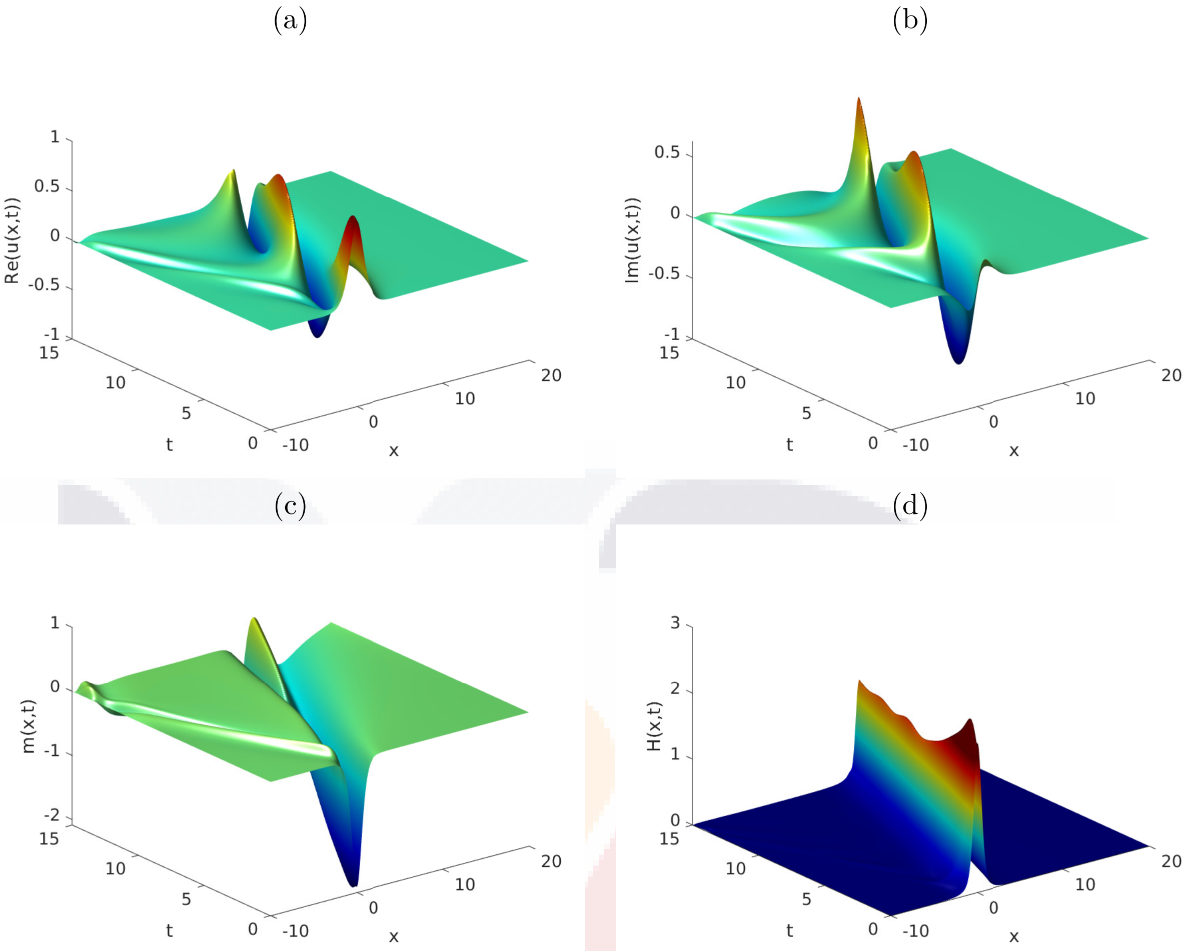

Figure 4.2: Results of approximating numerically the solution of the model (4.5) on the set $\Omega = (-10, 20) \times (0, 15)$, with initial data (4.99)–(4.102). In our numerical implementation, we used $h = 0.1$, $\tau = 0.05$, a tolerance of 1×10^{-8}, and a maximum number of iterations equal to 20. In this simulations, we employed $\alpha = \beta = 1.7$. The graphs provide the approximate behavior of (a) $\mathrm{Re}(u(x,t))$, (b) $\mathrm{Im}(u(x,t))$, (c) $m(x,t)$ and (f) $\mathcal{H}(x,t)$ as functions of $(x,t) \in \overline{\Omega}$.

to guarantee the existence and the uniqueness of solutions, and we will employ the approach followed in the proof of the theorem on the existence of solutions of our discrete model, namely, Theorem 4.3.10.

Beforehand, notice that the initial approximations U^0 and M^0 are provided explicitly through the initial conditions. In the general step, suppose that the approximations U^{n-1}, M^{n-1}, U^n and M^n have been constructed. As we pointed out in the proof of Theorem 4.3.10, the approximation M^{n+1} solves the linear system $A\Phi = b$, where $A = C - D^{(\beta)}$, and C, $D^{(\beta)}$ and b are given respectively by (4.43), (4.44) and (4.53). This solution exists by virtue of the fact that A is strictly diagonally dominant, so non-singular. To calculate the approximation U^{n+1}, notice that the first difference equation of the model (4.25) can be alternatively expressed, for each $j \in I_{J-1}$, as

$$\begin{aligned}\mathbb{A}_x U_j^{n+1} - \tfrac{\tau^2}{2}\left(\delta_x^{(\alpha)} U_j^{n+1} - (1 + M_j^n) U_j^{n+1}\right) = 2\mathbb{A}_x U_j^n - \mathbb{A}_x U_j^{n-1} + \tfrac{\tau^2}{2}\left[\delta_x^{(\alpha)} U_j^{n-1} - (1 + M_j^n) U_j^{n-1}\right] \\ - \tfrac{\tau^2}{4}\left(|U_j^{n+1}|^2 + |U_j^{n-1}|^2\right)\left(U_j^{n+1} + U_j^{n-1}\right). \end{aligned} \tag{4.94}$$

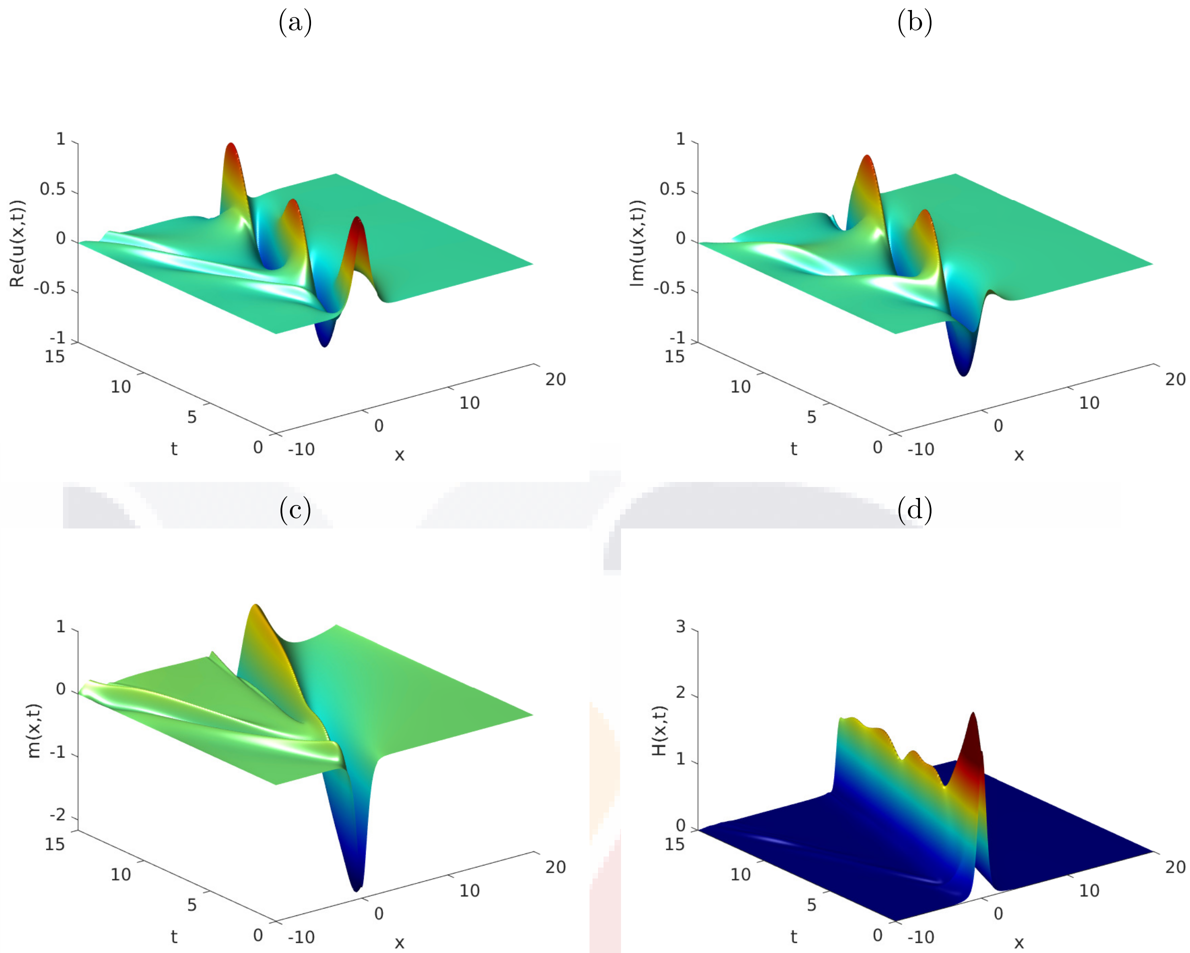

Figure 4.3: Results of approximating numerically the solution of the model (4.5) on the set $\Omega = (-10, 20) \times (0, 15)$, with initial data (4.99)–(4.102). In our numerical implementation, we used $h = 0.1$, $\tau = 0.05$, a tolerance of 1×10^{-8}, and a maximum number of iterations equal to 20. In this simulations, we employed $\alpha = \beta = 1.4$. The graphs provide the approximate behavior of (a) $\mathrm{Re}(u(x,t))$, (b) $\mathrm{Im}(u(x,t))$, (c) $m(x,t)$ and (f) $\mathcal{H}(x,t)$ as functions of $(x,t) \in \overline{\Omega}$.

For convenience, let us define now the matrix $F = C - D^{(\alpha)} + E$, where $D^{(\alpha)}$ is given as in (4.44) and E is the matrix of size $(J+1) \times (J+1)$ defined by

$$E = \frac{\tau^2}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 + M_1^n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 + M_2^n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 + M_3^n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 + M_{J-1}^n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.95}$$

In turn, let $c : \mathcal{V}_h \to \mathcal{V}_h$ be the function whose jth component is defined, for each $\Phi \in \mathcal{V}_h$, by

$$c_j(\Phi) = 2\mathbb{A}_x U_j^n - \mathbb{A}_x U_j^{n-1} + \tfrac{\tau^2}{2} \left[\delta_x^{(\alpha)} U_j^{n-1} - (1 + M_j^n) U_j^{n-1} \right] - \tau^2 \left(\mu_{t,\Phi}^{(1)} |U_j^n|^2 \right) \left(\mu_{t,\Phi}^{(1)} U_j^n \right), \tag{4.96}$$

whenever $j \in I_{J-1}$, and $c_j(\Phi) = 0$ when $j = 0, J$. The fact that U^{n+1} satisfies the system of equations (4.94) can be equivalently written in vector form as $FU^{n+1} = c(U^{n+1})$. Motivated

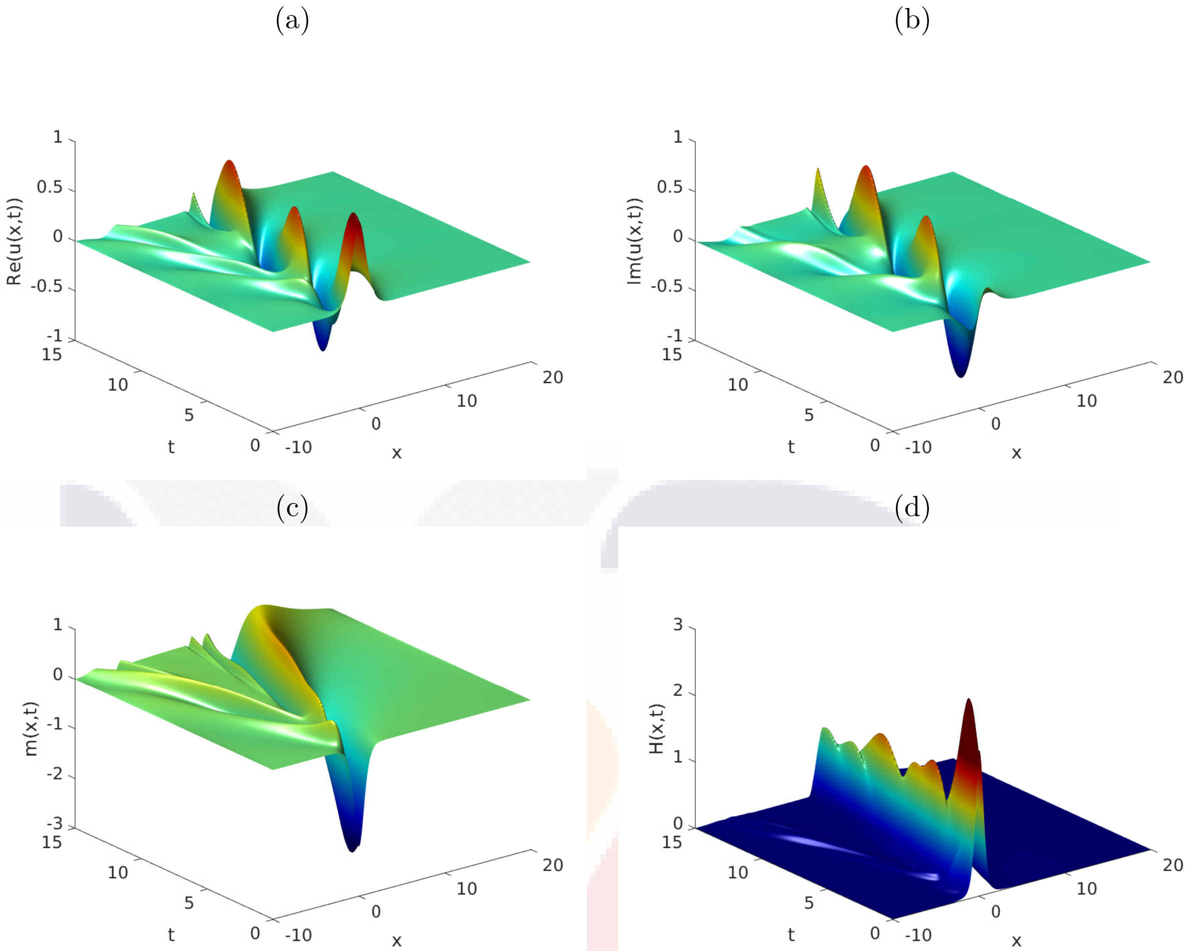

Figure 4.4: Results of approximating numerically the solution of the model (4.5) on the set $\Omega = (-10, 20) \times (0, 15)$, with initial data (4.99)–(4.102). In our numerical implementation, we used $h = 0.1$, $\tau = 0.05$, a tolerance of 1×10^{-8}, and a maximum number of iterations equal to 20. In this simulations, we employed $\alpha = \beta = 1.1$. The graphs provide the approximate behavior of (a) $\mathrm{Re}(u(x,t))$, (b) $\mathrm{Im}(u(x,t))$, (c) $m(x,t)$ and (f) $\mathcal{H}(x,t)$ as functions of $(x,t) \in \overline{\Omega}$.

by the proof of Theorem 4.3.10, we will solve this problem recursively. Suppose that an initial approximation $(U^{n+1})^{(0)}$ to the vector U^{n+1} has been provided. Inductively, assume that an approximation $(U^{n+1})^{(m)}$ to the vector U^{n+1} has been calculated, for some $m \in \mathbb{N} \cup \{0\}$. Then $(U^{n+1})^{(m+1)}$ is the vector satisfying the linear system $F((U^{n+1})^{(m+1)}) = c((U^{n+1})^{(m)})$. The procedure will continue until a stopping criterion is satisfied, say, when $\|(U^{n+1})^{(m+1)} - (U^{n+1})^{(m)}\|_2$ is smaller than some tolerance, or when m reaches the maximum number of iterations.

Finally, our implementation of the numerical model (4.25) will make use of the following recursive relation to calculate the values of the coefficients of the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$, for each $\alpha \in (1, 2]$.

Lemma 4.5.1 (Çelik and Duman [13]). *If $1 < \alpha \leq 2$, then the following recursive identities hold:*

$$g_0^{(\alpha)} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha/2 + 1)^2}, \tag{4.97}$$

$$g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha + 1}{\alpha/2 + k + 1}\right) g_k^{(\alpha)}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{4.98}$$

Before closing this section, we wish to provide some illustrative simulations in the fractional and non-fractional scenarios. To that end, we will let $B = (-15, 20)$ and $T = 15$, and define the functions u_0, m_0, u_1 and m_1 by

$$u_0(x) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x \in B, \tag{4.99}$$

$$m_0(x) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x \in B, \tag{4.100}$$

and

$$u_1(x) = \frac{\sqrt{10} - \sqrt{2}}{2}(\tanh x - 1) \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x \in B, \tag{4.101}$$

$$m_1(x) = -4\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x \in B. \tag{4.102}$$

This functions will be employed next as initial data in our simulations. It is worth noting that the exact solution of the model (4.5) on the set $\mathbb{R} \times \overline{\mathbb{R}}^+$, initial conditions (4.99)–(4.102) and $\alpha = \beta = 2$, is given by the system of functions

$$u(x,t) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right) \exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}, \tag{4.103}$$

and

$$m(x,t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}. \tag{4.104}$$

Let us consider the system (4.5) defined on the set $\Omega = (-10, 20) \times (0, 15)$, with initial data (4.99)–(4.102). We approximate the solutions of this model using our computer implementation of the finite-difference scheme (4.25), letting $h = 0.1$ and $\tau = 0.05$, using a tolerance of 1×10^{-8} and a maximum number of iterations equal to 20. Figure 4.1 shows the results of our simulations for the non-fractional case $\alpha = \beta = 2$. More precisely, the first five graphs provide surface plots of the numerical solutions of (a) $\mathrm{Re}(u(x,t))$, (b) $\mathrm{Im}(u(x,t))$, (c) $|u(x,t)|$, (d) $m(x,t)$ and (e) $\mathcal{H}(x,t)$ as functions of $(x,t) \in \overline{\Omega}$. Meanwhile, (f) provides the graph of total energy $\mathcal{E}(t)$ with respect to time $t \in [0, 15]$. It is worth pointing out that the numerical approximations are in good agreement with the exact solutions (4.103)–(4.104). Moreover, notice that the total energy of the system is approximately constant, in agreement with the theoretical results derived in this work. Finally, Figures 4.2, 4.3 and 4.4 provided illustrative simulations for the same experiment, using $\alpha = \beta = 1.7$, $\alpha = \beta = 1.4$ and $\alpha = \beta = 1.1$, respectively.

5. Two energy-preserving numerical models

5.1 Preliminaries

This manuscript considers a nonempty and bounded spatial interval of $\mathbb{R}$ of the form $B = (x_L, x_R)$, where $x_L, x_R \in \mathbb{R}$ satisfy $x_L < x_R$. Let $T > 0$ represent a finite period of time, and define $\Omega = B \times (0, T)$. In general, for each $S \subseteq \mathbb{R}^2$, we let $\overline{S}$ be the closure of S with respect to the standard topology of $\mathbb{R}^2$. Throughout, we will let $u : \overline{\Omega} \to \mathbb{C}$ and $m : \overline{\Omega} \to \mathbb{R}$ be sufficiently regular functions, and let $u_0, u_1 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be smooth functions. We will define all the relevant functions on $\overline{\Omega}$ and, for the sake of convenience, we will extend their definitions to the set $\mathbb{R} \times [0, T]$, assuming them to be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$.

Definition 5.1.1 (Podlubny [78])**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{R}$ satisfy $n - 1 < \alpha \leq n$. The *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is defined (when it exists) as

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{|x-\xi|^{\alpha+1-n}}, \tag{5.1}$$

where Γ is the usual Gamma function. In the case that $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, and n and α are as above, then the *Riesz fractional partial derivative* of u of order α with respect to x at the point $(x, t) \in \mathbb{R} \times [0, T]$ is given (if it exists) by

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi,t)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{5.2}$$

The Riesz fractional partial derivative of u of order α with respect to x is also denoted by $\partial^\alpha_{|x|} u$ in this work. In case that $\alpha = n \in \mathbb{N} \cup \{0\}$, then we agree that $\partial^\alpha_{|x|} u$ denotes the usual nth-order partial derivative operator with respect to x.

Definition 5.1.2. If $z \in \mathbb{C}$, then we will represent its complex conjugate by $\overline{z}$. Depending on the circumstances, we will use $\mathbb{F}$ to represent the fields $\mathbb{R}$ or $\mathbb{C}$. Let us define the set $L_{x,p}(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{F} : f(\cdot, t) \in L_p(\overline{B}), \text{ for each } t \in [0, T]\}$, where $p \in [1, \infty]$. If $p \in [1, \infty)$ and $f \in L_{x,p}(\overline{\Omega})$, then we convey that

$$\|f\|_{x,p} = \left(\int_B |f(x,t)|^p dx \right)^{1/p}, \quad \forall t \in [0, T]. \tag{5.3}$$

In the case when $p = \infty$, we set $\|f\|_{x,\infty} = \inf\{C \geq 0 : |f(x,t)| \leq C \text{ for almost all } x \in \overline{B}\}$. Obviously, $\|f\|_{x,p}$ is a function of $t \in [0, T]$ in any case. Moreover, for each pair $f, g \in L_{x,2}(\overline{\Omega})$, define the following function of t:

$$\langle f, g \rangle_x = \int_B f(x,t)\overline{g(x,t)}dx, \quad \forall t \in [0, T]. \tag{5.4}$$

In this work, we fix $\alpha, \beta \in (1, 2]$, and work with the fractional extension of the Klein–Gordon–Zakharov model given by the following coupled system of fractional differential equations with initial-boundary data:

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t) = 0, \quad \forall (x,t) \in \Omega, \\ \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta} = 0, \quad \forall (x,t) \in \Omega, \\ \text{subject to} \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), & \forall x \in \overline{B}, \\ \dfrac{\partial u(x,0)}{\partial t} = u_1(x), & \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, & \forall t \in [0,T]. \end{cases} \end{aligned} \tag{5.5}$$

It is important to point out that the case $\alpha = \beta = 2$ in (5.5) yields precisely the well-known Klein–Gordon–Zakharov system. Moreover, as we mention before, the Klein–Gordon–Zakharov system describes physical phenomena, specifically the interaction between Langmuir waves in a high-frequency plasma. Under this context, the function u represents the fast time-scale component of an electric field raised by electrons, and the function m is the deviation of ion density from its equilibrium. The model is given here in dimensionless form for the sake of convenience.

It is well known that the additive inverse of the Riesz fractional derivative of order α has a unique square-root operator over the space of sufficiently regular functions with compact support [46]. In fact, this unique operator is $\partial_{|x|}^{\alpha/2}$, and satisfies the following, for any two such functions u and v (see [27]):

$$\left\langle u, -\frac{\partial^\alpha v}{\partial |x|^\alpha} \right\rangle_x = \left\langle -\frac{\partial^\alpha u}{\partial |x|^\alpha}, v \right\rangle_x = \left\langle \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}, \frac{\partial^{\alpha/2} v}{\partial |x|^{\alpha/2}} \right\rangle_x, \quad \forall t \in [0,T]. \tag{5.6}$$

For convenience, we will employ the function $v : \overline{\Omega} \to \mathbb{R}$ defined as

$$\frac{\partial^\beta v(x,t)}{\partial |x|^\beta} = \frac{\partial m(x,t)}{\partial t}, \qquad \forall (x,t) \in \Omega. \tag{5.7}$$

Definition 5.1.3. Let u, m be a pair of functions satisfying the initial-boundary-value problem (5.5). We define the Hamiltonian of that fractional system as $\mathcal{H}(u(x,t), m(x,t)) = \mathcal{H}(x,t)$, where

$$\mathcal{H}(x,t) = \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right|^2 + |u|^2 + m|u|^2 + \frac{1}{2}\left|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right|^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \quad \forall (x,t) \in \Omega. \tag{5.8}$$

For simplicity, we obviated here the dependence of all the functions on the right-hand side of this identity with respect to (x,t). Then, the associated energy of the system at the time $t \in [0,T]$ is given by

$$\mathcal{E}(t) = \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2}\left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4. \tag{5.9}$$

Under this circumstances, we have the next results which were proved in [62].

Theorem 5.1.4 (Energy conservation)**.** *If u and m satisfy the problem* (5.5)*, then the function $\mathcal{E}$ is constant.* □

Theorem 5.1.5 (Boundedness). *Let u and m satisfy the initial-boundary-value problem* (5.5), *and let $u, \partial u/\partial x \in L_{x,2}(\overline{\Omega})$. Then there exist a constant C such that*

$$\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \|m\|_{x,2}^2 \leq C, \quad \forall t \in [0,T]. \tag{5.10}$$

Moreover, the constant function (5.9) *is nonnegative.* □

We provide next the numerical nomenclature to build the some discretizations of (5.5). In particular, we will focus on the concept of fractional centered differences, which allows us to discretize Riesz fractional derivatives. We have opted to use fractional centered differences for computational reasons. However, we must point out that there are other different numerical approaches that can be followed, like the the use of weighted-shifted Grünwald differences [92].

Definition 5.1.6 (Ortigueira [73]). Let $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ be the real sequence defined by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{N} \cup \{0\}, \tag{5.11}$$

and assume that $f : \mathbb{R} \to \mathbb{R}$ is any function. If $h > 0$ and $\alpha > -1$, then the *fractional-order centered difference* of order α of f at the point x is defined as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{5.12}$$

if the double series at the right-hand side of (5.12) converges.

Lemma 5.1.7 (Wang *et al.* [97]). *If $0 < \alpha \leq 2$ and $\alpha \neq 1$, then*

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ *for all $k \geq 1$, and*

(iii) $\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$. *As a consequence, it follows that* $g_0^{(\alpha)} = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} g_k^{(\alpha)} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |g_k^{(\alpha)}|$.

Theorem 5.1.8 (Wang *et al.* [97]). *Let $\alpha \in (1,2]$ and $h > 0$, and suppose that $f \in \mathcal{C}^5(\mathbb{R})$. If all the derivatives of f up to order five belong to $L^1(\mathbb{R})$, then*

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{5.13}$$

For the remainder, we let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$, for each $q \in \mathbb{N}$. Throughout, we let $J, N \in \mathbb{N}$ satisfy $J \geq 2$ and $N \geq 2$, and define the positive step-sizes $h = (x_R - x_L)/J$ and $\tau = T/N$. We consider uniform partitions of the intervals $[x_L, x_R]$ and $[0,T]$, respectively, of the forms

$$x_L = x_0 < x_1 < \ldots < x_j < \ldots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{5.14}$$

and

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T, \quad \forall n \in \overline{I}_N. \tag{5.15}$$

For each $(j,n) \in \overline{I}_J \times \overline{I}_N$, we let U_j^n and M_j^n represent numerical approximations to $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, respectively. Also, we let $\mathcal{R}_h = \{x_j : j \in \overline{I}_J\}$, and represent by $\mathring{\mathcal{V}}_h$ the vector space over $\mathbb{F}$ of all $\mathbb{F}$-valued functions on the grid space $\mathcal{R}_h$ which vanish at x_0 and x_J. If $V \in \mathring{\mathcal{V}}_h$, then we set $V_j = V(x_j)$, for each $j \in \overline{I}_J$. Moreover, we will let $U^n = (U_j^n)_{j\in\overline{I}_J} \in \mathring{\mathcal{V}}_h$ and $M^n = (M_j^n)_{j\in\overline{I}_J} \in \mathring{\mathcal{V}}_h$, and set $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$.

Definition 5.1.9. Let p be any number satisfying $1 \leq p < \infty$. The inner product $\langle \cdot, \cdot \rangle : \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h \to \mathbb{C}$ and the norms $\| \cdot \|_p, \| \cdot \|_\infty : \mathring{\mathcal{V}}_h \to \mathbb{R}$ are defined, respectively, by

$$\langle U, V \rangle = h \sum_{j \in \overline{I}_J} U_j \overline{V_j}, \quad \forall U, V \in \mathring{\mathcal{V}}_h, \tag{5.16}$$

$$\|U\|_p^p = h \sum_{j \in \overline{I}_J} |U_j|^p, \quad \forall U \in \mathring{\mathcal{V}}_h, \tag{5.17}$$

$$\|U\|_\infty = \max \left\{ |U_j| : j \in \overline{I}_J \right\}, \quad U \in \mathring{\mathcal{V}}_h. \tag{5.18}$$

Additionally, we let $|||V|||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}$, for each $V = (V^n)_{n \in \overline{I}_N} \subseteq \mathring{\mathcal{V}}_h$.

Definition 5.1.10. Let V represent any of the functions U or M, and suppose that $\alpha \in (1, 2]$. We will employ the linear difference operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j, n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{5.19}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j, n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{5.20}$$

and the linear average operators

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j, n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{5.21}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j, n) \in \overline{I}_J \times I_{N-1}. \tag{5.22}$$

If we omit the composition symbol for simplicity, we have the operators $\delta_x^{(2)} V_j^n = \delta_x \delta_x V_{j-1}^n$, $\delta_t^{(1)} V_j^n = \mu_t \delta_t V_j^{n-1}$, $\delta_t^{(2)} V_j^n = \delta_t \delta_t V_j^{n-1}$ and $\mu_t^{(2)} V_j^n = \mu_t \mu_t V_j^{n-1}$, for each $(j, n) \in I_{J-1} \times \overline{I}_N$. Moreover, using the notation in Definition 5.1.6, we define the discrete linear operator

$$\delta_x^{(\alpha)} V_j^n = -\frac{1}{h^\alpha} \sum_{k \in \overline{I}_J} g_{j-k}^{(\alpha)} V_k^n, \qquad \forall (j, n) \in I_{J-1} \times \overline{I}_N. \tag{5.23}$$

Lemma 5.1.11 (Macías-Díaz [49]). *If $\alpha \in (1, 2]$ and $U, V \in \mathring{\mathcal{V}}_h$, then the identity $\langle -\delta_x^{(\alpha)} U, V \rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V \rangle$ holds.* □

5.2 An implicit model

Using the nomenclature of the previous section, we have the following implicit model to approximate the solutions of (5.5), which is described by the algebraic system of difference equations:

$$\begin{aligned}
\delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n \left[1 + \mu_t^{(1)} M_j^n + \mu_t^{(1)} |U_j^n|^2 \right] = 0, \quad & \forall (j, n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\delta_t^{(2)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} |U_j^n|^2 = 0, \quad & \forall (j, n) \in I_{J-1} \times \overline{I}_{N-1}, \\
\text{subject to} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), \quad \forall j \in I_{J-1}, \\ \delta_t^{(1)} U_j^0 = u_1(x_j) & \delta_t^{(1)} M_j^0 = m_1(x_j), \quad \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases}
\end{aligned} \tag{5.24}$$

Note that the numerical model (5.24) is a three-step implicit nonlinear technique. Indeed, if the approximations at the times t_{n-1} and t_n are known, then the difference equations of (5.24) have the vectors U^{n+1} and M^{n+1} as unknowns. In the following and for the sake of convenience, we will let $\{V_j^n : (j,n) \in \overline{I}_J \times \overline{I}_N\}$ be such that

$$\delta_x^{(\beta)} V_j^n = \delta_t M_j^n, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{5.25}$$

$$V_0^n = V_J^n = 0, \quad \forall n \in \overline{I}_N. \tag{5.26}$$

Under these circumstances, (U, M) will denote a solution of (5.24), and $V = (V^n)_{n\in\overline{I}_N}$ will satisfy (5.25) and (5.26).

Definition 5.2.1. Let (U, M) be a solution of (5.24). We define the associated discrete energy density of the system at the point x_j and time t_n as $H(U_j^n, M_j^n) = H_j^n$, where

$$H_j^n = |\delta_t U_j^n|^2 + \mu_t|\delta_x^{(\alpha/2)} U_j^n|^2 + \mu_t|U_j^n|^2 + \frac{1}{2}\mu_t|U_j^n|^4 + \frac{1}{2}\mu_t|M_j^n|^2 + \frac{1}{2}|\delta_x^{(\beta/2)} V_j^n|^2 + \mu_t M_j^n|U_j^n|^2, \tag{5.27}$$

where $(j,n) \in I_{J-1} \times \overline{I}_{N-1}$. Meanwhile, the discrete energy of the system (5.24) at the time t_n and $\forall n \in \overline{I}_{N-1}$, is defined by

$$E^n = \|\delta_t U^n\|_2^2 + \mu_t\left(\|\delta_x^{(\alpha/2)} U^n\|_2^2 + \|U^n\|_2^2 + \tfrac{1}{2}\|U^n\|_4^4 + \tfrac{1}{2}\|M^n\|_2^2 + \langle M^n, |U^n|^2\rangle\right) + \tfrac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2. \tag{5.28}$$

Here, we employ the notation $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$.

Definition 5.2.2. Given any arbitrary $U, V \in \mathring{\mathcal{V}}_h$, we define their product point-wisely, that is, $UV = (U_j V_j)_{j\in\overline{I}_J}$.

Lemma 5.2.3. *Let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$ be sequences in $\mathring{\mathcal{V}}_h$, and assume that U is a sequence of complex functions while the functions of V are real. Suppose additionally that there exists $(V^n)_{n\in\overline{I}_N} \subseteq \mathring{\mathcal{V}}_h$ such that (5.25) holds. Then the following are satisfied for each $n \in I_{N-1}$:*

(a) $2\,\mathrm{Re}\langle \delta_t^{(2)} U^n, \delta_t^{(1)} U^n\rangle = \delta_t \|\delta_t U^{n-1}\|_2^2$,

(b) $2\,\mathrm{Re}\langle -\delta_x^{(\alpha)} \mu_t^{(1)} U^n, \delta_t^{(1)} U^n\rangle = \delta_t \mu_t \|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2$,

(c) $2\,\mathrm{Re}\langle \mu_t^{(1)} U^n, \delta_t^{(1)} U^n\rangle = \delta_t \mu_t \|U^{n-1}\|_2^2$,

(d) $2\,\mathrm{Re}\langle (\mu_t^{(1)} M^n)(\mu_t^{(1)} U^n), \delta_t^{(1)} U^n\rangle = \langle \mu_t^{(1)} M^n, \delta_t^{(1)} |U^n|^2\rangle$,

(e) $4\,\mathrm{Re}\langle (\mu_t^{(1)} |U^n|^2)(\mu_t^{(1)} U^n), \delta_t^{(1)} U^n\rangle = \delta_t \mu_t \|U^{n-1}\|_4^4$,

(f) $-2\langle \delta_t^{(2)} M^n, \mu_t V^{n-1}\rangle = \delta_t \|\delta_x^{(\beta/2)} V^{n-1}\|_2^2$,

(g) $2\langle \delta_x^{(\beta)} \mu_t^{(1)} M^n, \mu_t V^{n-1}\rangle = \delta_t \mu_t \|M^{n-1}\|_2^2$,

(h) $\langle \delta_x^{(\beta)} \mu_t^{(1)} |U^n|^2, \mu_t V^{n-1}\rangle = \langle \mu_t^{(1)} |U^n|^2, \delta_t^{(1)} M^n\rangle$.

Proof. The first five identities are trivial. For the remaining identities, notice that Lemma 5.1.11 and (5.25) imply that

$$-2\langle \delta_t^{(2)} M^n, \mu_t V^{n-1}\rangle = 2\langle \delta_t \delta_x^{(\beta/2)} V^{n-1}, \mu_t \delta_x^{(\beta/2)} V^{n-1}\rangle = \delta_t \|\delta_x^{(\beta/2)} V^{n-1}\|_2^2 \tag{5.29}$$

$$\langle \delta_x^{(\beta)} \mu_t^{(1)} M^n, \mu_t V^{n-1}\rangle = \langle \mu_t^{(1)} M^n, \mu_t \delta_x^{(\beta)} V^{n-1}\rangle = \langle \mu_t^{(1)} M^n, \delta_t^{(1)} M^n\rangle = \frac{1}{2}\delta_t \mu_t \|M^{n-1}\|_2^2, \tag{5.30}$$

$$\langle \delta_x^{(\beta)} \mu_t^{(1)} |U^n|^2, \mu_t V^{n-1}\rangle = \langle \mu_t^{(1)} |U^n|^2, \mu_t \delta_t M^{n-1}\rangle = \langle \mu_t^{(1)} |U^n|^2, \delta_t^{(1)} M^n\rangle. \tag{5.31}$$

This completes the proof of this result. □

Let $(U^n)_{n\in I_{N-1}}, (M^n)_{n\in I_{N-1}} \subseteq \mathring{\mathcal{V}}_h$. For convenience, we define $L = L_U \times L_M : \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h \to \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h$ by

$$L_U(U_j^n, M_j^n) = \delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n + \mu_t^{(1)} U_j^n \left[1 + \mu_t^{(1)} M_j^n + \mu_t^{(1)} |U_j^n|^2 \right], \quad \forall (j,n) \in I, \tag{5.32}$$

$$L_M(U_j^n, M_j^n) = \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} M_j^n - \delta_x^{(\beta)} \mu_t^{(1)} |U_j^n|^2, \quad \forall (j,n) \in I. \tag{5.33}$$

As expected, we define $L(U^n, M^n) = (L(U_j^n, M_j^n))_{j\in \overline{I}_J}$ for each $n \in I_{N-1}$, and let $L(U,M) = (L(U^n, M^n))_{n\in I_{N-1}}$. Let us introduce also the continuous operator $\mathcal{L} = \mathcal{L}_u \times \mathcal{L}_m$, defined for each (u,m) and $\forall (x,t) \in \Omega$, by

$$\mathcal{L}_u(u(x,t), m(x,t)) = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + u(x,t) + m(x,t)u(x,t) + |u(x,t)|^2 u(x,t), \tag{5.34}$$

$$\mathcal{L}_m(u(x,t), m(x,t)) = \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta}. \tag{5.35}$$

Also, for each $x \in \{x_L, x_R\}$ and $t \in [0,T]$, we let $\mathcal{L}(u(x,t), m(x,t)) = 0$. Let $\mathcal{L}(u^n, m^n) = (\mathcal{L}(u_j^n, m_j^n))_{j\in \overline{I}_J}$ for each $n \in I_{N-1}$, and define $\mathcal{L}(u,m) = (\mathcal{L}(u^n, m^n))_{n\in I_{N-1}}$. Similarly, let $L(u^n, m^n) = (L(u_j^n, m_j^n))_{j\in \overline{I}_J} \in \mathring{\mathcal{V}}_h$ for each $n \in I_{N-1}$, and $L(u,m) = (L(u^n, m^n))_{n\in I_{N-1}}$.

Theorem 5.2.4 (Energy conservation)**.** *If (U,M) is a solution of (5.5), then the quantities (5.28) are constant.*

Proof. We only need to apply Lemma 5.2.3 to obtain $\delta_t E^{n-1} = 0$. The conclusion follows by induction. □

Lemma 5.2.5 (Young's inequality)**.** *Let $a, b \in \mathbb{R}^+ \cup \{0\}$, and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For each $\epsilon > 0$, the following inequality holds:*

$$ab \leq \frac{|a|^p}{p\epsilon} + \frac{\epsilon |b|^q}{q}. \tag{5.36}$$

Definition 5.2.6. Let $(U^n)_{n\in \overline{I}_N}$ be any sequence in $\mathcal{V}_h$, let $\Phi \in \mathcal{V}_h$ and assume that $g : \mathbb{R} \to \mathbb{R}$ is a function. We define

$$\mu_{t,\Phi}^{(1)}[g(U_j^n)] = \tfrac{1}{2} \left[g(\Phi_j) + g(U_j^{n-1}) \right], \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}. \tag{5.37}$$

Lemma 5.2.7. *Let $U = (U^n)_{n\in \overline{I}_N}$ and $M = (M^n)_{n\in \overline{I}_N}$ be sequences in $\mathring{\mathcal{V}}_h$, and assume that U is a sequence of complex functions while the functions of M are real. The following identities are satisfied, for each $n \in I_{N-1}$ and $\Phi, \Psi \in \mathring{\mathcal{V}}_h$:*

(a) $4\,\mathrm{Re} \left\langle (\mu_{t,\Phi}^{(1)} |U^n|^2)(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1} \right\rangle = \|\Phi\|_4^4 - \|U^{n-1}\|_4^4.$

(b) $4\,\mathrm{Re} \left\langle (\mu_{t,\Psi}^{(1)} M^n)(\mu_{t,\Phi}^{(1)} U^n), \Phi - U^{n-1} \right\rangle = \langle \Psi + M^{n-1}, |\Phi|^2 - |U^{n-1}|^2 \rangle.$

(c) $2\,\mathrm{Re} \left\langle \mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1} \right\rangle = \|\Phi\|_2^2 - \|U^{n-1}\|_2^2.$

(d) $2\,\mathrm{Re} \left\langle -\delta_x^{(\alpha)} \mu_{t,\Phi}^{(1)} U^n, \Phi - U^{n-1} \right\rangle = \|\delta_x^{(\alpha/2)} \Phi\|_2^2 - \|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2.$

(e) $2 \left\langle -\delta_x^{(\beta)} \mu_{t,\Psi}^{(1)} M^n, \Psi - M^{n-1} \right\rangle = \|\delta_x^{(\beta/2)} \Psi\|_2^2 - \|\delta_x^{(\beta/2)} M^{n-1}\|_2^2.$

Additionally, the following inequalities are satisfied for each $\lambda \in [0,1]$:

(f) $\mathrm{Re}\langle -2\lambda U^n, \Phi - U^{n-1}\rangle \geq -\frac{\lambda}{3}\|\Phi\|_2^2 - C_1$.

(g) $\mathrm{Re}\langle \Phi + \lambda U^{n-1}, \Phi - U^{n-1}\rangle \geq \left(1 - \frac{\lambda+1}{6}\right)\|\Phi\|_2^2 - C_2$.

(h) $\langle \Psi - 2\lambda M^n + \lambda M^{n-1}, \Psi - M^{n-1}\rangle \geq \left[1 - \frac{1}{20}(3\lambda+1)\right]\|\Psi\|_2^2 - C_3$.

(i) $\lambda\tau^2\langle -\delta_x^{(\beta)}\mu_{t,\Phi}^{(1)}|U^n|^2, \Psi - M^{n-1}\rangle \geq -\frac{\lambda\tau^2 g_0^{(\beta)}}{\epsilon_1 h^{\beta-1}}\|\Phi\|_4^4 - \frac{\lambda\tau^2\epsilon_1 g_0^{(\beta)}}{h^{\beta-1}}\|\Psi\|_2^2 - C_4$, *for each* $\epsilon_1 > 0$.

(j) $\lambda\tau^2\,\mathrm{Re}\langle(\mu_{t,\Psi}^{(1)}M^n)(\mu_{t,\Phi}^{(1)}U^n), \Phi - U^{n-1}\rangle \geq -\frac{\lambda\tau^2}{8}\left(1 + \frac{1}{\epsilon_2}\right)\|\Psi\|_2^2 - \frac{\lambda\tau^2}{8}\left(\epsilon_2 + \frac{1}{2}\right)\|\Phi\|_4^4 - C_5$, *for each* $\epsilon_2 > 0$.

Here, the constants $C_1, \ldots, C_5 \in \mathbb{R}^+$ *depend only on* U^n, U^{n-1}, M^n *and* M^{n-1}. *Additionally,* C_4 *depends on* ϵ_1, *and* C_5 *depends on* ϵ_2.

Proof. The proofs of most of these relations are straightforward. To prove (i), notice that

$$\begin{aligned}
\lambda\tau^2\langle -\delta_x^{(2)}\mu_{t,\Phi}^{(1)}|U^n|^2, \Psi - M^{n-1}\rangle &\geq -\frac{\lambda\tau^2}{2}\left|\langle \delta_x^{(\beta/2)}(|\Phi|^2 + |U^{n-1}|^2), \delta_x^{(\beta/2)}(\Psi - M^{n-1})\rangle\right| \\
&\geq -\frac{\lambda\tau^2}{4\epsilon_1}\|\delta_x^{(\beta/2)}(|\Phi|^2 + |U^{n-1}|^2)\|_2^2 - \frac{\lambda\tau^2\epsilon_1}{4}\|\delta_x^{(\beta/2)}(\Psi - M^{n-1}))\|_2^2 \\
&\geq -\frac{\lambda\tau^2}{2\epsilon_1}\left[\|\delta_x^{(\beta/2)}|\Phi|^2\|_2^2 - \|\delta_x^{(\beta/2)}|U^{n-1}|^2\|_2^2\right] \\
&\quad - \frac{\lambda\tau^2\epsilon_1}{2}\left[\|\delta_x^{(\beta/2)}\Psi\|_2^2 - \|\delta_x^{(\beta/2)}M^{n-1}\|_2^2\right] \\
&\geq -\frac{\lambda\tau^2 g_0^{(\beta)}}{\epsilon_1 h^{\beta-1}}\|\Phi\|_4^4 - \frac{\lambda\tau^2\epsilon_1 g_0^{(\beta)}}{h^{\beta-1}}\|\Psi\|_2^2 - C_4,
\end{aligned} \tag{5.38}$$

where $C_4 = \frac{\lambda\tau^2 g_0^{(\beta)}}{h^{\beta-1}}\left(\frac{1}{\epsilon_1}\|U^{n-1}\|_4^4 + \epsilon_1\|M^{n-1}\|_2^2\right)$. □

Lemma 5.2.8 (Leray–Schauder fixed-point theorem)**.** *Let* X *be a Banach space, and let* $F : X \to X$ *be continuous and compact. If the set* $S = \{x \in X : \lambda F(x) = x$ *for some* $\lambda \in [0,1]\}$ *is bounded, then* F *has a fixed point.*

Theorem 5.2.9 (Solubility)**.** *The numerical model* (5.24) *is solvable for any set of initial conditions whenever*

$$\tau^2 < \min\left\{\frac{4}{5}, \frac{1}{5}\left(\frac{h^{\beta-1}}{4g_0^{(\beta)}}\right)^2\right\}. \tag{5.39}$$

Proof. The approximations (U^0, M^0) and (U^1, M^1) are defined through the initial data, so assume that (U^{n-1}, M^{n-1}) and (U^n, M^n) have been already obtained, for some $n \in I_{N-1}$. Let $X = \mathring{\mathcal{V}}_h \times \mathring{\mathcal{V}}_h$ and define the function $F : X \to X$ as $F = G \times H$, where $G, H : X \to \mathring{\mathcal{V}}_h$. In turn, for each $j \in I_{J-1}$ and $\Phi, \Psi \in \mathring{\mathcal{V}}_h$, we let

$$G_j(\Phi,\Psi) = 2U_j^n - U_j^{n-1} + \tau^2\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)}U_j^n - \tau^2\mu_{t,\Phi}^{(1)}U_j^n\left[1 + \mu_{t,\Psi}^{(1)}M_j^n + \mu_{t,\Phi}^{(1)}|U_j^n|^2\right], \tag{5.40}$$

$$H_j(\Phi,\Psi) = 2M_j^n - M_j^{n-1} + \tau^2\delta_x^{(\beta)}\mu_{t,\Psi}^{(1)}M_j^n + \tau^2\delta_x^{(\beta)}\mu_{t,\Phi}^{(1)}|U_j^n|^2. \tag{5.41}$$

In the case when $j \in \{0, J\}$, we let $G_j(\Phi,\Psi) = H_j(\Phi,\Psi) = 0$. It is obvious that F is a continuous and compact map from the Banach space X into itself. We will prove next that S of Lemma 5.2.8

is a bounded subset of X. Let $(\Phi,\Psi)\in X$ and $\lambda\in[0,1]$ satisfy $\lambda F(\Phi,\Psi)=(\Phi,\Psi)$. Equivalently, the following identities hold for each $n\in I_{N-1}$:

$$0=\Phi-2\lambda U^n+\lambda U^{n-1}-\lambda\tau^2\delta_x^{(\alpha)}\mu_{t,\Phi}^{(1)}U^n+\lambda\tau^2\mu_{t,\Phi}^{(1)}U^n+\lambda\tau^2\left(\mu_{t,\Phi}^{(1)}U^n\right)\left[\mu_{t,\Psi}^{(1)}M^n+\mu_{t,\Phi}^{(1)}|U^n|^2\right], \tag{5.42}$$

$$0=\Psi-2\lambda M^n+\lambda M^{n-1}-\lambda\tau^2\delta_x^{(\beta)}\mu_{t,\Psi}^{(1)}M^n-\lambda\tau^2\delta_x^{(\beta)}\mu_{t,\Phi}^{(1)}|U^n|^2, \tag{5.43}$$

Note now that (5.39) assures that $\frac{16g_0^{(\beta)}}{h^{\beta-1}}<\frac{h^{\beta-1}}{5\tau^2g_0^{(\beta)}}$ and $\frac{5\tau^2}{8}<\frac{1}{2}$. Take the real part of the inner product of both sides of the equation $0=\Phi-\lambda G(\Phi,\Psi)$ with $\Phi-U^{n-1}$. At the same time, take the inner product of both sides of the equation $0=\Psi-\lambda H(\Phi,\Psi)$ with $\Psi-M^{n-1}$. Add both results, use the identities and inequalities of Lemma 5.2.7 with $\epsilon_1\in\left(\frac{16g_0^{(\beta)}}{h^{\beta-1}},\frac{h^{\beta-1}}{5\tau^2g_0^{(\beta)}}\right)$ and $\epsilon_2\in\left(\frac{5\tau^2}{8},\frac{1}{2}\right)$, rearrange terms and simplify to obtain

$$\begin{aligned}0\geq{}&\frac{\lambda\tau^2}{4}\left(\|\Phi\|_4^4-\|U^{n-1}\|_4^4\right)+\frac{\lambda\tau^2}{2}\left(\|\Phi\|_2^2-\|U^{n-1}\|_2^2\right)+\frac{\lambda\tau^2}{2}\left(\|\delta_x^{(\alpha/2)}\Phi\|_2^2-\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2\right)\\&+\frac{\lambda\tau^2}{2}\left(\|\delta_x^{(\beta/2)}\Psi\|_2^2-\|\delta_x^{(\beta/2)}M^{n-1}\|_2^2\right)-\frac{\lambda\tau^2}{8}\left(\frac{1}{\epsilon_2}+1\right)\|\Psi\|_2^2-\frac{\lambda\tau^2}{8}\left(\epsilon_2+\frac{1}{2}\right)\|\Phi\|_4^4\\&+\left(1-\frac{\lambda+1}{6}\right)\|\Phi\|_2^2-\frac{\lambda}{3}\|\Phi\|_2^2+\left(1-\frac{3\lambda+1}{20}\right)\|\Psi\|_2^2-\frac{\lambda\tau^2g_0^{(\beta)}}{\epsilon_1h^{\beta-1}}\|\Phi\|_4^4-\frac{\lambda\tau^2\epsilon_1g_0^{(\beta)}}{h^{\beta-1}}\|\Psi\|_2^2-C_6,\end{aligned} \tag{5.44}$$

where $C_6=C_1+C_2+C_3+C_4+C_5$, and the constants $C_1,\ldots,C_5\in\mathbb{R}^+$ are those of Lemma 5.2.7. Rearranging terms and using the fact that $\lambda\in[0,1]$, we obtain that

$$K_1\|\Phi\|_2^2+K_2\|\Phi\|_4^4+K_3\|\Psi\|_2^2\leq C, \tag{5.45}$$

with

$$K_1=1+\frac{\lambda\tau^2}{2}-\frac{3\lambda+1}{6}\geq\frac{1}{3}, \tag{5.46}$$

$$K_2=\frac{\lambda\tau^2}{4}\left[1-\frac{4g_0^{(\beta)}}{\epsilon_1h^{\beta-1}}-\frac{1}{2}\left(\epsilon_2+\frac{1}{2}\right)\right]>\frac{\lambda\tau^2}{4}\left(1-\frac{1}{4}-\frac{1}{2}\right)=\frac{\lambda\tau^2}{16}, \tag{5.47}$$

$$K_3=1-\lambda\tau^2\left(\frac{\epsilon_1g_0^{(\beta)}}{h^{\beta-1}}+\frac{1}{8\epsilon_2}+\frac{1}{8}\right)-\frac{3\lambda+1}{20}\geq1-\tau^2\left(\frac{1}{5\tau^2}+\frac{1}{5\tau^2}+\frac{1}{8}\right)-\frac{1}{5}>\frac{3}{10}, \tag{5.48}$$

$$C=C_6+\frac{\tau^2}{2}\left(\frac{1}{2}\|U^{n-1}\|_4^4+\|U^{n-1}\|_2^2+\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2+\|\delta_x^{(\beta/2)}M^{n-1}\|_2^2\right). \tag{5.49}$$

The inequalities were obtained using the ranges of ϵ_1 and ϵ_2, and that $\lambda\in[0,1]$. It follows that K_1, K_2 and K_3 are positive and, moreover, that $\frac{1}{3}\|\Phi\|_2^2+\frac{3}{10}\|\Psi\|_2^2\leq C$. As a consequence, the set S is bounded, and the Leray–Schauder theorem guarantees that the system (5.24) has a solution (U^{n+1},M^{n+1}). The result follows now by induction. □

Theorem 5.2.10 (Boundedness). *Let $u_0,m_0\in H^1$ and $u_1,m_1\in L_2$, and suppose that (U,M) is the corresponding solution of* (5.24)*. Then there exists $C\in\mathbb{R}^+$ such that*

$$\max\left\{\|\delta_tU^n\|_2,\|\delta_x^{(\alpha/2)}U^n\|_2,\|U^n\|_2,\|M^n\|_2,\|U^n\|_4,\|\delta_x^{(\beta/2)}V^n\|_2\right\}\leq C,\quad\forall n\in\overline{I}_N. \tag{5.50}$$

Proof. By Theorem 5.2.4, for each $n \in I_{N-1}$, the quantities E^n are all equal to a constant $C_0 \in \mathbb{R}$. Notice that

$$C_0 \geq \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\mu_t \|U^n\|_4^4 + \frac{1}{2}\mu_t \|M^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 - \left|\mu_t \langle M^n, |U^n|^2\rangle\right|. \tag{5.51}$$

Applying Young's inequality, we have

$$\left|\mu_t \langle M^n, |U^n|^2\rangle\right| \leq \frac{1}{2}\mu_t \|M^n\|_2^2 + \frac{1}{2}\mu_t \|U^n\|_4^4 \quad \text{and} \quad \left|\mu_t \langle M^n, |U^n|^2\rangle\right| \leq \frac{1}{4}\mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4. \tag{5.52}$$

Using the first inequality of (5.52) in (5.51), implies

$$C_0 \geq \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2. \tag{5.53}$$

In particular, $\|U^n\|_2$ is bounded and, as a consequence, $\|U^n\|_4$ is also bounded. Now, removing some terms in (5.51) and using the second inequality of (5.52), it follows that $C \geq C_0 + \mu_t \|U^n\|_4^4 \geq \frac{1}{2}\mu_t \|M^n\|_2^2$. □

Theorem 5.2.11 (Consistency)**.** *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. Then there exists a constant C which is independent of τ and h, such that $|||(\mathcal{L} - L)(u, m)|||_\infty \leq C(\tau^2 + h^2)$ and $|||(\mathcal{H} - H)(u, m)|||_\infty \leq C(\tau^2 + h^2)$.*

Proof. Using the classical argument based on Taylor approximations, the mean value theorem and the smoothness of u, it is easy to prove that there exists constants $C_1, C_2 \in \mathbb{R}^+$, such that

$$\left|\frac{\partial^2 m(x,t)}{\partial x^2} - \delta_x^{(\beta)} \mu_t^{(1)} m_j^n\right| \leq C_1(\tau^2 + h^2), \quad \forall (j,n) \in \overline{I}_J \times I_{N-1} \tag{5.54}$$

$$\left|\frac{\partial^2 \left(|u(x,t)|^2\right)}{\partial x^2} - \delta_x^{(\beta)} \mu_t^{(1)} |u_j^n|^2\right| \leq C_2(\tau^2 + h^2), \quad \forall (j,n) \in \overline{I}_J \times I_{N-1}. \tag{5.55}$$

The conclusion is obtained in similar fashion as in [35], considering the above inequalities. □

Next, we prove the stability and convergence properties of (5.5). For that reason, in the following, (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ will represent two sets of initial conditions of (5.5), and we will assume that the initial data for (5.24) are provided exactly. Moreover, if $f : F \to F$ and $V \in \mathring{\mathcal{V}}_h$, then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$ and $F = \mathbb{R}, \mathbb{C}$. The following results will be crucial to establish the remaining numerical properties.

Lemma 5.2.12 (Macías-Díaz [49])**.** *If $V \in \mathcal{V}_h$ and $\alpha \in (1, 2]$ then*

(a) $\|\delta_x^{(\alpha/2)} V\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|V\|_2^2$,

(b) $\|\delta_x^{(\alpha)} V\|_2^2 = \|\delta_x^{(\alpha/2)} \delta_x^{(\alpha/2)} V\|_2^2$, *and*

(c) $\|\delta_x^{(\alpha)} V\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|\delta_x^{(\alpha/2)} V\|_2^2 \leq 4\left(g_0^{(\alpha)} h^{1-\alpha}\right)^2 \|V\|_2^2$.

Lemma 5.2.13 (Gronwall's inequality [105])**.** *Assume that $N \in \mathbb{N}$ with $N > 1$. Let $(\omega^n)_{n \in \overline{I}_N}$ and $(C_n)_{n \in I_N}$ be sequences of real numbers, and let A, B and C_n be nonnegative numbers, for each $n \in I_N$. Suppose that $\tau \in \mathbb{R}^+$, and that*

$$\omega^n - \omega^{n-1} \leq A\tau\omega^n + B\tau\omega^{n-1} + C_n\tau, \quad \forall n \in I_N. \tag{5.56}$$

If $(A + B)\tau \leq (N - 1)/(2N)$ then

$$\max_{n \in I_N} |\omega^n| \leq \left(\omega^0 + \tau \sum_{k \in I_N} C_k\right) e^{2(A+B)N\tau}. \tag{5.57}$$

Lemma 5.2.14. *Let $u_0, m_0, \widetilde{u}_0, \widetilde{m}_0 \in H^1(\overline{B})$ and $u_1, m_1, \widetilde{u}_1, \widetilde{m}_1 \in L_2(\overline{B})$. Suppose that (U, M) and $(\widetilde{U}, \widetilde{M})$ are the solutions of (5.24) corresponding to (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$, respectively. Let $\varepsilon^n = \widetilde{U}^n - U^n$ and $\zeta^n = \widetilde{M}^n - M^n$, for each $n \in \overline{I}_N$, and define*

$$\omega^n = \|\delta_t \varepsilon^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} \varepsilon^n\|_2^2 + \mu_t \|\varepsilon^n\|_2^2 + \|\delta_x^{(\beta/2)} \upsilon^n\|_2^2 + \mu_t \|\zeta^n\|_2^2, \quad \forall n \in I_{N-1}. \tag{5.58}$$

If τ is sufficiently small, then there exists a constant $C \in \mathbb{R}$ independent of h and τ, such that $\omega^n \leq \omega^0 \exp(CT)$, for each $n \in \overline{I}_{N-1}$.

Proof. It is easy to check that the sequence (ε, ζ) satisfies $\forall (j, n) \in I$, the following system

$$\begin{aligned} \delta_t^{(2)} \varepsilon_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} \varepsilon_j^n + \mu_t^{(1)} \varepsilon_j^n + \widetilde{\delta} \left[\left(\mu_t^{(1)} M_j^n \right) \left(\mu_t^{(1)} U_j^n \right) \right] + \widetilde{\delta} \left[\left(\mu_t^{(1)} |U_j^n|^2 \right) \left(\mu_t^{(1)} U_j^n \right) \right] &= 0, \\ \delta_t^{(2)} \zeta_j^n - \delta_x^{(\beta)} \mu_t^{(1)} \zeta_j^n - \widetilde{\delta} \left(\delta_x^{(\beta)} \mu_t^{(1)} |U_j^n|^2 \right) &= 0, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N. & \end{aligned} \tag{5.59}$$

For each $n \in \overline{I}_N$, define the difference $\upsilon^n = \widetilde{V}^n - V^n$ and, as a consequence, we have $\delta_x^{(\beta)} \upsilon^n = \delta_t \zeta^n$. The following identities are obtained using similar arguments to the Lemma 5.2.3

(i) $2 \operatorname{Re} \langle \delta_t^{(2)} \varepsilon^n, \delta_t^{(1)} \varepsilon^n \rangle = \delta_t \|\delta_t \varepsilon^{n-1}\|_2^2$,

(ii) $2 \operatorname{Re} \langle -\delta_x^{(\alpha)} \mu_t^{(1)} \varepsilon^n, \delta_t^{(1)} \varepsilon^n \rangle = \delta_t \mu_t \|\delta_x^{(\alpha/2)} \varepsilon^{n-1}\|_2^2$,

(iii) $2 \operatorname{Re} \langle \mu_t^{(1)} \varepsilon^n, \delta_t^{(1)} \varepsilon^n \rangle = \delta_t \mu_t \|\varepsilon^{n-1}\|_2^2$,

(iv) $-2 \langle \delta_t^{(2)} \zeta^n, \mu_t \upsilon^{n-1} \rangle = \delta_t \|\delta_x^{(\beta/2)} \upsilon^{n-1}\|_2^2$, and

(v) $2 \langle \delta_x^{(\beta)} \mu_t^{(1)} \zeta^n, \mu_t \upsilon^{n-1} \rangle = \delta_t \mu_t \|\zeta^{n-1}\|_2^2$.

Since the set $\{\|U^n\|_2, \|\widetilde{U}^n\|_2, \|M^n\|_2, \|\widetilde{M}^n\|_2 : n \in \overline{I}_{N-1}\}$ is bounded, we can show that there exist $C_1, C_2 \in \mathbb{R}^+$, such that

$$\begin{aligned} \operatorname{Re} \left\langle \widetilde{\delta} \left[\left(\mu_t^{(1)} M^n \right) \left(\mu_t^{(1)} U^n \right) \right], \delta_t^{(1)} \varepsilon^n \right\rangle &= \operatorname{Re} \left\langle \left(\mu_t^{(1)} \zeta^n \right) \left(\mu_t^{(1)} \widetilde{U}^n \right) + \left(\mu_t^{(1)} M^n \right) \left(\mu_t^{(1)} \varepsilon^n \right), \delta_t^{(1)} \varepsilon^n \right\rangle \\ &\leq \frac{1}{2} \left(\left\| \left(\mu_t^{(1)} \zeta^n \right) \left(\mu_t^{(1)} \widetilde{U}^n \right) \right\|_2^2 + \left\| \left(\mu_t^{(1)} M^n \right) \left(\mu_t^{(1)} \varepsilon^n \right) \right\|_2^2 \right) \\ &\quad + \|\delta_t^{(1)} \varepsilon^n\|_2^2 \\ &\leq \frac{C_1}{4} \left(\|\zeta^{n+1}\|_2^2 + \|\zeta^{n-1}\|_2^2 + \|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^{n-1}\|_2^2 \right) \\ &\quad + \frac{1}{2} \mu_t \|\delta_t \varepsilon^{n-1}\|_2^2 \\ &\leq C_2 \left(\mu_t \|\delta_t \varepsilon^{n-1}\|_2^2 + \mu_t^{(1)} \left[\|\zeta^n\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_x^{(\alpha/2)} \varepsilon^n\|_2^2 \right] \right). \end{aligned} \tag{5.60}$$

In similar fashion, it is also easy to prove the existence of $C_3, C_4 \in \mathbb{R}^+$, such that

$$\operatorname{Re} \left\langle \widetilde{\delta} \left[\left(\mu_t^{(1)} |U^n|^2 \right) \left(\mu_t^{(1)} U^n \right) \right], \delta_t^{(1)} \varepsilon^n \right\rangle \leq C_3 \left(\mu_t^{(1)} \|\varepsilon^n\|_2^2 + \mu_t \|\delta_t \varepsilon^{n-1}\|_2^2 \right), \tag{5.61}$$

$$\left\langle \widetilde{\delta} \left(\delta_x^{(\beta)} \mu_t^{(1)} |U^n|^2 \right), \mu_t \upsilon^{n-1} \right\rangle \leq C_4 \left(\mu_t^{(1)} \|\varepsilon^n\|_2^2 + \mu_t \|\delta_t \zeta^{n-1}\|_2^2 \right). \tag{5.62}$$

Take the real part of the inner product between the first equation in (5.59) and $2\delta_t^{(1)} \varepsilon^n$, and compute the inner product of the second equation with $2\delta_t \zeta^{n-1}$. Using identities (i)–(v) and inequalities (5.60)–(5.62), simplify algebraically and rearrange terms we show that there exists $C_5 \in \mathbb{R}^+$ such that the following inequalities are satisfied for each $n \in I_{N-1}$:

$$\delta_t \|\delta_t \varepsilon^{n-1}\|_2^2 + \delta_t^{(1)} \left[\|\delta_x^{(\alpha/2)} \varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2 \right] \leq C_5 \left(\mu_t^{(1)} \left[\|\zeta^n\|_2^2 + \|\delta_x^{(\alpha/2)} \varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2 \right] + \mu_t \|\delta_t \varepsilon^{n-1}\|_2^2 \right) \tag{5.63}$$

$$\delta_t \|\delta_x^{(\beta/2)} \upsilon^{n-1}\|_2^2 + \delta_t^{(1)} \|\zeta^n\|_2^2 \leq C_5 \left(\mu_t^{(1)} \|\varepsilon^n\|_2^2 + \mu_t \|\delta_t \zeta^{n-1}\|_2^2 \right). \tag{5.64}$$

Adding these last inequalities, we obtain that for each $n \in I_N$,

$$\begin{aligned} \omega^n - \omega^{n-1} &\leq C_5 \tau \left(\mu_t^{(1)} \|\zeta^n\|_2^2 + \mu_t^{(1)} \|\delta_x^{(\alpha/2)} \varepsilon^n\|_2^2 + 2\mu_t^{(1)} \|\varepsilon^n\|_2^2 + \mu_t \|\delta_t \varepsilon^{n-1}\|_2^2 + \mu_t \|\delta_t \zeta^{n-1}\|_2^2 \right) \\ &\leq C_6 \tau \left(\omega^n + \omega^{n-1} \right) \end{aligned} \tag{5.65}$$

where $\max \left\{ 2, 2g_0^{(\beta)} h^{1-\beta} \right\} = C_6 / C_5$. If τ is sufficiently small, namely, $C_6 \tau \leq (N-1)/(4N)$, then Gronwall's inequality establishes the conclusion of this result with $C = 4C_6$. □

The following results are immediate consequences of Lemma 5.2.14.

Theorem 5.2.15 (Stability). *If the initial data satisfy $u_0, m_0 \in H^1(\overline{B})$ and $u_1, m_1 \in L_2(\overline{B})$, then the solutions of the numerical model* (5.24) *are unconditionally stable.* □

Theorem 5.2.16 (Uniqueness). *Assume that the hypotheses of Theorem 5.2.15 are satisfied. If τ is sufficiently small, then the numerical model* (5.24) *is uniquely solvable.* □

To establish the quadratic order of convergence of our finite-difference method, we take the local truncation errors of the equations in (5.24) at the point (x_j, t_n), this is

$$\rho_j^n = \delta_t^{(2)} u_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n + \mu_t^{(1)} u_j^n + \left(\mu_t^{(1)} m_j^n \right) \left(\mu_t^{(1)} u_j^n \right) + \left(\mu_t^{(1)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right), \quad \forall (j,n) \in I, \tag{5.66}$$

$$\sigma_j^n = \delta_t^{(2)} m_j^n - \delta_x^{(\beta)} \mu_t^{(1)} m_j^n - \delta_x^{(\beta)} \mu_t^{(1)} |u_j^n|^2, \quad \forall (j,n) \in I. \tag{5.67}$$

By Theorem 5.2.11, it is clear that $|\rho_j^n| + |\sigma_j^n| = \mathcal{O}(\tau^2 + h^2)$. Again, (u, m) will represent a solution of the continuous problem (5.5), while (U, M) will denote the solution of (5.24) corresponding to the same set of initial data. Under these circumstances, let $\epsilon_j^n = u_j^n - U_j^n$, $\eta_j^n = m_j^n - M_j^n$ and $\delta_x^{(\beta)} \theta_j^n = \delta_t \eta_j^n$, for each $(j,n) \in I$. Also, if $f : F \to F$ is a function and $V \in \mathring{\mathcal{V}}_h$, then we define $\widehat{\delta}(f(v_j)) = f(v_j) - f(V_j)$, for each $j \in I_{J-1}$ and $F = \mathbb{R}, \mathbb{C}$.

Theorem 5.2.17 (Convergence). *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. Then the solution of the problem* (5.24) *converges to that of* (5.5) *with order $\mathcal{O}(\tau^2 + h^2)$ in the L_∞ norm for $(U^n)_{n \in \overline{I}_N}$, and in the L_2 norm for $(M^n)_{n \in \overline{I}_N}$, when the numerical initial conditions are exact.*

Proof. In light of the exactness of the initial conditions, notice firstly that (ϵ, η) satisfies $\forall (j,n) \in I$, the discrete system

$$\begin{aligned} \delta_t^{(2)} \epsilon_j^n - \delta_x^{(\alpha)} \mu_t^{(1)} \epsilon_j^n + \mu_t^{(1)} \epsilon_j^n + \widehat{\delta} \left[\left(\mu_t^{(1)} m_j^n \right) \left(\mu_t^{(1)} u_j^n \right) \right] + \widehat{\delta} \left[\left(\mu_t^{(1)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right) \right] &= \rho_j^n, \\ \delta_t^{(2)} \eta_j^n - \delta_x^{(\beta)} \mu_t^{(1)} \eta_j^n - \widehat{\delta} \left(\delta_x^{(\beta)} \mu_t^{(1)} |u_j^n|^2 \right) &= \sigma_j^n, \\ \text{subject to} \begin{cases} \epsilon^0 = \eta^0 = \epsilon^1 = \eta^1 = 0, \\ \epsilon_0^n = \epsilon_J^n = \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N. \end{cases} \end{aligned} \tag{5.68}$$

Let $n \in I_{N-1}$. Following the argument used in the proof of Lemma 5.2.14, there exists a common $C_1 \in \mathbb{R}^+$ such that

$$\operatorname{Re}\langle \rho^n, \delta_t^{(1)} \epsilon^n \rangle \leq C_1 \left(\|\rho^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \right), \tag{5.69}$$

$$\operatorname{Re}\left\langle \widehat{\delta}\left[\left(\mu_t^{(1)} m^n\right)\left(\mu_t^{(1)} u^n\right)\right], \delta_t^{(1)} \epsilon^n \right\rangle \leq C_1 \left(\mu_t^{(1)} \|\eta^n\|_2^2 + \mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \right), \tag{5.70}$$

$$\operatorname{Re}\left\langle \widehat{\delta}\left[\left(\mu_t^{(1)} |u^n|^2\right)\left(\mu_t^{(1)} u^n\right)\right], \delta_t^{(1)} \epsilon^n \right\rangle \leq C_1 \left(\mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 + \mu_t^{(1)} \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 \right), \tag{5.71}$$

$$\langle \sigma^n, \mu_t \theta^{n-1} \rangle \leq C_1 \left(\|\sigma^n\|_2^2 + \mu_t \|\theta^{n-1}\|_2^2 \right), \tag{5.72}$$

$$\langle \widehat{\delta}\left(\delta_x^{(\beta)} \mu_t^{(1)} |u_j^n|^2 \right), \mu_t \theta^{n-1} \rangle \leq C_1 \left(\mu_t^{(1)} \|\epsilon^n\|_2^2 + \mu_t \|\delta_t \eta^{n-1}\|_2^2 + \mu_t^{(1)} \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 \right). \tag{5.73}$$

Again, we take the inner product of the first vector equation of (5.68) with $2\delta_t^{(1)} \epsilon^n$ and use the inequalities (5.69)–(5.71). On the other hand, take the inner product of the second vector equation with $2\mu_t \theta^{n-1}$ and use (5.72) and (5.73). In such way, one can show that there exists a constant $C_2 \in \mathbb{R}^+$ such that

$$\begin{aligned} \delta_t \|\delta_t \epsilon^{n-1}\|_2^2 + \delta_t^{(1)} \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 + \delta_t^{(1)} \|\epsilon^n\|_2^2 \leq C_2 \Big[& \|\rho^n\|_2^2 + \mu_t \|\delta_t \epsilon^{n-1}\|_2^2 \\ & + \mu_t^{(1)} \left(\|\eta^n\|_2^2 + \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 + \|\epsilon^n\|_2^2 \right) \Big], \end{aligned} \tag{5.74}$$

$$\begin{aligned} \delta_t \|\delta_x^{(\beta/2)} \theta^{n-1}\|_2^2 + \delta_t^{(1)} \|\eta^n\|_2^2 \leq C_2 \Big[& \|\sigma^n\|_2^2 + \mu_t^{(1)} \left(\|\epsilon^n\|_2^2 + \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 \right) \\ & + \mu_t \left(\|\delta_t \eta^{n-1}\|_2^2 + \|\theta^{n-1}\|_2^2 \right) \Big]. \end{aligned} \tag{5.75}$$

respectively. Multiply (5.74) and (5.75) by τ, and adding both new inequalities, it follows that there exists $C_3 \in \mathbb{R}^+$ such that $\xi^n - \xi^{n-1} \leq C_3 \tau (\|\rho^n\|_2^2 + \|\sigma^n\|_2^2) + C_3 \tau (\xi^n + \xi^{n-1})$, for each $n \in I_{N-1}$. Here,

$$\xi^n = \|\delta_t \epsilon^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} \epsilon^n\|_2^2 + \mu_t \|\epsilon^n\|_2^2 + \|\delta_x^{(\beta/2)} \theta^n\|_2^2 + \mu_t \|\eta^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{5.76}$$

Finally, as a consequence of Lemma 5.2.13, it follows that there exists $C \in \mathbb{R}^+$ such that $\xi^n \leq C(\tau^2 + h^2)$. In particular, note that $\|\epsilon^n\|_2, \|\eta^n\|_2 \leq C(\tau^2 + h^2)$, for each $n \in \overline{I}_{N-1}$, as desired. □

5.3 Explicit Model

Now, in this section, we are going to study the finale scheme. This time, an explicit model to approximate the solutions of (5.5), given by the following algebraic system, for each $\forall (j, n) \in I$:

$$\begin{gathered} \delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} U_j^n + \mu_t^{(1)} U_j^n + M_j^n \mu_t^{(1)} U_j^n + \left(\mu_t^{(1)} |U_j^n|^2 \right) \left(\mu_t^{(1)} U_j^n \right) = 0, \\ \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2 = 0, \\ \text{such that} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), & \forall j \in \overline{I}_J, \\ \delta_t^{(1)} U_j^0 = u_1(x_j) & \delta_t^{(1)} M_j^0 = m_1(x_j), & \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, & \forall n \in \overline{I}_N. \end{cases} \end{gathered} \tag{5.77}$$

Note that the numerical model (5.77) is a three-step explicit technique. The first equation of that system yields an expression with complex parameters in which the only unknown is U_j^{n+1}. Moreover, the second equation of (5.77) is a fully explicit difference equation which can be easily solved for M_j^{n+1}, for each $(j, n) \in I$. Using then the initial data, we readily obtain that for each

$j \in I_{J-1}$, the following identities hold:

$$\frac{2U_j^1 - 2u_0(x_j) - 2\tau u_1(x_j)}{\tau^2} = \delta_x^{(\alpha)} U_j^0 - (U_j^1 - \tau u_1(x_j)) \left[1 + M_j^0 + \frac{1}{2} \left(|U_j^1|^2 + |U_j^1 - 2\tau u_1(x_j)|^2 \right) \right], \tag{5.78}$$

$$M_j^1 = m_0(x_j) + \tau m_1(x_j) + \frac{\tau^2}{2} \delta_x^{(\beta)} \left(M_j^0 + |U_j^0|^2 \right). \tag{5.79}$$

Again, we will let $(V^n)_{n \in \overline{I}_N}$ be a sequence in $\mathring{\mathcal{V}}_h$ such that

$$\delta_x^{(\beta)} V_j^n = \delta_t M_j^n, \quad \forall (j,n) \in I_{J-1} \times \overline{I}_{N-1}, \tag{5.80}$$

Under these circumstances, (U, M) will denote a solution of (5.77), and $V = (V^n)_{n \in \overline{I}_N}$ will satisfy (5.80).

Definition 5.3.1. Let (U, M) be a solution of (5.77). The *discrete energy density* at the time t_n is given by

$$\begin{aligned} H_j^n = {} & |\delta U_j^n|^2 - U_j^n \delta_x^{(\alpha)} U_j^{n+1} + \mu_t |U_j^n|^2 + \frac{1}{2} |\delta_x V_j^n|^2 \\ & + \frac{1}{2} M_j^{n+1} M_j^n + \frac{1}{2} \mu_t |U_j^n|^4 + \frac{1}{2} \left[M_j^n |U_j^{n+1}|^2 + M_j^{n+1} |U_j^n|^2 \right], \quad \forall j \in I_{J-1}. \end{aligned} \tag{5.81}$$

The *total discrete energy* at the time t_n is defined, for each $n \in \overline{I}_{N-1}$, by

$$\begin{aligned} E^n = {} & \|\delta_t U^n\|_2^2 + \mathrm{Re} \langle \delta_x^{(\alpha/2)} U^{n+1}, \delta_x^{(\alpha/2)} U^n \rangle + \mu_t \|U^n\|_2^2 + \frac{1}{2} \|\delta_x^{(\beta/2)} V^n\|_2^2 \\ & + \frac{1}{2} \langle M^{n+1}, M^n \rangle + \frac{1}{2} \mu_t \|U^n\|_4^4 + \frac{1}{2} \left[\langle M^n, |U^{n+1}|^2 \rangle + \langle M^{n+1}, |U^n|^2 \rangle \right], \end{aligned} \tag{5.82}$$

Here, we employ the notation $|U^n|^2 = (|U_j^n|^2)_{j \in \overline{I}_J}$, for each $n \in \overline{I}_N$.

There are equal terms in both schemes (5.24) and (5.77), nevertheless, we need to estimate some new quantities in view to establish the energy conservation for (5.77).

Lemma 5.3.2. *The following are satisfied for each* $n \in I_{N-1}$*,*

(a) $2\,\mathrm{Re} \langle -\delta_x^{(\alpha)} U^n, \delta_t^{(1)} U^n \rangle = \delta_t \,\mathrm{Re} \langle \delta_x^{(\alpha/2)} U^n, \delta_x^{(\alpha/2)} U^{n-1} \rangle$,

(b) $2\,\mathrm{Re} \langle M^n \mu_t^{(1)} U^n, \delta_t^{(1)} U^n \rangle = \langle M^n, \delta_t^{(1)} |U^n|^2 \rangle$,

(c) $2 \langle M^n, \delta_t^{(1)} |U^n|^2 \rangle + 2 \langle |U^n|^2, \delta_t^{(1)} M^n \rangle = \delta_t \left[\langle M^{n-1}, |U^n|^2 \rangle + \langle M^n, |U^{n-1}|^2 \rangle \right]$,

(d) $\mathrm{Re} \langle U^{n+1}, U^n \rangle = \mu_t \|U^n\|_2^2 - \frac{1}{2} \tau^2 \|\delta_t U^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}$,

(e) $|\langle M^n, |U^{n+1}|^2 \rangle + \langle M^{n+1}, |U^n|^2 \rangle| \leq \mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4$.

(f) $2 \langle \delta_x^{(\beta)} M^n, \mu_t V^{n-1} \rangle = \delta_t \langle M^n, M^{n-1} \rangle$,

(g) $\langle \delta_x^{(\beta)} |U^n|^2, \mu_t V^{n-1} \rangle = \langle |U^n|^2, \delta_t^{(1)} M^n \rangle$.

Proof. The identities (a)–(d) are straightforward and (e) is obtained applying Young's inequality. Finally, to prove (f) and (g), we use the property (5.80) (see [60]). □

Theorem 5.3.3 (Energy conservation)**.** *If* (U, M) *is solution of* (5.77)*, then the quantities* (5.82) *are constant. Moreover, if* $g_0^{(\gamma)} \tau^2 h^{1-\gamma} < 1$ *holds for* $\gamma = \alpha$*, and* $\gamma = \beta$*, then the constants are nonnegative.*

Proof. The first part of the proof is very similar to that in [60], but this time using the quantities with the term β, then, we will focus only in the second part. By Lemma 5.3.2 (d) and (e), we have

$$E^n \geq \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 - \frac{1}{2}\tau^2 \|\delta_x^{(\alpha/2)} \delta_t U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 - \frac{1}{4}\tau^2 \|\delta_t M^n\|_2^2, \tag{5.83}$$

and, from Lemma 5.2.12, it follows that $\|\delta_x^{(\alpha)} \delta_t U^n\|_2^2 \leq 2g_0^{(\alpha)} h^{1-\alpha} \|\delta_t U^n\|_2^2$ and $\|\delta_t M^n\|_2^2 \leq 2g_0^{(\beta)} h^{1-\beta} \|\delta_x^{(\beta/2)} V^n\|_2^2$. Using these inequalities in (5.83), we conclude that $\forall n \in \overline{I}_{N-1}$,

$$E^n \geq \left(1 - \frac{g_0^{(\alpha)}\tau^2}{h^{\alpha-1}}\right) \|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\left(1 - \frac{g_0^{(\beta)}\tau^2}{h^{\beta-1}}\right) \|\delta_x^{(\beta/2} V^n\|_2^2 \geq 0. \tag{5.84}$$

□

For this explicit scheme, also we have consistency of order two. In fact, the proof is similar as in Section 5.2, but this time, we are going to use

$$L_U(U_j^n, M_j^n) = \delta_t^{(2)} U_j^n - \delta_x^{(\alpha)} U_j^n + \mu_t^{(1)} U_j^n + M_j^n \mu_t^{(1)} U_j^n + \left(\mu_t^{(1)} |U_j^n|^2\right)\left(\mu_t^{(1)} U_j^n\right), \quad \forall (j,n) \in I, \tag{5.85}$$

$$L_M(U_j^n, M_j^n) = \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2, \quad \forall (j,n) \in I. \tag{5.86}$$

where $L(U^n, M^n) = (L(U_j^n, M_j^n))_{j \in \overline{I}_J}$ for each $n \in I_{N-1}$, and $L(U, M) = (L(U^n, M^n))_{n \in I_{N-1}}$. In similar fashion, the continuous operator $\mathcal{L} = \mathcal{L}_u \times \mathcal{L}_m$, defined for each (u, m) as in (5.34) and (5.35). Also, for each $x \in \{x_L, x_R\}$ and $t \in [0, T]$, we let $\mathcal{L}(u(x,t), m(x,t)) = 0$. Let $\mathcal{L}(u^n, m^n) = (\mathcal{L}(u_j^n, m_j^n))_{j \in \overline{I}_J}$ for each $n \in I_{N-1}$, and define $\mathcal{L}(u, m) = (\mathcal{L}(u^n, m^n))_{n \in I_{N-1}}$. Similarly, let $L(u^n, m^n) = (L(u_j^n, m_j^n))_{j \in \overline{I}_J} \in \mathring{\mathcal{V}}_h$ for each $n \in I_{N-1}$, and introduce $L(u, m) = (L(u^n, m^n))_{n \in I_{N-1}}$.

Theorem 5.3.4 (Consistency)**.** *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. Then there exists a constant C which is independent of τ and h, such that $|||(\mathcal{L} - L)(u, m)|||_\infty \leq C(\tau^2 + h^2)$ and $|||(\mathcal{H} - H)(u, m)|||_\infty \leq C(\tau + h^2)$.* □

Theorem 5.3.5 (Boundedness)**.** *Let $u_0, m_0 \in H^1$ and $u_1, m_1 \in L^2(\overline{B})$, and suppose that (U, M) is the solution of (5.77) corresponding to the initial data u_0, m_0, u_1 and m_1. If $g_0^{(\gamma)} \tau^2 h^{1-\gamma} < 1$ for $\gamma = \alpha, \beta$, then there is a common bound $C \in \mathbb{R}^+$ for $(\|\delta_t U^n\|_2)_{n \in \overline{I}_{N-1}}$, $(\|\delta_x^{(\alpha/2)} U^n\|_2)_{n \in \overline{I}_{N-1}}$, $(\|U^n\|_2)_{n \in \overline{I}_{N-1}}$, $(\|\delta_x^{(\beta/2)} V^n\|_2)_{n \in \overline{I}_{N-1}}$, $(\|M^n\|_2)_{n \in \overline{I}_{N-1}}$ and $(\|U^n\|_4)_{n \in \overline{I}_{N-1}}$.*

Proof. Let $n \in \overline{I}_{N-1}$ and $\epsilon > 1$. Proceeding as in [60] is easy to see that

$$\|\delta_t U^n\|_2^2 + \mathrm{Re}\langle \delta_x^{(\alpha/2)} U^{n+1}, \delta_x^{(\alpha/2)} U^n \rangle \geq \frac{1}{\epsilon}\left(\|\delta_t U^n\|_2^2 + \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2\right), \tag{5.87}$$

and

$$\|\delta_x^{(\beta/2)} V^n\|_2^2 + \langle M^{n+1}, M^n \rangle \geq \frac{1}{\epsilon}\left(\|\delta_x^{(\beta/2)} V^n\|_2^2 + \mu_t \|M^n\|_2^2\right). \tag{5.88}$$

On the other hand, applying Young's inequality we have

$$|\langle M^n, |U^{n+1}|^2 \rangle| \leq \|M^n\|_2^2 + \tfrac{1}{4}\|U^{n+1}\|_4^4 \quad \text{and} \quad |\langle M^{n+1}, |U^n|^2 \rangle| \leq \tfrac{1}{4\epsilon}\|M^{n+1}\|_2^2 + \epsilon \|U^n\|_4^4, \tag{5.89}$$

for each $n \in \overline{I}_{N-1}$. By Theorem 5.3.3, we know that the quantities (5.82) are equal to some $C_0 \geq 0$. Then, let $C_0^n \geq 0$ be a common bound of $\|M^n\|_2^2$ and $\|U^n\|_4^4$, and define $C_1^n = C_0 + \frac{1}{2}(\epsilon+1)C_0^n$. Using inequalities (5.87), (5.88) and (5.89), we obtain

$$\begin{aligned} C_1^n \geq & \frac{1}{\epsilon}\left(\|\delta_t U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2 + \tfrac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \tfrac{1}{2}\mu_t\|M^n\|_2^2\right) + \mu_t\|U^n\|_2^2 + \tfrac{1}{2}\mu_t\|U^n\|_4^4 \\ & - \tfrac{1}{2}\|M^n\|_2^2 - \tfrac{1}{8}\|U^{n+1}\|_4^4 - \tfrac{1}{8\epsilon}\|M^{n+1}\|_2^2 - \tfrac{\epsilon}{2}\|U^n\|_4^4 + \tfrac{1}{2}(\epsilon+1)\left(\|M^n\|_2^2 + \|U^n\|_4^4\right) \\ \geq & \frac{1}{\epsilon}\left[\|\delta_t U^n\|_2^2 + \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2\right] + \tfrac{1}{2\epsilon}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \mu_t\|U^n\|_2^2 + \tfrac{1}{4\epsilon}\mu_t\|M^n\|_2^2 + \tfrac{1}{4}\mu_t\|U^n\|_4^4 \end{aligned} \tag{5.90}$$

for each $n \in \overline{I}_{N-1}$. As a consequence, there exists $C_2^n \geq 0$ such that

$$\|\delta_t U^n\|_2^2, \|\delta_x^{(\alpha/2)}U^{n+1}\|_2^2, \|U^{n+1}\|_2^2, \|\delta_x^{(\beta)}V^n\|_2^2, \|M^{n+1}\|_2^2, \|U^{n+1}\|_4^4 \leq C_2^n, \tag{5.91}$$

for each $n \in \overline{I}_{N-1}$. Moreover, by hypotheses there exists a common bound $C_0^0 \geq 0$ for $\|\delta_t U^{-1}\|_2^2$, $\|\delta_x^{(\alpha/2)}U^0\|_2^2$, $\|U^0\|_2^2$, $\|\delta_x^{(\beta/2)}V^{-1}\|_2^2$, $\|M^0\|_2^2$, $\|U^0\|_4^4$ and $\|U^0\|_2^2$. Let $n \in \overline{I}_{N-1}$ and suppose that

$$\|\delta_t U^{n-1}\|_2^2, \|\delta_x^{(\alpha/2)}U^n\|_2^2, \|U^n\|_2^2, \|\delta_x^{(\beta/2)}V^{n-1}\|_2^2, \|M^n\|_2^2, \|U^n\|_4^4, \|U^n\|_2^2 \leq C_0^n, \tag{5.92}$$

for some $C_0^n \geq 0$. The proof follows by induction using $C = C_0^{n+1} = C_2^n$. In fact, it is possible to show that there exists $C_3^n \geq 0$ such that $\|U^{n+1}\|_\infty \leq C_3^n$ (see [35]) and take $C_0^{n+1} = \max\{C_2^n, C_3^n\}$
. □

In the following, (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ will represent two sets of initial conditions of (5.5), and we will assume that the initial data for (5.77) are provided exactly. Again, if $f : F \to F$ and $V \in \mathring{\mathcal{V}}_h$, then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$ and $F = \mathbb{R}, \mathbb{C}$. One more time, we are going to use Lemma 5.2.13 and proceed as in Lemma 5.2.14.

Theorem 5.3.6 (Stability)**.** *The method* (5.77) *is stable under the hypotheses of Theorem 5.3.5.*

Proof. Let (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ be sets of initial data of (5.5), and observe that the assumptions of Theorem 5.3.5 hold for both (U, V) and $(\widetilde{U}, \widetilde{V})$. On the other hand, note that (ε, ζ) satisfies the following with $(j, n) \in I$:

$$\begin{gathered} \delta_t^{(2)}\varepsilon_j^n - \delta_x^{(\alpha)}\varepsilon_j^n + \mu_t^{(1)}\varepsilon_j^n + \widetilde{\delta}\left[M_j^n\left(\mu_t^{(1)}U_j^n\right) + \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0, \\ \delta_t^{(2)}\zeta_j^n - \delta_x^{(\beta)}\zeta_j^n - \widetilde{\delta}\left(\delta_x^{(\beta)}|U_j^n|^2\right) = 0, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N. \end{gathered} \tag{5.93}$$

We want to compute the real part of the inner product between the first difference equation of (5.93) and $2\delta_t^{(1)}\varepsilon^n$, and the inner product of second difference equation of (5.93) with $2\mu_t v^{n-1}$, where $v^n = \widetilde{V}^n - V^n$ and $\delta_x^{(\beta)}v^n = \delta_t\zeta^n$, for each $n \in \overline{I}_{N-1}$. As we mention before, this is the same idea that was used in Lemma 5.2.14, but with some different quantities, for instance, notice that

$$\left|\left\langle\widetilde{\delta}\left[M^n\left(\mu_t^{(1)}U^n\right)\right], \delta_t^{(1)}\varepsilon^n\right\rangle\right| \leq C\left(\mu_t^{(1)}\|\zeta^n\|_2^2 + \mu_t^{(1)}\|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 + \mu_t^{(1)}\|\varepsilon^n\|_2^2 + \mu_t\|\delta_t\varepsilon^{n-1}\|_2^2\right), \tag{5.94}$$

and

$$\left|\left\langle\widetilde{\delta}\left(\delta_x^{(\beta)}|U^n|^2\right), \mu_t v^{n-1}\right\rangle\right| \leq C\left(\mu_t\|\delta_x^{(\beta/2)}v^{n-1}\|_2^2 + \|\delta_x^{(\alpha/2)}\varepsilon^n\|_2^2 + \|\varepsilon^n\|_2^2\right) \tag{5.95}$$

which are obtained in the same way that they were calculated in (5.60). Also, recall (5.87) and (5.88) to compute the terms $\mathrm{Re}\langle\delta_x^{(\alpha/2)}\varepsilon^n, \delta_x^{(\alpha/2)}\varepsilon^{n-1}\rangle_x$ and $\langle\zeta^n, \zeta^{n-1}\rangle_x$, respectively. The conclusion is follows using again, Gronwall's inequality of Lemma 5.2.13. In fact, the quantity ω^n is defined as in (6.71) to obtain the inequality (5.65). □

The argument to prove our final proposition is similar to that of stability, like in Theorem 5.2.17.

Theorem 5.3.7 (Convergence)**.** *If $u, m \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$ solves* (5.5)*, then the solution of* (5.77) *converges to that of the continuous problem with order $\mathcal{O}(\tau^2 + h^2)$ in L^∞ for $(U^n)_{n\in\overline{I}_N}$, and in L^2 for $(M^n)_{n\in\overline{I}_N}$.* □

The remainder of this section will be devoted to study computationally the performance of the methods proposed in this work. Our study will focus firstly on the capability of the schemes to preserve the discrete energy throughout time. In all of our simulations, we let $B = (-20, 20)$, and define the functions u_0, m_0, u_1 and m_1 by

$$u_0(x) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x \in B, \tag{5.96}$$

$$m_0(x) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x \in B, \tag{5.97}$$

$$u_1(x) = \frac{\sqrt{10} - \sqrt{2}}{2}(\tanh x - 1) \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \quad \forall x \in B, \tag{5.98}$$

$$m_1(x) = -4\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \quad \forall x \in B, \tag{5.99}$$

where $i^2 = -1$. As a matter of fact, it is worth pointing out that the exact solution of (5.5) when $\alpha = \beta = 2$ is given by the following exact traveling-wave solution on $\mathbb{R} \times \overline{\mathbb{R}^+}$ (see [45, 44]):

$$u(x,t) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right) \exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}, \tag{5.100}$$

$$m(x,t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}. \tag{5.101}$$

We consider the system (5.5) with $\alpha = \beta$ and the initial data (5.96)–(5.99). Computationally, let us fix $h = 0.025$ and let $\tau = 0.004$. This study concentrates on the capability of the schemes to preserve the energy throughout time. To that end, each experiment will determine the deviation of the total energy of the system with respect to the energy at the initial time, and the results will be presented as a function of the discrete time. The results of our simulations are shown in Figure 5.1 for $\alpha = \beta = 1.2$ (top row), $\alpha = \beta = 1.5$ (middle row) and $\alpha = \beta = 1.8$ (bottom row). The graphs on the left column were obtained using the implicit scheme (5.24), while those on the right column correspond to the explicit model (5.77). The results show that both methods approximately preserve the energy of their systems. This is in agreement with the theoretical results derived in this work. It is worth pointing out that the variation in the total energy associated to the explicit scheme is smaller. However, we are convinced that this phenomenon is due to the computational implementation of our techniques.

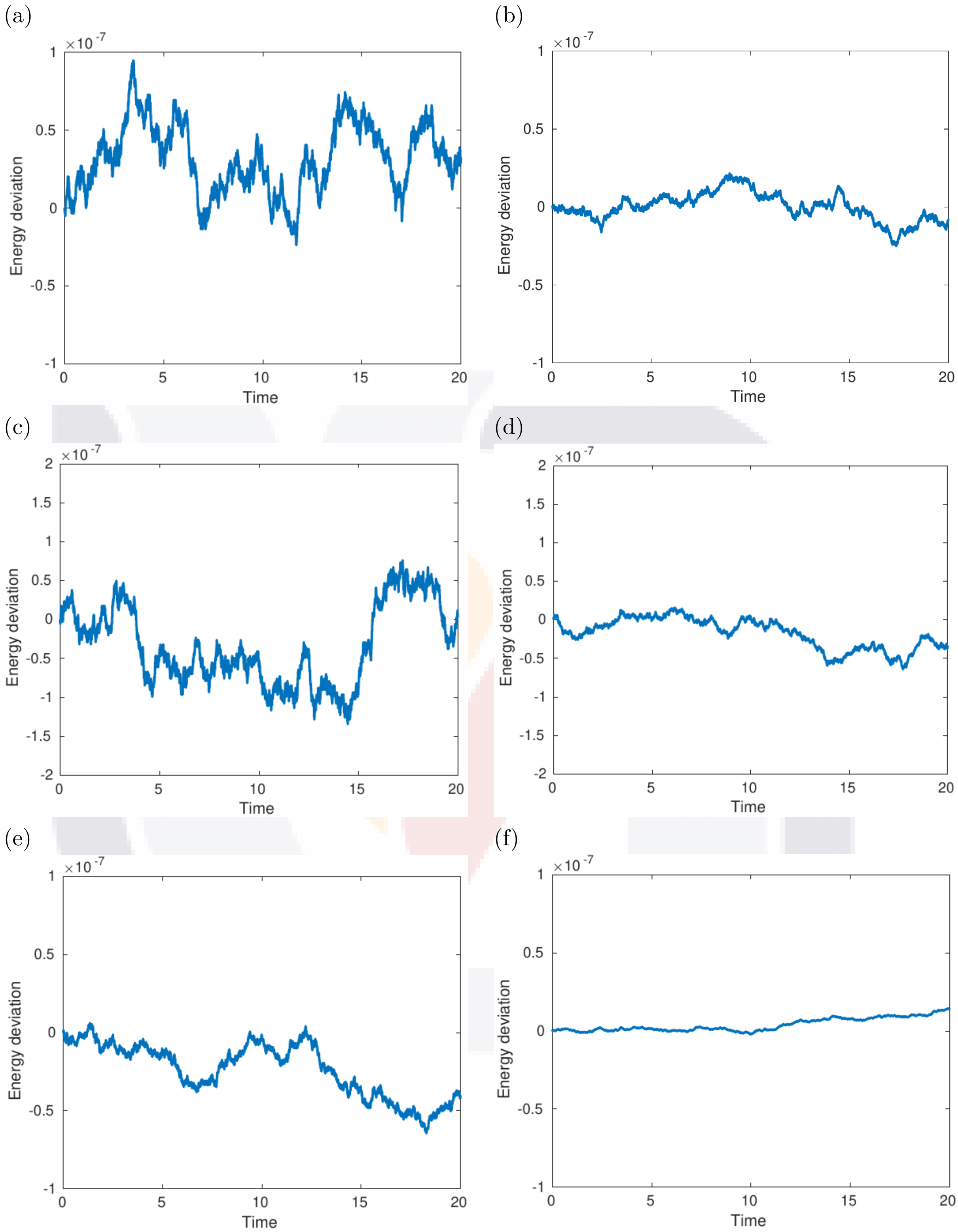

Figure 5.1: Graphs of the numerical deviation of discrete energy versus discrete time. The simulations correspond to numerical solution of (5.5) subjected to the initial data (5.96)–(5.99), on the spatial domain $\Omega = (20, 20)$. Left column: implicit model (5.24); right column: explicit model (5.77). We used $\alpha = \beta = 1.2$ (top row), $\alpha = \beta = 1.5$ (middle row) and $\alpha = \beta = 1.8$ (bottom row). The deviations were calculated with respect to the initial energy. Computationally, we used $h = 0.025$ and $\tau = 0004$.

6. Zakharov equations

6.1 Preliminaries

Let $I_q = \{1, \dots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$ for $q \in \mathbb{N}^+$. We use the symbol $\overline{X}$ to denote the closure of a set $X \subseteq \mathbb{R}^p$ under the usual topology of $\mathbb{R}^p$, where $p \in \mathbb{N}^+$ is a fixed natural number. For the remainder, we will suppose that $T > 0$ represents a fixed period of time, and $B = (x_L, x_R)$ is a nonempty interval in $\mathbb{R}$. Define $\Omega = B \times (0, T)$, and agree that all the functions of this study will be defined on the set $\overline{\Omega} \subseteq \mathbb{R}^2$. Moreover, we may extend the domain of definition of our functions to $\mathbb{R} \times [0, T]$ whenever needed, and by allowing them to be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$.

Definition 6.1.1 (Podlubny [78])**.** Let Γ denote the usual Gamma function which extends the factorial function. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any function, and assume that n is a non-negative integer and α is a real number, with the property that $n - 1 < \alpha \leq n$ is satisfied. Whenever it exists, the *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is given by

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{6.1}$$

When $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, the *Riesz fractional partial derivative* of u of order α with respect to x at $(x, t) \in \mathbb{R} \times [0, T]$ is given by (when it exists)

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi,t)d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{6.2}$$

For any $z \in \mathbb{C}$, we will represent its complex conjugate using the standard notation $\overline{z}$. Let us define the set $L_{x,p}(\overline{\Omega}) = \{f : \overline{\Omega} \to F : f(\cdot, t) \in L_p(\overline{B}),$ for each $t \in [0, T]\}$, where $p \in [1, \infty)$ and $F = \mathbb{R}, \mathbb{C}$. On the other hand, for any $f \in L_{x,p}(\overline{\Omega})$, we convey that

$$\|f\|_{x,p} = \left(\int_{\overline{B}} |f(x,t)|^p dx \right)^{1/p}, \quad \forall t \in [0, T], \tag{6.3}$$

which is a function of $t \in [0, T]$. Moreover, for each pair $f, g \in L_{x,2}(\overline{\Omega})$, define the following function of t:

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x,t)\overline{g(x,t)}dx, \quad \forall t \in [0, T]. \tag{6.4}$$

For the remainder of this work, we fix $\alpha, \beta \in (1, 2]$. Assume that u and m are a complex- and a real-valued functions, respectively, whose domains are both equal to $\overline{\Omega}$. Moreover, let $u_0 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions. Under these circumstances, the fractional extension of the Zakharov problem investigated in this work is given by the system

$$
\begin{aligned}
i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} - u(x,t) - m(x,t)u(x,t) - |u(x,t)|^2 u(x,t) = 0, \quad \forall (x,t) \in \Omega, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta} = 0, \quad \forall (x,t) \in \Omega, \\
\text{subject to } \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), \quad \forall x \in \overline{B}, \\ \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, \quad \forall t \in [0,T]. \end{cases}
\end{aligned}
\tag{6.5}
$$

Notice that the case $\alpha = \beta = 2$ is precisely the well-known Zakharov system [102, 103]. For convenience, we define the function $v : \overline{\Omega} \to \mathbb{R}$ in such way that

$$
\frac{\partial^\beta v(x,t)}{\partial |x|^\beta} = \frac{\partial m(x,t)}{\partial t}, \quad \forall (x,t) \in \Omega. \tag{6.6}
$$

Definition 6.1.2. Let u, m be a pair of functions satisfying the initial-boundary-value problem (6.5). The mass density of the system is given by the expression $\mathcal{M}(x,t) = |u(x,t)|^2$, for each $(x,t) \in \Omega$. In turn, the total mass at the time $t \in [0,T]$ is calculated through $\mathcal{M}(t) = \|u\|_{x,2}^2$. Let us define the Hamiltonian of our fractional Zakharov equations as

$$
\mathcal{H}(x,t) = \left|\frac{\partial u}{\partial t}\right|^2 + \mathcal{H}_F(x,t), \quad \forall (x,t) \in \Omega. \tag{6.7}
$$

Here,

$$
\mathcal{H}_F(x,t) = \left|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right|^2 + |u|^2 + m|u|^2 + \frac{1}{2}\left|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right|^2 + \frac{1}{2}m^2 + \frac{1}{2}|u|^4, \quad \forall (x,t) \in \Omega \tag{6.8}
$$

denotes the Higgs' free local energy density component, and v satisfies Equation (6.6). For the sake of simplification of the nomenclature, we obviated the dependence of all the functions on the right-hand side of this identity with respect to (x,t). In turn, the associated total energy of the system at the time $t \in [0,T]$ is provided then by

$$
\mathcal{E}(t) = \int_{-\infty}^{\infty} \mathcal{H}(x,t)dx = \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \mathcal{E}_F(t), \tag{6.9}
$$

where

$$
\mathcal{E}_F = \left\|\frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}\right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2\rangle_x + \frac{1}{2}\left\|\frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}}\right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4 \tag{6.10}
$$

represents the Higgs' free energy at the time t.

Theorem 6.1.3 (Conservation of mass)**.** *If u and m satisfy the problem* (6.5)*, then the total mass is conserved.*

Proof. Take the imaginary part of the inner product between the first equation of (6.5) with u to obtain that

$$
0 = \operatorname{Im}\left\langle i\frac{\partial u}{\partial t} + \frac{\partial^\alpha u}{\partial |x|^\alpha} - u - mu - |u|^2 u, u\right\rangle_x = \frac{1}{2}\frac{d}{dt}\|u\|_{x,2}^2, \quad \forall t \in (0,T). \tag{6.11}
$$

The property of conservation of mass readily follows now from these identities. □

Theorem 6.1.4 (Conservation of free energy)**.** *If u and m satisfy* (6.5)*, then the free energy is non-negative and constant.*

Proof. Using the first equation of (6.5), it follows that $\forall t \in (0,T)$,

$$\begin{aligned} 0 &= \operatorname{Re}\left\langle i\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right\rangle_x = \operatorname{Re}\left\langle -\frac{\partial^\alpha u}{\partial |x|^\alpha} + u + mu + |u|^2 u, \frac{\partial u}{\partial t}\right\rangle_x \\ &= \frac{1}{2}\frac{d}{dt}\left(\left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2\rangle_x + \frac{1}{2}\left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4 \right). \end{aligned} \tag{6.12}$$

We conclude from this that $\mathcal{E}_F'(t) = 0$, for each $t \in [0,T]$, as desired. The non-negativity of the function $\mathcal{E}_F$ readily follows from its definition and the fact that $\langle m, |u|^2\rangle_x \leq \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4$ by Young's inequality. □

Corollary 6.1.5 (Boundedness)**.** *Assume that u and m satisfy the initial-boundary-value problem* (6.5)*. Suppose also that $u, \partial^2 u/\partial x^2 \in L_{x,2}(\overline{\Omega})$. Then there exist a constant C which depends only on the initial conditions, such that*

$$\left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \|m\|_{x,2}^2 \leq C, \quad \forall t \in [0,T]. \tag{6.13}$$

Moreover, the functions (6.9) *and* (6.10) *are both non-negative.*

Proof. Notice that Theorem 6.1.4 assures that there exists a constant $C_0 \in \mathbb{R}$ such that $\mathcal{E}_F(t) = C_0$, for each $t \in [0,T]$. It is worth pointing out that $C_0 = \mathcal{E}_F(0)$, which is entirely expressed in terms of the initial conditions. On the other hand, observe that $|\langle m, |u|^2\rangle| \leq \frac{1}{2}(\|m\|_2^2 + \|u\|_4^4)$ holds for all $t \in [0,T]$. Therefore, it follows that

$$\begin{aligned} 3C_0 &\geq \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 - \left|\langle m, |u|^2\rangle_x\right| + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \frac{3}{2}\|m\|_{x,2}^2 + \frac{3}{2}\|u\|_{x,4}^4 \\ &\geq \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \|m\|_{x,2}^2 + \|u\|_{x,4}^4. \end{aligned} \tag{6.14}$$

Finally, we readily reach the conclusion of this result by letting $C = 3C_0$. □

Theorem 6.1.6 (Dissipation of energy)**.** *The total energy of the system* (6.5) *is dissipated.*

Proof. We compute firstly the derivative of the first equation of (6.5). Next, we take the imaginary part of the inner product between that derivative and $\frac{\partial u}{\partial t}$ to obtain, for each $t \in [0,T]$, that

$$\begin{aligned} 0 &= \operatorname{Im}\left\langle i\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{\partial^\alpha u_t}{\partial |x|^\alpha} - u - mu - |u|^2 u\right), \frac{\partial u}{\partial t}\right\rangle_x \\ &= \frac{1}{2}\frac{d}{dt}\left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 - \operatorname{Im}\left\langle u\left(\frac{\partial m}{\partial t} + u\frac{\partial \overline{u}}{\partial t}\right), \frac{\partial u}{\partial t}\right\rangle_x. \end{aligned} \tag{6.15}$$

Using the property on the conservation of free energy and the last identity, we notice that

$$\mathcal{E}'(t) = 2\operatorname{Im}\left\langle u\left(\frac{\partial m}{\partial t} + u\frac{\partial \overline{u}}{\partial t}\right), \frac{\partial u}{\partial t}\right\rangle_x, \quad \forall t \in (0,T). \tag{6.16}$$

We conclude that the total energy of the system (6.5) is dissipated, as desired. □

Before we close this section, we introduce the concept of fractional centered differences which will be the cornerstone to provide consistent a discretization for Riesz-type fractional partial derivatives. For the remainder, we will employ the discrete spatial step-size $h = (x_R - x_L)/J$.

Definition 6.1.7 (Ortigueira [73])**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function, and let α and h be real numbers such that $\alpha \in (0,1) \cup (1,2]$ and $h > 0$. Let $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ be the two-sided infinite sequence given by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{Z}. \tag{6.17}$$

When it exists, the *fractional-order centered difference* of order α of f at the point x is defined as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{6.18}$$

It is well known [97] that the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ satisfies the following properties when $\alpha \in (0,1) \cup (1,2]$:

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ for all $k \geq 1$, and

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$.

Moreover, if all the derivatives of the function $f : \mathbb{R} \to \mathbb{R}$ up to the order five are integrable over $\mathbb{R}$ and $h > 0$, then the following consistency property holds true [97]:

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{6.19}$$

6.2 Numerical model

For the remainder, we will use the symbol $\mathbb{F}$ to denote any of $\mathbb{R}$ or $\mathbb{C}$. Let J and N be arbitrary natural numbers, and introduce the computational constant $\tau = T/N$. Fix regular partitions of $[x_L, x_R]$ and $[0, T]$, respectively, in the following way:

$$x_L = x_0 < x_1 < \ldots < x_j < \ldots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{6.20}$$

and

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T, \quad \forall n \in \overline{I}_N. \tag{6.21}$$

Let us set $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, for each $(j,n) \in \overline{I}_J \times \overline{I}_N$, and agree that U_j^n and M_j^n denote computational estimates for the exact values of u_j^n and m_j^n, respectively. We employ the notation $\mathcal{V}_h$ to represent the vector space of all $\mathbb{F}$-valued functions defined on the set $\{x_j : j \in \overline{I}_J\}$ which vanish at x_0 and x_J. If $V \in \mathcal{V}_h$, we agree that $V_j = V(x_j)$, for each $j \in \overline{I}_J$. Finally, set $U^n = (U_j^n)_{j \in \overline{I}_J} \in \mathcal{V}_h$ and $M^n = (M_j^n)_{j \in \overline{I}_J} \in \mathcal{V}_h$, and let $U = (U^n)_{n \in \overline{I}_N}$ and $M = (M^n)_{n \in \overline{I}_N}$.

Definition 6.2.1. Let $1 \leq q < \infty$. The functions $\langle \cdot, \cdot \rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{C}$ and $\| \cdot \|_q, \| \cdot \|_\infty : \mathcal{V}_h \to \mathbb{R}$ are defined by

$$\langle U, V \rangle = h \sum_{j \in \overline{I}_J} U_j \overline{V_j}, \quad \forall U, V \in \mathcal{V}_h, \tag{6.22}$$

$$\|U\|_q^q = h \sum_{j \in \overline{I}_J} |U_j|^q, \quad \forall U \in \mathcal{V}_h, \tag{6.23}$$

$$\|U\|_\infty = \max \left\{ |U_j| : j \in \overline{I}_J \right\}, \quad U \in \mathcal{V}_h. \tag{6.24}$$

Moreover, for any $V = (V^n)_{n \in \overline{I}_N} \subseteq \mathcal{V}_h$ we define $|||V|||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}$.

Definition 6.2.2. Let V be U or M, and assume $\alpha \in (0,1) \cup (1,2]$. Introduce the discrete operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j,n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{6.25}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{6.26}$$

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j,n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{6.27}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j,n) \in \overline{I}_J \times I_{N-1}. \tag{6.28}$$

Using these definitions, we introduce the operators $\delta_x^{(2)} V_j^n = \delta_x \circ \delta_x V_{j-1}^n$, $\delta_t^{(1)} V_j^n = \mu_t \circ \delta_t V_j^{n-1}$, $\delta_t^{(2)} V_j^n = \delta_t \circ \delta_t V_j^{n-1}$ and $\mu_t^{(2)} V_j^n = \mu_t \circ \mu_t V_j^{n-1}$, for each $(j,n) \in I_{J-1} \times \overline{I}_N$. Moreover, let

$$\delta_x^{(\alpha)} V_j^n = -\frac{1}{h^\alpha} \sum_{k \in \overline{I}_J} g_{j-k}^{(\alpha)} V_k^n, \qquad \forall (j,n) \in I_{J-1} \times \overline{I}_N. \tag{6.29}$$

Lemma 6.2.3 (Macías-Díaz [49])**.** *Assume that* $\alpha \in (1,2]$ *and* $U, V \in \mathcal{V}_h$. *Then* $\langle -\delta_x^{(\alpha)} U, V \rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V \rangle$. □

With this nomenclature, the discrete model proposed in the present manuscript to approximate the solutions of (6.5) is summarized as the following coupled system of algebraic equations:

$$\begin{gathered} i\delta_t^{(1)} U_j^n + \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n - \mu_t^{(1)} U_j^n - M_j^n \mu_t^{(1)} U_j^n - \left(\mu_t^{(1)} |U_j^n|^2 \right) \left(\mu_t^{(1)} U_j^n \right) = 0, \quad \forall (j,n) \in I, \\ \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2 = 0, \quad \forall (j,n) \in I, \\ \text{such that} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), & \forall j \in \overline{I}_J, \\ \mu_t^{(1)} U_j^0 = u_0(x_j), & \delta_t^{(1)} M_j^0 = m_1(x_j), & \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, & \forall n \in \overline{I}_N. \end{cases} \end{gathered} \tag{6.30}$$

The first equation of this system yields an expression with complex parameters in which the only unknown is U_j^{n+1}. Moreover, the second equation of (6.30) is a fully explicit difference equation which can be easily solved for M_j^{n+1}, for each $(j,n) \in I$. Using then the initial data, we readily obtain that for each $j \in I_{J-1}$, the following identities hold:

$$U_j^1 = u_0(x) + i\tau \left[\delta_x^{(\alpha)} u_0(x_j) - u_0(x_j) \left(1 + M_j^0 + \frac{1}{2} \left(|U_j|^2 + |2u_0(x_j) - U_j^1|^2 \right) \right) \right], \tag{6.31}$$

$$M_j^1 = m_0(x_j) + \tau m_1(x_j) + \frac{\tau^2}{2} \delta_x^{(\beta)} \left(m_0(x_j) + |u_0(x_j)|^2 \right). \tag{6.32}$$

For the remainder of this manuscript, we will employ the sequence $(V^n)_{n\in\overline{I}_N}$ in $\mathring{\mathcal{V}}_h$ which satisfies $\delta_x^{(\beta)}V_j^n = \delta_t M_j^n$, for each $(j,n) \in I_{J-1} \times \overline{I}_{N-1}$. Under these circumstances, (U,M) will denote a solution of (6.30).

Lemma 6.2.4 (Macías-Díaz [49]). *If $V \in \mathcal{V}_h$ and $\alpha \in (1,2]$ then*

(a) $\|\delta_x^{(\alpha/2)}V\|_2^2 \leq 2g_0^{(\alpha)}h^{1-\alpha}\|V\|_2^2$,

(b) $\|\delta_x^{(\alpha)}V\|_2^2 = \|\delta_x^{(\alpha/2)}\delta_x^{(\alpha/2)}V\|_2^2$,

(c) $\|\delta_x^{(\alpha)}V\|_2^2 \leq 2g_0^{(\alpha)}h^{1-\alpha}\|\delta_x^{(\alpha/2)}V\|_2^2 \leq 4\left(g_0^{(\alpha)}h^{1-\alpha}\right)^2\|V\|_2^2$.

In a first stage, we will prove the existence of solutions for the numerical model (6.30). The cornerstone in our proof will be the following fixed-point result from the standard literature.

Lemma 6.2.5 (Browder fixed-point [26]). *Let $(H, \langle\cdot,\cdot\rangle)$ be a finite-dimensional inner-product space, let $\|\cdot\| : H \to H$ be the norm induced by $\langle\cdot,\cdot\rangle$, and suppose that $F : H \to H$ is continuous. Assume that there exists $\lambda > 0$ such that $\operatorname{Re}\langle F(z), z\rangle > 0$, for all $z \in H$ with $\|z\| = \lambda$. Then, there is $z^* \in H$, such that $F(z^*) = 0$ and $\|z^*\| \leq \lambda$.*

Theorem 6.2.6 (Solubility). *The model* (6.30) *is solvable for any set of initial conditions.*

Proof. Notice that the approximation (U^0, M^0) is defined by the initial conditions. Proceeding inductively, suppose that (U^{n-1}, M^{n-1}) and (U^n, M^n) have been already obtained for some $n \in I_{N-1}$. In a first stage, observe that the second equation of (6.30) can be written as $A\Psi = b$, where Ψ is the unknown vector of approximations at time t_{n+1}, and the matrix A and the vector b, are given by

$$A = \frac{1}{\tau^2}\begin{pmatrix} \tau^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tau^2 \end{pmatrix} \tag{6.33}$$

and

$$b = \begin{pmatrix} 0 \\ \delta_x^{(\beta)}|U_1^n|^2 + \delta_x^{(\beta)}M_1^n + \frac{1}{\tau^2}\left(2M_1^n - M_1^{n-1}\right) \\ \delta_x^{(\beta)}|U_2^n|^2 + \delta_x^{(\beta)}M_2^n + \frac{1}{\tau^2}\left(2M_2^n - M_2^{n-1}\right) \\ \vdots \\ \delta_x^{(\beta)}|U_{J-1}^n|^2 + \delta_x^{(\beta)}M_{J-1}^n + \frac{1}{\tau^2}\left(2M_{J-1}^n - M_{J-1}^{n-1}\right) \\ 0 \end{pmatrix}. \tag{6.34}$$

Since A is nonsingular, there exists a vector M^{n+1} which satisfies the system consisting of all the second difference equations of (6.30) at time t_n. On the other hand, observe that we can rewrite the first equation of (6.30) as

$$\mu_t^{(1)}U_j^n - U_j^{n-1} + i\tau\left[-\delta_x^{(\alpha)}\mu_t^{(1)}U_j^n + \mu_t^{(1)}U_j^n + M_j^n\mu_t^{(1)}U_j^n + \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0. \tag{6.35}$$

Now, let us consider the function $F : \mathcal{V}_h \to \mathcal{V}_h$, where each of the component functions of F is given by

$$F_j(\eta) = \eta_j^n - U_j^{n-1} + i\tau\left[-\delta_x^{(\alpha)}\eta_j^n + \eta_j^n + M_j^n\eta_j^n + \left(\mu_t^{(1)}|U_j^n|^2\right)\eta_j^n\right], \quad \forall \eta \in \mathcal{V}_h, j \in I_{J-1}. \tag{6.36}$$

Then, taking the real part of the inner product between above identity and η, we obtain

$$\operatorname{Re}\langle F(\eta),\eta\rangle = \|\eta\|_2^2 - \operatorname{Re}\langle U^{n-1},\eta\rangle \geq \|\eta\|_2^2 - |\langle U^{n-1},\eta\rangle| \geq \frac{1}{2}\left(\|\eta\|_2^2 - \|U^{n-1}\|_2^2\right). \tag{6.37}$$

Applying Browder's fixed-point theorem with $\lambda = \|U^{n-1}\|_2^2 + 1$, it follows that there exists a vector $U^{n+1} \in \mathcal{V}_h$ which satisfies the remaining equation of (6.30). □

Definition 6.2.7. Let (U, M) be a solution of (6.30). The discrete mass density of (6.30) at the point x_j and time t_n is given by $\mu_t|U_j^n|^2$, for each $(j,n) \in I_{J-1} \times I_{N-1}$. In turn, the total discrete mass of the system at time t_n is given by $\mu_t\|U^n\|_2^2$, for each $n \in I_{N-1}$. The discrete energy density at the point x_j and time t_n is given by

$$\begin{aligned} H_j^n &= |\delta_t U_j^n|^2 + \mu_t|\delta_x^{(\alpha/2)}U_j^n|^2 + \mu_t|U_j^n|^2 + \frac{1}{2}|\delta_x^{(\beta/2)}V_j^n|^2 \\ &\quad + \frac{1}{2}M_j^{n+1}M_j^n + \frac{1}{2}\mu_t|U_j^n|^4 + \frac{1}{2}\left[M_j^n|U_j^{n+1}|^2 + M_j^{n+1}|U_j^n|^2\right], \quad \forall (j,n) \in I_{J-1} \times I_{N-1}. \end{aligned} \tag{6.38}$$

In turn, the total discrete energy at the time t_n is defined, for each $n \in \overline{I}_{N-1}$, by

$$E^n = h\sum_{j\in\overline{J}} H_j^n = \|\delta_t U^n\|_2^2 + E_F^n, \tag{6.39}$$

where

$$\begin{aligned} E_F^n &= \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n\rangle + \frac{1}{2}\mu_t\|U^n\|_4^4 \\ &\quad + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2\rangle\right], \end{aligned} \tag{6.40}$$

Here, the nomenclature $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$ is observed, for each $n \in \overline{I}_N$.

Theorem 6.2.8 (Conservation of discrete mass)**.** *If (U, M) is a solution of (6.30), then the total discrete mass is conserved with respect to the discrete time.*

Proof. Rewrite the first difference equation of the discrete model (6.30), compute the inner product on both sides with $\mu_t^{(1)}U_j^n$ and take imaginary parts. As a consequence, we readily check that

$$0 = \operatorname{Im}\left\langle -i\delta_t^{(1)}U^n, \mu_t^{(1)}U^n\right\rangle = \frac{1}{2}\delta_t^{(1)}\|U^n\|_2^2 = \frac{1}{2}\delta_t\mu_t\|U^n\|_2^2, \quad \forall n \in I_{N-1}, \tag{6.41}$$

which yields what we wanted to prove. □

Theorem 6.2.9 (Conservation of discrete free energy)**.** *Suppose that (U, M) is a solution of (6.30). Then the discrete free energy E_F^n is constant. Moreover, if $\tau^2 g_0^{(\beta)}h^{1-\beta} < 1$, then $E_F^n \geq 0$ and $E^n \geq 0$, for each $n \in I_{N-1}$.*

Proof. Notice that, for each $n \in I_{N-1}$, the following identities are satisfied:

$$0 = \operatorname{Re}\left\langle i\delta_t^{(1)}U^n, \delta_t^{(1)}U^n\right\rangle = \operatorname{Re}\left\langle -\delta_x^{(\alpha)}\mu_t^{(1)}U^n + \mu_t^{(1)}U^n\left(1 + M^n + \mu_t^{(1)}|U^n|^2\right), \delta_t^{(1)}U^n\right\rangle, \tag{6.42}$$

$$0 = \langle -\delta_t^{(2)}M^n + \delta_x^{(\beta)}M^n + \delta_x^{(\beta)}|U^n|^2, \mu_t V^{n-1}\rangle + \langle |U^n|^2\delta_t^{(1)}M^n\rangle \tag{6.43}$$

$$+ \frac{1}{2}\delta_t\left(\|\delta_x^{(\beta/2)}V^{n-1}\|_2^2 + \langle M^n, M^{n-1}\rangle\right). \tag{6.44}$$

Using identities of [63] and calculating the right-hand side of the first of these identities, we obtain that

$$\delta_t\mu_t\left(\|\delta_x^{(\alpha/2)}U^{n-1}\|_2^2+\|U^{n-1}\|_2^2+\frac{1}{2}\|U^{n-1}\|_4^4\right)+\langle M^n,\delta_t^{(1)}U^n\rangle=0,\quad\forall n\in I_{N-1}. \tag{6.45}$$

Observe that the identity $\langle M^n,\delta_t^{(1)}U^n\rangle+\langle |U^n|^2,\delta_t^{(1)}M^n\rangle=\frac{1}{2}\delta_t\left[\langle M^{n-1},|U^n|^2\rangle+\langle M^n,|U^{n-1}|^2\rangle\right]$ is satisfied, for each $n\in I_{N-1}$. Finally, sum (6.44) and (6.45) to reach $\delta_t E_F^{n-1}=0$. For the second part, notice that Lemma 6.2.4(c) implies that $h^{\beta-1}\|\delta_t M^n\|_2^2\leq 2g_0^{(\beta)}\|\delta_x^{(\beta/2)}V^n\|_2^2$. Moreover, rearranging terms, using some algebraic simplifications and applying the Cauchy–Schwarz inequality, we may observe that

$$\langle M^{n+1},M^n\rangle=\mu_t\|M^n\|_2^2-\frac{\tau^2}{2}\|\delta_t M^n\|_2^2,\quad\forall n\in I_{N-1}, \tag{6.46}$$

$$\langle M^n,|U^{n+1}|^2\rangle+\langle M^{n+1},|U^n|^2\rangle|\leq\mu_t\|M^n\|_2^2+\mu_t\|U^n\|_4^4,\quad\forall n\in I_{N-1}. \tag{6.47}$$

As a consequence,

$$E_F^n\geq\mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2+\mu_t\|U^n\|_2^2+\frac{1}{4}\left(\frac{h^{\beta-1}}{g_0^{(\beta)}}-\tau^2\right)\|\delta_t M^n\|_2^2\geq 0,\quad\forall n\in I_{N-1}. \tag{6.48}$$

Moreover, $E^n=\|\delta_t U^n\|_2^2+E_F^n\geq 0$, whence the conclusion of this result readily follows. □

Theorem 6.2.10 (Boundedness)**.** *Let $u_0,m_0\in H^1$ and $u_1,m_1\in L^2(\overline{B})$, and suppose that (U,M) is the solution of (6.30) corresponding to the initial data u_0, m_0, u_1 and m_1. If $g_0^{(\beta)}\tau^2h^{1-\beta}<1$, then the sequences $(\|\delta_x^{(\alpha/2)}U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|\delta_x^{(\beta/2)}V^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|M^n\|_2)_{n\in\overline{I}_{N-1}}$ and $(\|U^n\|_4)_{n\in\overline{I}_{N-1}}$ are bounded by a common constant.*

Proof. Proceeding as in Theorem 6.2.9, we have $\forall n\in\overline{I}_{N-1}$,

$$\frac{1}{2}\mu_t\|M^n\|_2^2=\frac{\tau^2}{4}\|\delta_x^{(\beta)}V^n\|_2^2+\frac{1}{2}\langle M^{n+1},M^n\rangle\leq\frac{1}{2}\tau^2g_0^{(\beta)}h^{1-\beta}\left\|\delta_x^{(\beta/2)}V^n\right\|_2^2+\frac{1}{2}\langle M^{n+1},M^n\rangle. \tag{6.49}$$

From the previous theorem, we know that there is a constant C_0 such that $E_F^n=C_0$, for all $n\in\overline{I}_{N-1}$. Then

$$\begin{aligned}C_0\geq{}&\mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2+\mu_t\|U^n\|_2^2+\frac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2+\frac{1}{2}\langle M^{n+1},M^n\rangle\\&+\frac{1}{2}\mu_t\|U^n\|_4^4-\frac{1}{2}\left|\langle M^n,|U^{n+1}|^2\rangle\right|-\frac{1}{2}\left|\langle M^{n+1},|U^n|^2\rangle\right|\end{aligned} \tag{6.50}$$

Applying Young's inequality two times, we obtain

$$\frac{1}{2}\left|\langle M^n,|U^{n+1}|^2\rangle\right|+\frac{1}{2}\left|\langle M^{n+1},|U^n|^2\rangle\right|\leq\frac{1}{2}\mu_t\|M^n\|_2^2+\frac{1}{2}\mu_t\|U^n\|_4^4,\quad\forall n\in\overline{I}_{N-1}, \tag{6.51}$$

$$\frac{1}{2}\left|\langle M^n,|U^{n+1}|^2\rangle\right|+\frac{1}{2}\left|\langle M^{n+1},|U^n|^2\rangle\right|\leq\frac{1}{4}\mu_t\|M^n\|_2^2+\mu_t\|U^n\|_4^4,\quad\forall n\in\overline{I}_{N-1}. \tag{6.52}$$

Using (6.51) in (6.50) and simplifying algebraically, it is easy to see that

$$\mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2+\mu_t\|U^n\|_2^2+\frac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2+\frac{1}{2}\langle M^{n+1},M^n\rangle-\frac{1}{2}\mu_t\|M^n\|_2^2\leq C_0,\quad\forall n\in\overline{I}_{N-1}. \tag{6.53}$$

Now, take the sum between both sides of (6.49) and (6.53) and simplify again. As a consequence, it follows that

$$C_0\geq\mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2+\mu_t\|U^n\|_2^2+\frac{1}{2}\left(1-\tau^2g_0^{(\beta)}h^{1-\beta}\right)\|\delta_x^{(\beta/2)}V^n\|_2^2,\quad\forall n\in\overline{I}_{N-1}. \tag{6.54}$$

Using Lemmas 4.3 and 4.4 of [60], it readily follows that

$$\|\delta_x^{(\beta/2)}V^n\|_2^2 + \langle M^{n+1}, M^n\rangle \leq \frac{3}{5}\left(\|\delta_x^{(\beta/2)}V^n\|_2^2 + \mu_t\|M^n\|_2^2\right). \tag{6.55}$$

Removing now the first two terms of right-hand side of (6.50), and using (6.52) and the previous remark yields

$$C_0 \geq \frac{3}{10}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \frac{3}{10}\mu_t\|M^n\|_2^2 - \frac{1}{4}\|M^n\|_2^2 - \frac{1}{2}\mu_t\|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{6.56}$$

Since $\mu_t\|U^n\|_2$ is bounded, then $\mu_t\|U^n\|_4$ is also bounded. Therefore, there is a constant C_1 such that

$$C_1 \geq \frac{1}{2}\mu_t\|U^n\|_4^4 + C_0 \geq \frac{1}{20}\mu_t\|M^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{6.57}$$

The conclusion of this theorem is obtained now by letting $C = C_0 + 20C_1$. □

6.3 Numerical properties

In this section, we establish the main properties of the finite-difference method (6.30). More precisely, we prove the consistency, the stability and the convergence of our numerical model. Some additional nomenclature will be required to that end. For example, we will employ the continuous differential operators

$$\mathcal{L}_u(x,t) = i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} - u(x,t) - m(x,t)u(x,t) - |u(x,t)|^2 u(x,t), \quad \forall (x,t) \in \Omega, \tag{6.58}$$

$$\mathcal{L}_m(x,t) = \frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta}, \quad \forall (x,t) \in \Omega. \tag{6.59}$$

Set $\mathcal{L}(x,t) = (\mathcal{L}_u(x,t), \mathcal{L}_m(x,t))$, for each $(x,t) \in \Omega$. Moreover, define $\mathcal{L}_j^n = \mathcal{L}(x_j, t_n)$, for each $(j,n) \in \overline{I}_J \times \overline{I}_N$. For the sake of convenience, we let $\mathcal{L}^n = (\mathcal{L}_j^n)_{j\in\overline{I}_J}$, for each $n \in \overline{I}_N$, and convey $\mathcal{L} = (\mathcal{L}^n)_{n\in\overline{I}_N}$.

On the other hand, let us introduce the discrete difference operators

$$L_U(x_j,t_n) = i\delta_t^{(1)}U_j^n + \delta_x^{(\alpha)}\mu_t^{(1)}U_j^n - \mu_t^{(1)}U_j^n\left(1 + M_j^n + \mu_t^{(1)}|U_j^n|^2\right) = 0, \quad \forall (j,n) \in I, \tag{6.60}$$

$$L_M(x_j,t_n) = \delta_t^{(2)}M_j^n - \delta_x^{(\beta)}M_j^n - \delta_x^{(\beta)}|U_j^n|^2 = 0, \quad \forall (j,n) \in I. \tag{6.61}$$

As in the continuous case, we agree that $L(x_j,t_n) = (L_U(x_j,t_n), L_M(x_j,t_n))$, for each $(j,n) \in \overline{I}_J \times \overline{I}_N$, and define $L_j^n = L(x_j,t_n)$. Let us set $L^n = (L_j^n)_{j\in\overline{I}_J}$, for each $n \in \overline{I}_N$, and let $L = (L^n)_{n\in\overline{I}_N}$.

Theorem 6.3.1 (Consistency). *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. Then there exist constants C and C' which are independent of τ and h, such that $|||\mathcal{L} - L|||_\infty \leq C(\tau^2 + h^2)$ and $|||\mathcal{H} - H|||_\infty \leq C'(\tau^2 + h^2)$.*

Proof. Using Taylor's theorem, the mean value theorem and the regularity of the functions u and m, it is possible to show that there are constants $C_i \in \mathbb{R}^+$ independent of τ and h, for each $i \in I_5$, such that

$$\left|\frac{\partial u(x_j,t_n)}{\partial t} - \delta_t^{(1)} u_j^n\right| \leq C_1(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{6.62}$$

$$\left|\frac{\partial^\alpha u(x_j,t_n)}{\partial |x|^\alpha} - \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n\right| \leq C_2(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{6.63}$$

$$\left|u(x_j,t_n) - \mu_t^{(1)} u_j^n\right| \leq C_3\tau^2, \quad \forall (j,n) \in I, \tag{6.64}$$

$$\left|m(x_j,t_n)u(x_j,t_n) - m_j^n \mu_t^{(1)} u_j^n\right| \leq C_4\tau^2, \quad \forall (j,n) \in I, \tag{6.65}$$

$$\left||u(x_j,t_n)|^2 u(x_j,t_n) - \left(\mu_t^{(1)} |u_j^n|^2\right)\left(\mu_t^{(1)} u_j^n\right)\right| \leq C_5\tau^2, \quad \forall (j,n) \in I. \tag{6.66}$$

From the triangle inequality, there exists a constant $C^* \in \mathbb{R}^+$ which is independent of τ and h, with the property that $|||\mathcal{L}_U - L_U|||_\infty < C^*(\tau^2 + h^2)$. In similar fashion, there exist constants $C_6, C_7, C_8 \in \mathbb{R}^+$ which are independent of both τ and h, for which the inequalities

$$\left|\frac{\partial^2 m(x_j,t_n)}{\partial t^2} - \delta_t^{(2)} m_j^n\right| \leq C_6(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{6.67}$$

$$\left|\frac{\partial^\beta m(x_j,t_n)}{\partial |x|^\beta} - \delta_x^{(\beta)} m_j^n\right| \leq C_7(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{6.68}$$

$$\left|\frac{\partial^\beta \left(|u(x_j,t_n)|^2\right)}{\partial |x|^\beta} - \delta_x^{(\beta)} |u_j^n|^2\right| \leq C_8 h^2, \quad \forall (j,n) \in I, \tag{6.69}$$

are satisfied. Again, we use the triangle inequality to show that there is a constant $C^{**} \in \mathbb{R}^+$ which is independent of τ and h, with the property that $|||\mathcal{L}_M - L_M|||_\infty < C^{**}(\tau^2 + h^2)$. The conclusion is follows letting C as the maximum of C^* and C^{**}. The second inequality of this result can be obtained in similar fashion. □

In view to show the stability and convergence properties of (6.5), assume that (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ are two sets of initial conditions of (6.5). Moreover, suppose that the initial data for (6.30) are provided exactly.

Definition 6.3.2. If $f : \mathbb{F} \to \mathbb{F}$ and $V \in \mathcal{V}_h$ then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$.

Lemma 6.3.3 (Pen-Yu [75])**.** *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative real numbers, assume that $\tau > 0$ and suppose that there exists $C \geq 0$ such that*

$$\omega^k \leq \rho^k + C\tau \sum_{n=0}^{k} \omega^n, \quad \forall k \in \overline{I}_N. \tag{6.70}$$

If τ is sufficiently small then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in \overline{I}_N$.

Theorem 6.3.4 (Stability)**.** *Let $u_0, m_0, \widetilde{u}_0, \widetilde{m}_0 \in H^1(\overline{B})$ and $u_1, m_1, \widetilde{u}_1, \widetilde{m}_1 \in L_2(\overline{B})$. Suppose that (U, M) and $(\widetilde{U}, \widetilde{M})$ are the solutions of (6.30) corresponding to (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$, respectively. Let $\varepsilon^n = \widetilde{U}^n - U^n$, $\zeta^n = \widetilde{M}^n - M^n$ and $\upsilon^n = \widetilde{V}^n - V^n$, for each $n \in \overline{I}_N$, and define*

$$\omega^n = \mu_t \left(\|\varepsilon^n\|_2^2 + \|\zeta^n\|_2^2\right) + \|\delta_x^{(\beta/2)} \upsilon^n\|_2^2, \quad \forall n \in I_{N-1}. \tag{6.71}$$

For τ sufficiently small, there exists $C \in \mathbb{R}^+$ independent of h and τ, such that $\omega^n \leq \omega^0 e^{Cn\tau}$, for each $n \in \overline{I}_{N-1}$.

Proof. Clearly the sequence (ε, ζ) satisfies the system

$$\begin{aligned} i\delta_t^{(1)}\varepsilon_j^n + \delta_x^{(\alpha)}\mu_t^{(1)}\varepsilon_j^n - \mu_t^{(1)}\varepsilon_j^n - \widetilde{\delta}\left[\left(M_j^n + \mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0, \quad \forall (j,n) \in I, \\ \delta_t^{(2)}\zeta_j^n - \delta_x^{(\beta)}\zeta_j^n - \widetilde{\delta}\left(\delta_x^{(\beta)}|U_j^n|^2\right) = 0, \quad \forall (j,n) \in I, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N. \end{aligned} \tag{6.72}$$

Solving the first equation of (6.72) for $i\delta_t^{(1)}\varepsilon_j^n$, computing the inner product on both sides of that identity with $2\mu_t^{(1)}\varepsilon^n$, taking imaginary parts, and using algebraic arguments, there exists $C_1 > 0$ such that, for each $n \in \overline{I}_{N-1}$,

$$\begin{aligned} \mu_t\delta_t\|\varepsilon^{n-1}\|_2^2 &= 2\,\mathrm{Im}\left\langle \widetilde{\delta}\left[\left(M_j^n + \mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right], \mu_t^{(1)}\varepsilon^n \right\rangle \\ &\leq C_1\left(\|\varepsilon^{n-1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\varepsilon^{n+1}\|_2^2 + \|\zeta^{n-1}\|_2^2 + \|\zeta^n\|_2^2 + \|\zeta^{n+1}\|_2^2\right). \end{aligned} \tag{6.73}$$

Now, since $\delta_t\zeta^n = \delta_x^{(\beta)}\upsilon^n$ for each $n \in \overline{I}_{N-1}$, is easy to check that

$$\begin{aligned} 2\left[\langle -\delta_t^{(2)}\zeta^n, \mu_t\upsilon^{n-1}\rangle + \langle \delta_x^{(\beta)}\zeta^n, \mu_t\upsilon^{n-1}\rangle\right] &= \delta_t\left(\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \langle \zeta^{n-1}, \zeta^n\rangle\right) \\ &\geq \frac{1}{2}\delta_t\left(\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \mu_t\|\zeta^{n-1}\|_2^2\right). \end{aligned} \tag{6.74}$$

Take the inner product between the second equation of (6.72) and $\mu_t\upsilon^{n-1}$, use the above inequality and the fact that $\|\delta_t\zeta^n\|_2^2 \leq 2g_0^{(\beta)}h^{1-\beta}\|\delta_x^{(\beta/2)}\upsilon^n\|_2^2$. It is possible to show then that there is a constant $C_2 > 0$ such that

$$\mu_t\delta_t\|\zeta^{n-1}\|_2^2 + \delta_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 \leq C_2\left(\|\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right). \tag{6.75}$$

Adding (6.73) and (6.75), and taking the sum from $n = 1$ to m on both sides of the resulting inequality, we obtain that

$$\begin{aligned} \mu_t\left(\|\varepsilon^m\|_2^2 + \|\zeta^m\|_2^2\right) + \|\delta_x^{(\beta/2)}\upsilon^m\|_2^2 &\leq \mu_t\left(\|\varepsilon^0\|_2^2 + \|\zeta^0\|_2^2\right) + \|\delta_x^{(\beta/2)}\upsilon^0\|_2^2 \\ &\quad + C_2\tau\sum_{n=1}^m\left(\|\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right) \\ &\quad + C_1\tau\sum_{n=1}^m\left(\|\varepsilon^{n-1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\varepsilon^{n+1}\|_2^2\right. \\ &\qquad \left. + \|\zeta^{n-1}\|_2^2 + \|\zeta^n\|_2^2 + \|\zeta^{n+1}\|_2^2\right) \\ &\leq (1 + (6C_1 + 2C_2)\tau)\left[\mu_t\left(\|\varepsilon^0\|_2^2 + \|\zeta^0\|_2^2\right) + \|\delta_x^{(\beta/2)}\upsilon^0\|_2^2\right] \\ &\quad + (6C_1 + 2C_2)\tau\sum_{n=1}^m\left[\mu_t\left(\|\varepsilon^n\|_2^2 + \|\zeta^n\|_2^2\right) + \|\delta_x^{(\beta/2)}\upsilon^n\|_2^2\right]. \end{aligned} \tag{6.76}$$

Let now $C = 6C_1 + 2C_2$ and $\rho = (1 + (6C_1 + 2C_2)\tau)\left[\mu_t\left(\|\varepsilon^0\|_2^2 + \|\zeta^0\|_2^2\right) + \|\delta_x^{(\beta/2)}\upsilon^0\|_2^2\right]$, and apply Lemma 6.3.3 to reach the conclusion of this theorem. □

The following is a straight-forward consequence from the stability property of (6.30).

Corollary 6.3.5 (Uniqueness). *Let (u_0, u_1, m_0, m_1) be a set of initial conditions satisfying $u_1, \tilde{u}_1 \in L_2$ and $u_0, \tilde{u}_0, m_0 \in H^1$. For sufficiently small values of τ, the finite-difference scheme (6.30) is uniquely solvable.* □

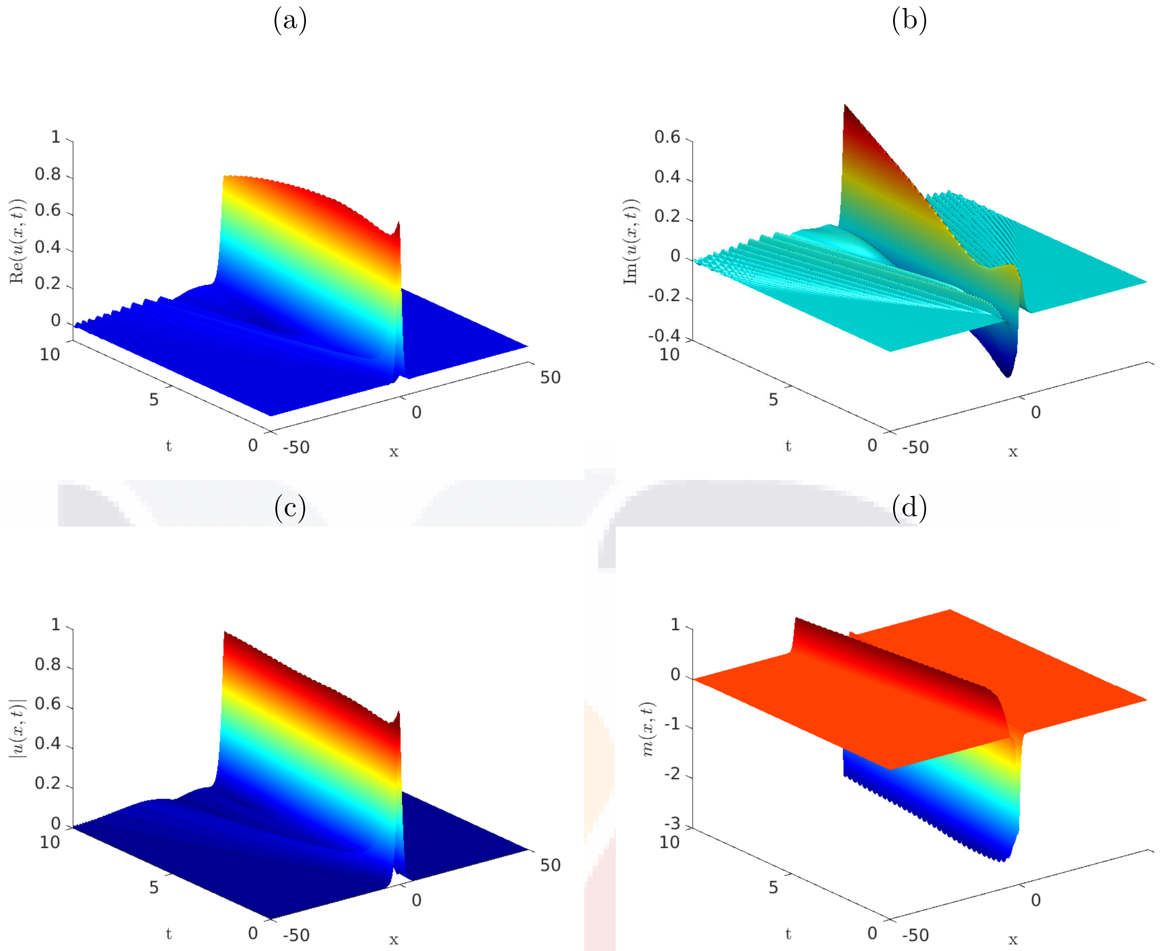

Figure 6.1: Approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. The approximations were obtained using the finite-difference method (6.30) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$ and $\alpha = \beta = 2$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

Definition 6.3.6. If $f : \mathbb{F} \to \mathbb{F}$ and $V \in \mathcal{V}_h$ then we define $\widehat{\delta}(f(v_j)) = f(v_j) - f(V_j)$, for each $j \in I_{J-1}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

We establish next the convergence property of our numerical model.

Theorem 6.3.7 (Convergence)**.** *Suppose that* $u, m \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$. *Then the solution of the problem* (6.30) *converges to that of* (6.5) *with order* $\mathcal{O}(\tau^2 + h^2)$.

Proof. Consider the local truncation errors of the finite-difference system (6.30) at (x_j, t_n), given by

$$\begin{aligned} \rho_j^n &= i\delta_t^{(1)} u_j^n + \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n - \mu_t^{(1)} u_j^n - m_j^n \mu_t^{(1)} u_j^n - \left(\mu_t^{(1)} |u_j^n|^2\right)\left(\mu_t^{(1)} u_j^n\right), \quad \forall (j,n) \in I, \\ \sigma_j^n &= \delta_t^{(2)} m_j^n - \delta_x^{(\beta)} m_j^n - \delta_x^{(\beta)} |u_j^n|^2 = 0, \quad \forall (j,n) \in I. \end{aligned} \tag{6.77}$$

By Theorem 6.3.1, we know that $|\rho_j^n| + |\sigma_j^n| = \mathcal{O}(\tau^2 + h^2)$. Then, let (u, m) be a solution of 6.5 and (U, M) a solution of 6.30, and define $\epsilon_j^n = u_j^n - U_j^n$, $\eta_j^n = m_j^n - M_j^n$ and $\theta_j^n = v_j^n - V_j^n$,

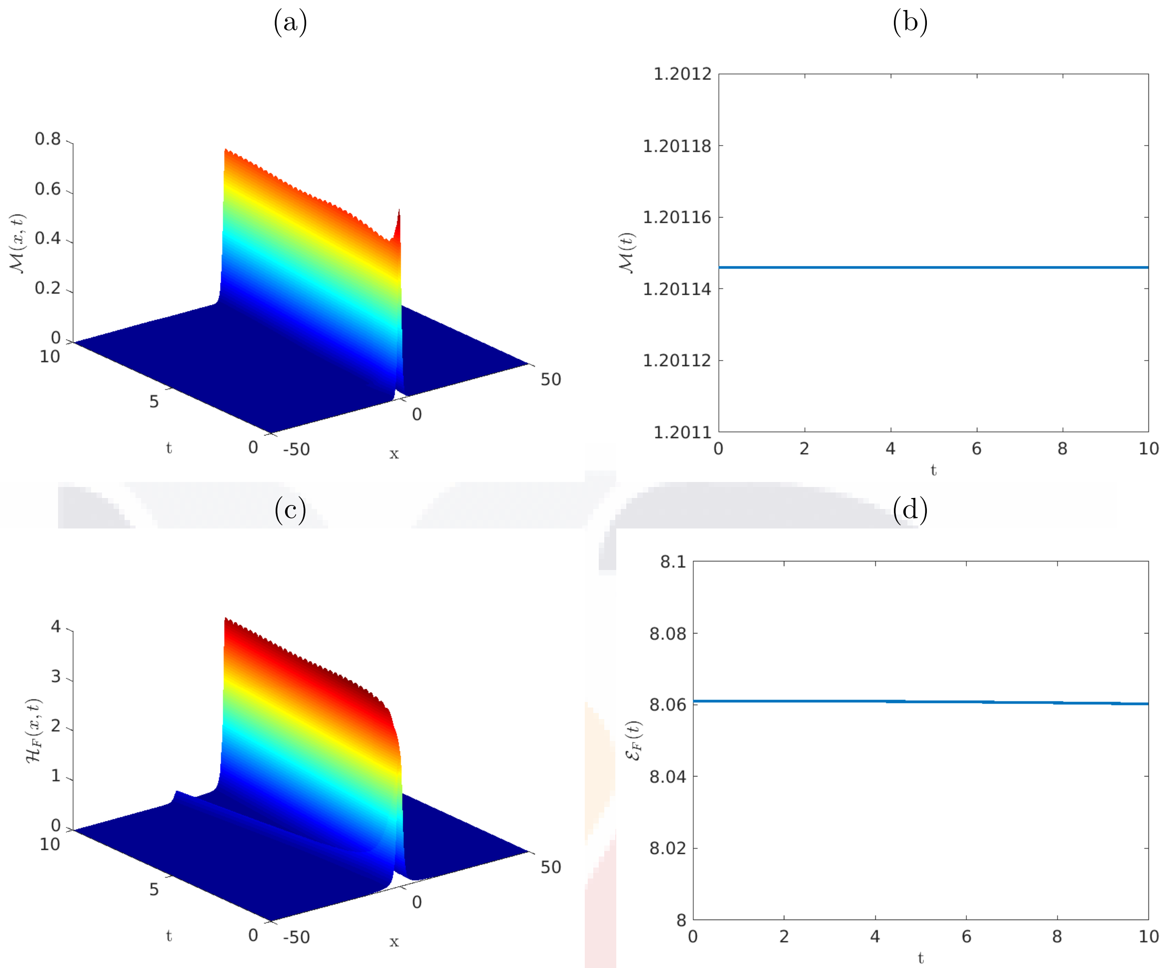

Figure 6.2: Left column: approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t. Right column: approximate solutions for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$. The approximations were obtained using the finite-difference method (6.30) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$ and $\alpha = \beta = 2$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

$\forall (j,n) \in I$. Notice that, $\delta_x^{(\beta)} \theta_j^n = \delta_t \eta_j^n$, $\forall (j,n) \in I$. Moreover, the pair (ϵ, η) satisfies the system

$$\begin{aligned} i\delta_t^{(1)} \epsilon_j^n + \delta_x^{(\alpha)} \mu_t^{(1)} \epsilon_j^n - \mu_t^{(1)} \epsilon_j^n - \widehat{\delta} \left[\left(m_j^n + \mu_t^{(1)} |u_j^n|^2 \right) \left(\mu_t^{(1)} u_j^n \right) \right] &= \rho_j^n, \quad \forall (j,n) \in I, \\ \delta_t^{(2)} \eta_j^n - \delta_x^{(\beta)} \eta_j^n - \widehat{\delta} \left(\delta_x^{(\beta)} |u_j^n|^2 \right) &= \sigma_j^n, \quad \forall (j,n) \in I, \\ \text{subject to } \epsilon_0^n = \epsilon_J^n = 0 \text{ and } \eta_0^n = \eta_J^n = 0, \quad \forall n \in \overline{I}_N. & \end{aligned} \tag{6.78}$$

Proceeding as in Theorem 6.3.4, we can check that there exist constants C_3 and C_4 such that

$$\mu_t \delta_t \|\epsilon^{n-1}\|_2^2 \leq C_3 \left(\|\rho^n\|_2^2 + \|\epsilon^{n-1}\|_2^2 + \|\epsilon^n\|_2^2 + \|\epsilon^{n+1}\|_2^2 + \|\eta^{n-1}\|_2^2 + \|\eta^n\|_2^2 + \|\eta^{n+1}\|_2^2 \right), \tag{6.79}$$

and

$$\mu_t \delta_t \|\eta^{n-1}\|_2^2 + \delta_t \|\delta_x^{(\beta/2)} \theta^{n-1}\|_2^2 \leq C_4 \left(\|\sigma^n\|_2^2 + \|\epsilon^n\|_2^2 + \mu_t \|\delta_x^{(\beta/2)} \theta^{n-1}\|_2^2 \right). \tag{6.80}$$

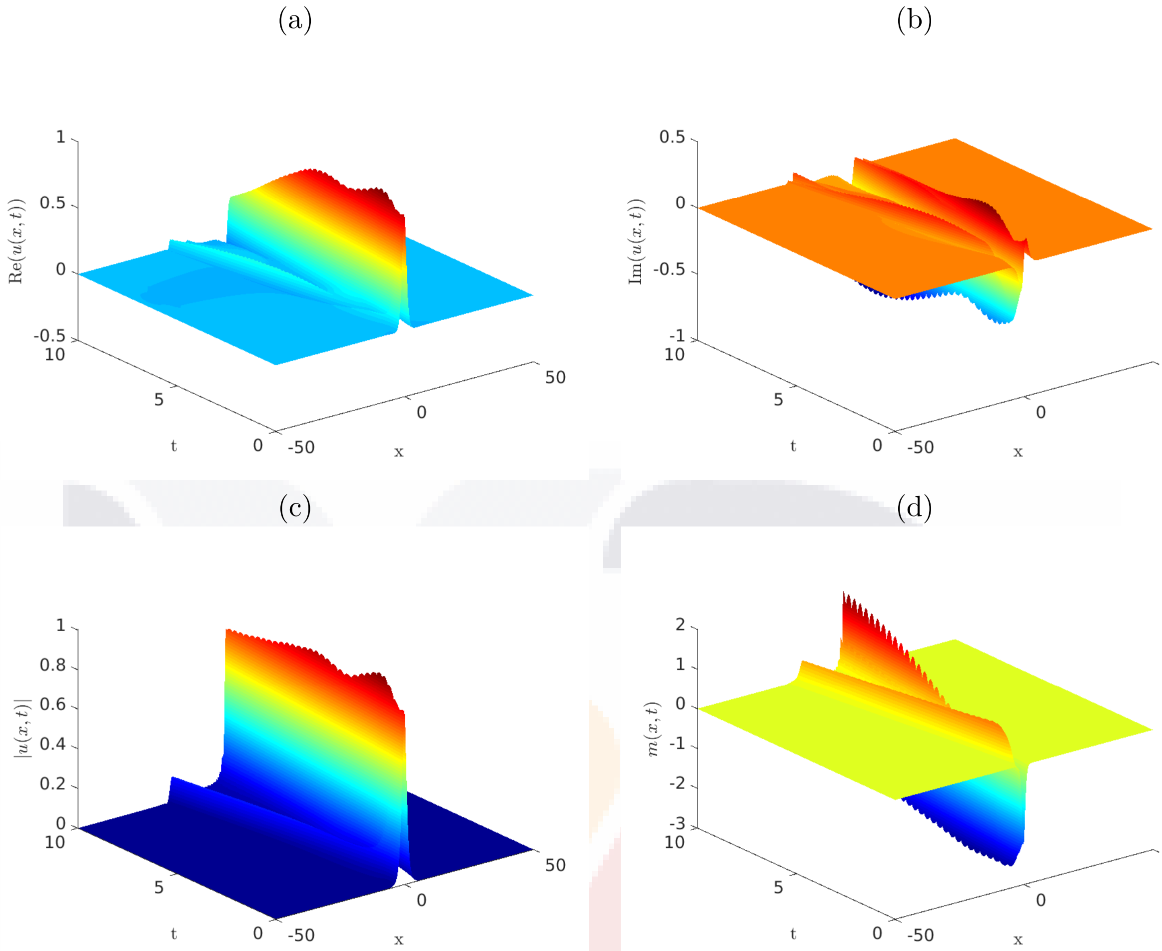

Figure 6.3: Approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. The approximations were obtained using the finite-difference method (6.30) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$, $\alpha = 1.2$ and $\beta = 1.8$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

For each $k \in I_{N-1}$, take $\omega^k = \mu_t \left(\|\epsilon^k\|_2^2 + \|\eta^k\|_2^2 \right) + \|\delta_x^{(\beta/2)} \theta^k\|_2^2$ and

$$\rho^k = (1 + (6C_3 + 2C_4)\tau) \left[\mu_t \left(\|\epsilon^0\|_2^2 + \|\eta^0\|_2^2 \right) + \|\delta_x^{(\beta/2)} \theta^0\|_2^2 \right] + (C_3 + C_4)\tau \sum_{n=0}^{m} \left(\|\rho^n\|_2^2 + \|\sigma^n\|_2^2 \right). \tag{6.81}$$

It follows that there exists a constant $C \geq 0$, with the property that $\omega^k \leq C\rho^k$, for each $k \in I_{N-1}$. As a consequence, $\|\epsilon^n\|_2, \|\eta^n\|_2 \leq \sqrt{C}(\tau^2 + h^2)$, which implies that the solutions of (6.30) converge quadratically to those of (6.5). $\square$

6.4 Computer simulations

The purpose of this section is to provide computer simulations using a Matlab implementation of the numerical model (6.30) to solve the Zakharov system (6.5). The computer code was employed a fixed-point approach to approximate the solution of the first discrete equation of (6.30) at each iteration. Meanwhile, the second equation of our numerical model was solved explicitly and

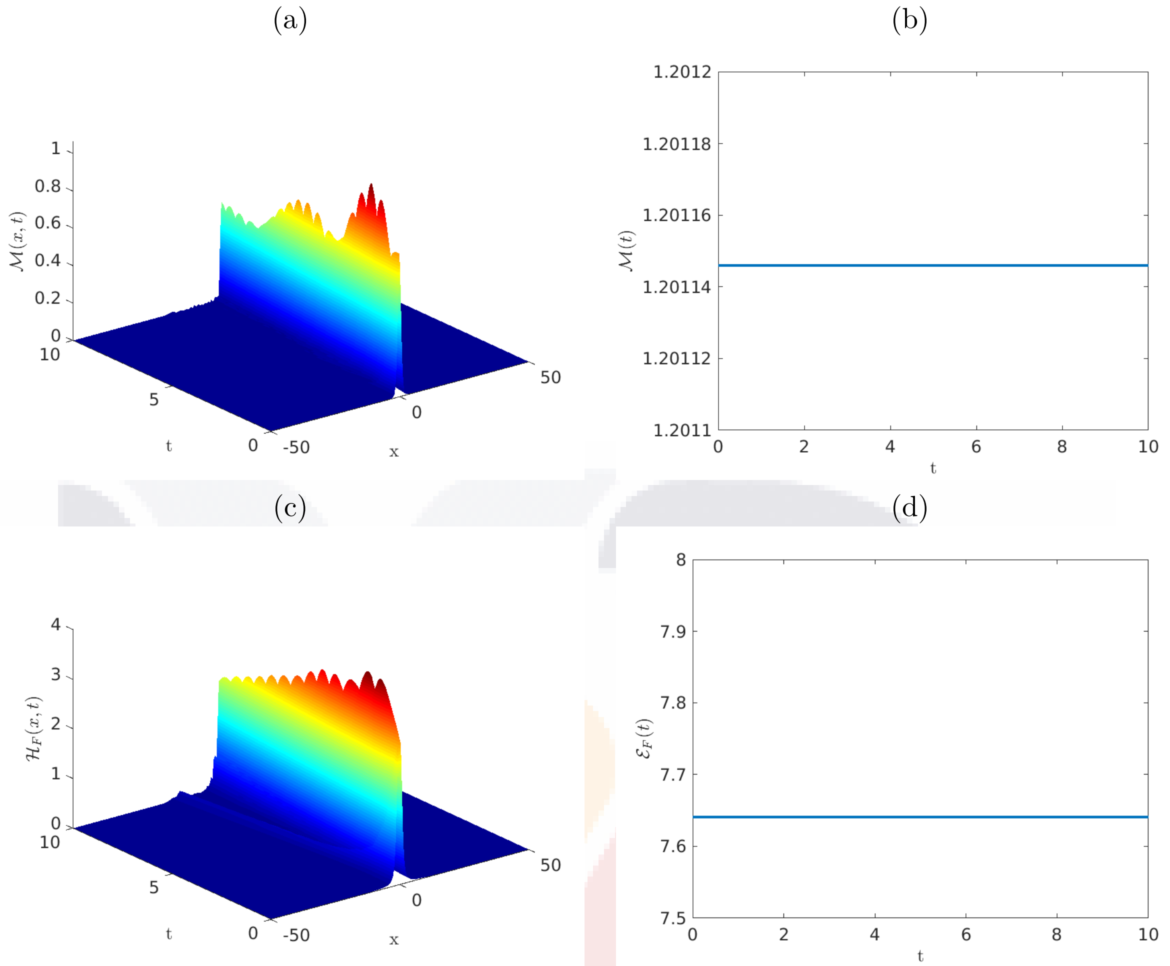

Figure 6.4: Left column: approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t. Right column: approximate solutions for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$. The approximations were obtained using the finite-difference method (6.30) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$, $\alpha = 1.2$ and $\beta = 1.8$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

exactly.

To produce our simulations, we will impose homogeneous Neumann conditions on the boundary of B, along with the following set of initial conditions:

$$u_0(x) = \frac{\sqrt{10}-\sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \tag{6.82}$$

$$m_0(x) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \tag{6.83}$$

$$m_1(x) = -4\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right). \tag{6.84}$$

As a matter of fact, it is worth pointing out that these functions are initial conditions for an exact solution of the well-known Klein–Gordon–Zakharov equations which describe the propagation of Langmuir waves in plasma physics. That exact solution is actually provided by the set of functions (see [45, 44])

$$u(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2}\operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right)\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \quad \forall (x,t) \in \mathbb{R}\times\overline{\mathbb{R}^+}, \tag{6.85}$$

$$m(x,t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \quad \forall (x,t) \in \mathbb{R}\times\overline{\mathbb{R}^+}. \tag{6.86}$$

In a first approach, we consider the system (6.5) with $\alpha = \beta = 2$, and defined over the space-time domain $\Omega = (-50, 50) \times (0, 10)$. Computationally, we let $h = 0.5$ and $\tau = 0.01$. As we mentioned previously, the mathematical model will be solved using the finite-difference scheme (6.30), which will require a computational implementation of a fixed-point method to solve the first difference equation at each iteration. To that end, we will set a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30. In the absence of a known exact solution for the Zakharov system, we will obtain the first approximations of our methodology using the exact solutions (6.85)–(6.86). Under these circumstances, Figure 6.1 provides the approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. In turn, Figure 6.2 shows graphs of the approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t, and for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$ versus t. From these results, we can readily observe that the total mass and the Higgs' free energy are approximately conserved in the discrete domain, in agreement with the theoretical results presented in this work.

Before closing this section, we will provide a new set of simulations using now $\alpha = 1.2$ and $\beta = 1.8$. All the initial and boundary conditions along with the model and computational parameters are as before. With these conventions, Figure 6.3 shows the approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. On the other hand, Figure 6.4 shows graphs of the approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t, and for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$ versus t. The results show again the capability of the finite-difference scheme to preserve the total mass of the system and the Higgs' free energy in the discrete scenario. Again, this is in agreement with the theoretical results provided in this work.

Conclusions

Chapter 1 In this work, we investigated the numerical solution of a fractional extension of the Klein–Gordon–Zakharov equations from plasma physics. The model considers the presence of space-fractional derivatives of the Riesz type, together with homogeneous Dirichlet data at the boundary and initial conditions. The fractional model has an invariant energy functional, and we propose an explicit numerical model to approximate the solutions using fractional-order centered differences. A discrete energy functional is also proposed in this work and we prove rigorously that, as its continuous counterpart, it is preserved at each iteration and, in that sense, the present work reports on a conservative finite-difference scheme to approximate the solutions of hyperbolic systems [36, 42, 38]. Among the most important numerical properties established in this work, we show that the model is a consistent, stable and convergent technique. Additionally, we propose some bounds for the numerical solutions, and provide some computer simulations which illustrate the fact that the numerical model is quadratically convergent.

Chapter 2 In this note, we have provided an accurate statement of an existence of solutions of the implicit finite-difference model (2.2), along with its proof. This result corrects the statement and the proof of [35, Theorem 5.3] of our published manuscript. It is worth mentioning that the actual statement to guarantee the existence of solutions requires additional hypotheses, and that the proof is more complicated that the wrong demonstration given in our former article. We took this opportunity to introduce an explicit variation of the finite-difference model investigated in [35]. In accordance with the aims of this letter, we establish the unconditional existence of solution for the new scheme. The argument of the proof is similar to that of the implicit scheme, that is why we only provide a shortened demonstration in this manuscript. The authors would like to apologize for any convenience caused by the wrong proof in our previous paper. Finally, we would like to point out that the first author was who first pointed out the error in the proof.

Chapter 3 In this work, we investigated the numerical solution of a fractional extension of the Klein–Gordon–Zakharov equations from plasma physics. The model considers the presence of space-fractional derivatives of the Riesz type, together with homogeneous Dirichlet data at the boundary and initial conditions. We proved in this work that the fractional model has a positive invariant quantity, which we identify as the energy of the system. Motivated by this fact, we propose a numerical model to approximate the solutions of the Klein–Gordon–Zakharov equations, which is based on the use of fractional-order centered differences. A discrete energy functional is also proposed in this work and we prove rigorously that, as its continuous counterpart, is preserved at each iteration. In that sense, the present approach reports on a structure-preserving technique for a complex system [55, 24]. Among the most important numerical properties established in this work, we show that the model is a consistent, stable and convergent technique. We establish also that the numerical model has solutions for any set of initial conditions, and that they are unique for sufficiently small values of the temporal step-size. The existence is a consequence of the Leray–Schauder fixed-point theorem, while the uniqueness follows from the stability of the method. Additionally, we propose some bounds for the numerical solutions, and provide some computer simulations which illustrate the fact that the numerical model is quadratically convergent. It is worth pointing out that the advantage of the discretization proposed in this work lies in that the difference equations to solve the component equations are decoupled. This implies that the numerical schemes can be solved separately at each temporal step. Obviously, pertinent physical applications are expected as a consequence of the completion of this work [53].

Chapter 4 In this work, we investigated numerically a generalization of the well-known Klein–Gordon–Zakharov system which considers two fractional derivatives of the Riesz type. The fractional derivatives are not required to be of the same order, but they belong to the interval $(1, 2]$. It is known that the system considered in this work has a conserved energy-like quantity, and that the solutions are uniformly bounded. Motivated by these facts, we proposed a finite-difference discretization of the the system under investigation which has associated conserved energy-like quantities. In that sense, the present work proposes a structure-preserving scheme to solve a system of partial differential equations [54]. The discretization is based on the use of fractional-order centered differences to approximate the Riesz-type fractional derivatives. In turn, we prove thoroughly that the energy-like quantities are conserved throughout time, and that they are non-negative as their continuous counterparts. As a corollary, we establish the uniform boundedness of the numerical approximations. The properties of consistency, stability and convergence were theoretically established, also. In particular, we show that our approach yields second-order consistent approximations to the exact solutions of the continuous model. Using a discrete form of Gronwall's inequality, we prove that the finite-difference model is stable and quadratically convergent. As a corollary of stability, we proved that the numerical model is uniquely solvable. Some numerical simulations were presented using a computer implementation of our numerical methodology.

Chapter 5 In this work, we investigated the numerical solution of a multi-fractional extension of the Klein–Gordon–Zakharov equations from plasma physics. The continuous model considers the presence of space-fractional derivatives of the Riesz type, together with homogeneous Dirichlet data at the boundary and initial conditions. It is well known that the system investigated in this work has an energy functional which conserved through time. Two nonlinear finite-difference schemes were proposed in this work to solve the mathematical model, one is implicit and the other is explicit. Our discretizations are both based on the use of fractional-order centered differences. Discrete energy functionals are proposed for both models, and we prove prove rigorously that, as their continuous counterpart, they are preserved throughout the discrete time. Among the most important numerical properties established in this work, we show that both models are consistent, stable and convergent techniques. We establish also that the numerical model has solutions for any set of initial conditions under suitable circumstances, and that they are unique for sufficiently small values of the temporal step-size. The existence properties are consequences of the Leray–Schauder fixed-point theorem, while the uniqueness follows from the stability of the methods. Additionally, we propose some bounds for the numerical solutions, and provide some computer simulations which illustrate the fact that the numerical models are quadratically convergent, and are capable of preserving their energy functionals.

Chapter 6 A space-fractional extension of the Zakharov system was introduced and investigated in this study from analytical and numerical points of views. The system consists of two partial differential equations with nonlinear coupling, and initial and boundary conditions are imposed on a bounded interval of real numbers. It was proven that the fractional system is capable of preserving the mass and Higgs' free energy throughout time, and that the total energy is dissipated. Moreover, the total mass, the total free energy and the total energy are non-negative functions of time. Consequently, the boundedness of the solutions of the system we established. Motivated by these results, we proposed a finite-difference scheme to solve this system via fractional-order central difference approximations. The discrete model proposed is a three-level scheme whose implementation was implemented by using both vector equations and fixed-point techniques. The existence of solutions was proven rigorously through Browder's fixed-point theorem, and proposed discrete expressions for the total mass, Higgs' free energy and the total energy. It was shown theoretically that the numerical model is capable of preserving the discrete mass and the discrete Higgs' free energy. Moreover, the positivity of the mass, the free energy and the total energy was also verified. From the numerical analysis point of view, we proved systematically properties of consistency, stability and convergence of the algorithm. As a consequence of these investigations, the uniqueness of the numerical solutions was also validated. Computer simulations based on the discrete model were presented. The computational experiments illustrate important properties of our numerical solution, including its capability to preserve the mass and Higgs' free energy.

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