UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

ON THE ASYMPTOTIC BEHAVIOR OF A SEMILINEAR NON-HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

PRESENTA

Jorge Sigfrido Macías Medina

PARA OBTENER EL GRADO DE MAESTRO EN CIENCIAS EN MATEMÁTICAS APLICADAS

TUTOR

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COMITÉ TUTORAL

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Aguascalientes, Ags., 24 de Marzo de 2018

UNIVERSIDAD AUTONOMA
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FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
DECANO DEL CENTRO DE CIENCIAS BÁSICAS
PRESENTE

Por medio de la presente, en mi calidad de tutor designado del estudiante **JORGE SIGFRIDO MACÍAS MEDINA** con ID 115773 quien realizó la tesis titulada: **ON THE ASYMPTOTIC BEHAVIOR OF A SEMILINEAR NON-HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

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"Se Lumen Proferre"
Aguascalientes, Ags., a 20 de Mayo de 2019

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JORGE SIGFRIDO MACIAS MEDINA
MAESTRIA EN CIENCIAS CON OPCION A LA
COMPUTACION, MATEMATICAS APLICADAS

Estimado alumno:

Por medio de este conducto me permito comunicar a Usted que habiendo recibido los votos aprobatorios de los revisores de su trabajo de tesis y/o caso práctico titulado: **"ON THE ASYMPTOTIC BEHAVIOR OF A SEMILINEAR NON-HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION"** hago de su conocimiento que puede imprimir dicho documento y continuar con los trámites para la presentación de su examen de grado.

Sin otro particular me permito saludarle muy afectuosamente.

A T E N T A M E N T E
Aguascalientes, Ags., a 27 de Mayo de 2019
"Se lumen proferre"
EL DECANO

M. en C. JOSÉ DE JESÚS RUÍZ GALLEGOS

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A todos, muchas gracias!

Jorge Sigfrido Macías Medina

Contents

Resumen

En este trabajo estudiamos el comportamiento asintótico de las soluciones de una ecuación diferencial parcial no homogénea semilineal. Comenzamos probando la existencia de una solución suave para nuestra ecuación de interés haciendo uso del teorema del punto fijo de Banach. También demostramos que nuestra solución es acotada. A continuación, establecemos la regularidad temporal de nuestra solución utilizando el teorema del valor medio, la desigualdad de Gronwall y dando explícitamente la primera derivada. Para finalizar el capítulo 2, demostramos que nuestra solución es, de hecho, una solución clásica, probamos que nuestra solución es positiva y, además, demostramos que nuestra solución local es una solución global. En el capítulo 3 determinamos las condiciones bajo las cuales nuestra solución es integrable. Finalmente, estudiamos el comportamiento asintótico de nuestra solución y discutimos brevemente las consecuencias y la importancia de nuestros resultados finales.

Abstract

In this work we study the asymptotic behavior of the solutions of a semilinear non-homogeneous partial differential equation. We begin by proving the existence of a mild solution for our equation of interest making use of Banach's fixed point theorem. We also prove our solution is bounded. Next, we establish the temporal regularity of our solution using the mean value theorem, Gronwall's inequality and giving explicitly the first derivative. To finish chapter 2, we prove that our mild solution is in fact a classical solution, we prove that our solution is positive and moreover, we prove that our local solution is a global solution. In chapter 3 we determine conditions under which our solution is integrable. Finally, we study the asymptotic behavior of our solution and we briefly discuss the consequences and importance of our final results.

Introduction

Background

In recent years there has been an increase in interest in the study of partial differential equations that involve the fractional Laplacian. One reason for this is the fact that unlike the classical operators, the fractional Laplacian is a non-local operator. As a result, new analytical techniques are developed to be able to study fractional partial differential equations. Moreover, the fractional Laplacian models a great number of phenomena in molecular biology, mathematical finance, statistical physics and hydrodynamics, for example.

The existence of global solutions of (1) has been proved in some particular cases. For example, in the classical scenario ($\alpha = 2$), a stochastic method was used in [10] to estimate the solutions of (1) when $g = \varphi = 0$. Also, the existence of a unique bounded solution of an elliptic equation similar to (2) was proved when $g = 0$. In that case, the author uses a Dirichlet problem to establish the existence. Similar results were obtained in [12, 19] for a more general source term f. It is worth mentioning that the number of reports for the classical case is very large. On the other hand, the number of works dealing with positive solutions of (1) when $\alpha \in (0, 2)$ has increased in recent years, though most of them study the existence of positive radial solutions (see [7, 16] as examples).

Other existing works on parabolic semilinear fractional partial differential equations are scarce. For example, in [13] the authors study a model similar to (1) for a fractional Laplacian with $\alpha \in (0, 2]$. The existence of solutions is established but no asymptotic behavior is studied. Some finite-element methods were introduced in [15] to approximate the solutions of an extended form of a model similar ours. Meanwhile, the convergence of solutions of a fractional heat equation is investigated in [21] considering $\alpha \in (0, 1)$, and a more general fractional heat equation is studied analytically in [11]. However, from our point of view, the study reported in this manuscript has not been performed previously in the literature.

The purpose of this thesis is to study the temporal monotonicity of the positive solutions of the parabolic semilinear partial differential equation of interest. We additionally use our results to establish the existence of solutions for the elliptic equation. The proofs regarding the existence of solutions for our model of interest and regarding the temporal regularity of our solutions are fairly standard in literature. The novelty of this work commences in the proof of the positivity of our solutions. When we try to apply the technical approach to our particular model we are faced with various technical difficulties. In this case the novelty lies in the use of Gronwall's inequality to prove the uniform integrability of the solution. This result was then employed to prove that the local minimum is global. The rest of the proof follows as in the classic case. In the proof that the elliptic equation has a solution for the case $\alpha = 2$, $d \in \{1, 2\}$, to the best of our knowledge, our approach is entirely new. The idea consists of using the asymptotic properties of the solutions of an initial value problem which we will mention in following chapters. This leads to proving that the solution of our elliptic equation is in fact in the domain of the fractional Laplacian. Finally, we show that integrability depends entirely on our source term f.

The results studied in this work can be used in several branches. One example can be found in probability theory. Firstly, it is worth recalling that there are some Markov process in probability theory which are characterized through their Laplace functionals [5, 17]. Such processes are measure-valued and, intuitively, they represent the continuous-state of clouds of certain branching phenomenon. The study of their path properties is based mainly on the Laplace functionals and, consequently, on the properties of the solutions of equations like that studied in the present paper. For example, when the diffusion is a stable process and the branching mechanism f is $x^{1+\beta}$, then we obtain the (α, d, β)-superprocess.

There is an extensive literature on path behavior properties of superprocesses. Using mild solutions of the equation $\partial u(t,x)/\partial t = \Delta_\alpha u(t,x) - (u(t,x))^{1+\beta}$, one of the authors studied the self-intersection local time of (α, d, β)-superprocess [17].

Aims and scope

This thesis is sectioned as follows:

- Chapter 1 begins by providing various important inequalities which will be used throughout this work, such as Young's inequality for convolutions and Gronwall's inequality. Furthermore, we list important results and theorems which are of great importance throughout this work. Next, we proceed to provide a definition for semigroups and we state the definition of its infinitesimal generator. Additionally, we list some properties of semigroups and generators, and we finish the section stating important results on semigroups. In the next section we introduce the concept of α-stable densities and list a few results and properties of these densities. We close this chapter by discussing the definition of the fractional Laplacian, a key concept in this work.

- Chapter 2 starts proving the existence of a mild solution for our model of interest. To achieve this goal we prove the temporal continuity of an auxiliary function, we prove this function is a contraction and finally we apply Banach's contraction principle to obtain a local mild solution for our model of interest. In the following section we seek to prove the temporal regularity of our solution. To prove this we give explicitly the first derivative of our solution and prove that it is in fact the function we seek employing diverse inequalities, Gronwall's inequality and the mean value theorem. Afterwards, we give a brief demonstration proving that the mild solution we found in the previous section is in fact a classical solution. In the third section of this chapter we start by briefly discussing the uniform continuity of our solution. We then proceed to prove the positivity of our solution by assuming our solution is not positive and reaching a contradiction, thus our solution is non-negative. To end this chapter we use the positivity of our solution along with a few other results to prove that the maximum time of existence of our solution is not finite.

- In Chapter 3, we commence by using the time-monotonicity of our solution to obtain a solution for our semilinear elliptic equation. We consider the case in which $d > \alpha$ and the case $\alpha = 2$, $d \in \{1, 2\}$. After proving this result we define a new function and we use this function to determine conditions that guarantee the integrability of our solutions.

- This thesis closes with a section of conclusions for each chapter and a list of relevant references.

The Problem

In this work, we study the temporal monotonicity of the positive solutions of the parabolic semilinear partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \Delta_\alpha u(t,x) - g(x)f(u(t,x)) + \varphi(x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= \psi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where Δ_α, is the fractional Laplacian with $\alpha \in (0, 2]$, the initial datum ψ is a nonnegative function in the domain of Δ_α, the source term f is a convex function with $f(0) = 0$, and both g and φ are nonnegative continuous functions. The purpose of the present work is to prove that (1) has a unique positive and bounded global solution $u(t, x)$. Moreover, we will see that u is monotone in time, in which case it makes sense to define $u_\infty(x) = \lim_{t\to\infty} u(t, x)$. As a result, we obtain that u_∞ is a solution of the semilinear elliptic equation

$$\Delta_\alpha v(x) = g(x)f(v(x)) - \varphi(x), \quad x \in \mathbb{R}^d, \tag{2}$$

with boundary condition $\lim_{||x||\to\infty} v(x) = 0$.

1. Preliminaries

In this chapter, we will state some important results that will be used throughout this work. We will also discuss basic properties of semigroups and the α stable density.

1.1 Elementary results

Theorem 1.1 (Leibniz rule)**.** *Let $f(t,x), a(t), b(t)$ be differentiable functions in $t_0 \leq t \leq t_1$ with $a, b \in x_0 \leq x \leq x_1$. Thus*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \int_{a(t)}^{b(t)} \frac{\partial f(t,x)}{\partial t}dx + f(t,b)\frac{db}{dt}(t) - f(t,a)\frac{da}{dt}(t).$$

The formula has the following interpretation: the first term on the right yields the change in the integral, due to the fact that the function is changing with respect to time. The second term takes into consideration the area gained by moving the upper limit of integration in the direction of the positive axis, and the third term takes into consideration the area lost from moving the lower limit of integration. The proof can be found in [8].

We will list now various inequalities and results which will be used throughout this work. Most of these results along with their respective proofs can be found in [1].

Theorem 1.2 (Bellman-Gronwall inequality)**.** *Suppose $\varphi \in L^1[a,b]$ and that φ satisfies*

$$\varphi(t) \leq f(t) + \beta \int_a^t \varphi(s)ds, \tag{1.1}$$

$\forall t \in [a,b]$, and β being a positive constant. Thus

$$\varphi(t) \leq f(t) + \beta \int_a^t f(s)\exp(\beta(t-s))ds. \tag{1.2}$$

Moreover, when $f(t)$ is a constant α, we have

$$\varphi(t) \leq \alpha \exp(\beta(t-s)).$$

Theorem 1.3 (Minkowski's inequality)**.** *Let $f \in L^p(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$, for $1 \leq p \leq \infty$. Thus*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p . \tag{1.3}$$

Theorem 1.4 (Young's convolution inequality)**.** *Assume that $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ and suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, for $1 \leq p, q, r \leq \infty$. Thus*

$$\|f * g\|_p \leq \|f\|_p \|g\|_p , \tag{1.4}$$

with the convolution of two functions defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy. \tag{1.5}$$

Theorem 1.5 (Dominated convergence theorem)**.** *Let f_n be a sequence of measurable functions in the measure space (X, Σ, μ) such that $f_n \to f$ pointwise. Suppose that there exists g integrable such that $|f_n| \leq g$ for all n. Then f is integrable and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu. \tag{1.6}$$

Theorem 1.6 (Mean value theorem)**.** *Suppose that the function f is continuous on a closed interval $I = [a, b]$ and that the derivative f' exists at every point of (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b-a). \tag{1.7}$$

Theorem 1.7 (Maximum principal)**.** *Let Ω be a bounded domain. Suppose $u \in C_1^2(\Omega) \cap C(\Omega)$, that L is strictly elliptic, $c \leq 0$ and $Lu \geq 0$ in Ω. Then either $u \equiv \sup_\Omega u$ or u does not attain a nonnegative maximum in Ω.*

Lemma 1.8 (Fatou's lemma)**.** *Let (X, Σ, μ) be a measure space and consider $f_n : X \to [0, \infty]$ a sequence of nonnegative measurable functions. Then $\liminf_{n\to\infty} f_n$ is measurable and*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu. \tag{1.8}$$

Theorem 1.9 (L'Hopital's rule)**.** *Let f, g be differentiable functions on an open interval I except in a point c. Suppose that the following conditions are satisfied:*

(a) $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ *or* $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$,

(b) $g'(x) \neq 0$ *for all* $x \in I \setminus \{c\}$, *and*

(c) $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ *exists.*

Then we have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}. \tag{1.9}$$

1.2 Subfunctions and superfunctions

Theorem 1.10. *Consider the boundary value problem* $y'' = f(x,y)$, $y(x_1) = y_1$, $y(x_2) = y_2$, *and let* $I_1 = \mathbb{R}^+$ *and* $I_2 = \{y : |y| < \infty\}$. *If* $f(x,y)$ *satisfies*

(a) $f(x,y)$ *is continuous on* $S = \{(x,y) : x \in I_1, y \in I_2\}$ *and*

(b) $f(x,y)$ *is nondecreasing in* y *for fixed* $x \in I_1$

on S, *then for any* x_1, $x_2 \in I$, $x_1 \neq x_2$, *and any* y_1 *and* y_2, *the boundary value problem* $y'' = f(x.y)$, $y(x_1) = y_1$, $y(x_2) = y_2$, *has a unique solution of class* $C^{(2)}$ *on* $[x_1, x_2]$.

The proof of this theorem is standard and motivates the following:

Definition 1.11. A real valued function S defined on an interval $I_0 \subset I$ is a superfunction with respect to the solutions of (1.10) on I_0 if $y(x) \leq S(x)$ on any subinterval $[x_1, x_2] \subset I$ and any solution $y(x)$ of (1.10) on $[x_1, x_2]$ with $y(x_1) \leq S(x_1)$ and $y(x_2) \leq S(x_2)$. The definition of subfunctions is given similarly in terms of inequalities in the opposite direction [2].

The following are some properties regarding subfunctions and superfunctions.

(*i*) If $s(x)$ is a subfunction on $I_0 \subset I$, then $s(x)$ is continuous on the interior of I_0.

(*ii*) If $\{s_\alpha : \alpha \in A\}$ is any collection of subfunctions on $I_0 \subset I$ which is bounded above at each point of I_0, then s defined by

$$s(x) = \sup\{s_\alpha(x) : \alpha \in A\} \tag{1.10}$$

is a subfunction on I_0.

(*iii*) If s is continuous on $I_0 \subset I$ and of class $C^{(2)}$ on the interior of I_0, then s is a subfunction on I_0 if and only if $f(x,s) \leq s''$ on the interior of I_0. If $s'' > f(x,s)$ on $[x_1, x_2]$ and $y(x)$ is a solution of (1.10) on $[x_1, x_2]$ with $s(x_1) = y(x_1)$, $s(x_2) = y(x_2)$, then $s(x) < y(x)$ for $x \in (x_1, x_2)$.

(*iv*) Let s_1 be a subfunction on $I_0 \subset I$ and s_2 a subfunction on $[x_1, x_2] \subset \bar{I_0}$. Assume also that $s_2(x_i) \leq s_1(x_i)$ at points x_i, $i = 1, 2$, contained in the interior of I_0. Then s defined on I_0 by $s(x) = s_1(x)$ for $x \notin [x_1, x_2]$ and $s(x) = \max[s_1(x), s_2(x)]$ for $x \in [x_1, x_2]$ is a subfunction on I_0.

1.3 Existence

Lemma 1.12. *Let* $y_0(x)$ *be the solution of the boundary value problem* $y'' = f(x,y)$, $y(0) = -\alpha$, $y(1) = 0$. *Then the function* $\Psi_0(x)$ *defined by*

$$\begin{cases} \Psi_0(x) = y_0(x), 0 \leq x \leq 1, \\ \Psi_0(x) = 0, x \geq 1, \end{cases} \tag{1.11}$$

is a continuous superfunction on $[0, +\infty)$.

The proof of this lemma follows from properties of superfunctions given in [2]. We will also make use of the following definition.

Definition 1.13. Let $\{\varphi\}$ denote the collection of every continuous subfunction on $[0,+\infty)$ such that $\varphi(x) \leq \Psi_0(x)$ on $[0,+\infty)$. We define the function ω as $\omega(x) = \sup\{\varphi(x) : \varphi \in \{\varphi\}\}$.

We note that our function ω is bounded and continuous on $[0,+\infty)$ and, moreover, is both a subfunction and a superfunction on $[0,+\infty)$ (see [2]). These results will be used in the proof of the following result

Theorem 1.14. *Consider the following boundary problem:*

$$\begin{cases} y'' = f(x,y), \\ y(0) = -\alpha, \alpha > 0, \\ y'(x) \geq 0, y(x) \leq 0, \end{cases} \tag{1.12}$$

If $f(x,y)$ satisfies

(a) *$f(x,y)$ is continuous on $S = \{(x,y) : x \in I_1, y \in I_2\}$ where I_1 and I_2 are the intervals associated with the boundary value problem,*

(b) *$f(x,y)$ is nondecreasing in y for fixed $x \in I_1$, and*

(c) *$f(x,0) \equiv 0$ in I_1.*

on $S_1 = \{(x,y) : x \geq 0, y \leq 0\}$ then our boundary problem has a unique solution.

Proof. We shall now show that the function $\omega(x)$ is the unique solution of (1.12). Using Theorem 2.1 from [2], it follows that for any $b > 0$ the boundary value problem $y'' = f(x,y)$, $y(0) = -\alpha$, $y(b) = \omega(b)$ has a $C^{(2)}$ solution y_b on $[0,b]$. From a previous lemma, we know that ω is both a subfunction and a superfunction on $[0,b]$, this implies that $y_b(x) = \omega(x)$ on $[0,b]$. Hence, ω is a solution of $y'' = f(x,y)$ on $[0.+\infty)$ with $\omega(0) = -\alpha$, and $\omega(x) \leq \Psi_0(x) \leq 0$ on $[0,+\infty)$. It follows that if $\omega'(x_0) < 0$ for some $x_0 > 0$, then $\lim_{x\to+\infty} \omega(x) = -\infty$, which is a contradiction due to the fact that ω is bounded. Thus we conclude $\omega'(x) \geq 0$ on $[0.+\infty)$ and additionally, ω is a solution of (1.12). To prove the uniqueness of our solution, we will begin by assuming there exists another distinct solution $u(x)$ of the equation (1.12). Due to the uniqueness of the solution for the the boundary problem $y'' = f(x,y)$ on finite intervals, there exists $x_0 \geq 0$ such that $u(x) = \omega(x)$ on $[0,x_0]$ and $u(x) > \omega(x)$ for $x > x_0$. This implies that $u''(x) - \omega''(x) \geq 0$ for $x \geq x_0$. As a consequence $u(x) - \omega(x) \to +\infty$ as $x \to +\infty$ which is impossible since u and ω are both bounded. Hence, ω is the unique solution of (1.12). $\square$

Corollary 1.15. *Let $h : [0,\infty) \to \mathbb{R}$ be a continuous, non-decreasing function. The initial value problem*

$$\begin{cases} y''(r) = h(y(r)), \quad r > 0, \\ y(0) = 1, \end{cases} \tag{1.13}$$

subject to

$$y(r) > 0,\ y'(r) < 0, \quad r > 0, \tag{1.14}$$

has a unique solution χ that satisfies $\lim_{r\to\infty} \chi(r) = 0$ and $\lim_{r\to\infty} \chi'(r) = 0$.

Proof. The proof of this corollary proceeds similarly. Since our function $h : [0,\infty) \to \mathbb{R}$ is a continuous non - decreasing function, we can use (1.10). As a result we have that the boundary problem $y''(r) =$

$h(y(r))$, $y(0) = 1$, $y(1) = 0$ has a unique solution $y_0(r)$. Taking this solution we can define the following function $\Psi_0(r)$ as

$$\begin{cases} \Psi_0(x) = y_0(x), & 0 \leq r \leq 1, \\ \Psi_0(x) = 0, & r \geq 1, \end{cases} \tag{1.15}$$

In this case, $\Psi_0(r)$ is a continuous subfunction on $[0, +\infty)$.

Let $\{\varphi\}$ denote the collection of every continuous superfunction on $[0, +\infty)$ such that $\varphi(r) \geq \Psi_0(r)$ on $[0, +\infty)$. We define the function χ as $\chi(r) = \inf\{\varphi(r) : \varphi \in \{\varphi\}\}$. As in our previous proof, our function χ is continuous on $[0, +\infty)$ and is both a superfunction and subfunction. Using (1.10), for any $b > 0$ the boundary problem $y''(r) = h(y(r))$, $y(0) = 1$, $y(b) = \chi(b)$ has a $C^{(2)}$ solution y_b on $[0, b]$. Arguing as in (1.14) we see that χ is a solution of (1.15). □

1.4 Semigroups

A semigroup is a family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on a Banach space L that satisfy:

(i) $S(0) = I$.

(ii) $S(t + s) = S(t)S(s), t, s \geq 0$.

We say a semigroup $S(t)$ is strongly continuous on L if $\lim_{t\to 0} S(t)f = f$ for all $f \in L$. We say $S(t)$ is a contraction semigroup if $\|S(t)\| \leq 1$ for all $t \geq 0$. We will also define a couple of concepts regarding semigroups (see [6]):

(a) Suppose $S(t)$ is a strongly continuous semigroup on L that satisfies $\|S(t)\| \leq M$ for some $M \geq 1$. Then we can define the norm $\|\cdot\|$ on L by

$$\|f\| = \sup_{t\geq 0} \|S(t)f\|. \tag{1.16}$$

(b) We define A the infinitesimal generator of a semigroup $S(t)$ on L as the linear operator defined by:

$$Af = \lim_{t\to 0} \frac{1}{t}(S(t)f - f). \tag{1.17}$$

(c) The domain of A, $\mathcal{D}(A)$ is the space of all $f \in L$ for which the previous limit exists.

(d) A linear operator A on L is a linear mapping whose domain $\mathcal{D}(A)$ is a subspace of L and whose range resides in L. The graph of A is defined as

$$\mathcal{B}(A) = \{(f, Af) : f \in \mathcal{D}(A)\} \subset L \times L. \tag{1.18}$$

A is said to be closed if $\mathcal{B}(A)$ is a closed subspace of $L \times L$.

The infinitesimal generator of a semigroup will be employed a couple of times throughout this work. To further be able to work with infinitesimal generators we will give a few results regarding generators.

Proposition 1.16. *Let $S(t)$ be a strongly continuous semigroup on L and let A be the corresponding infinitesimal generator. Then we have the following results:*

(a) *Suppose* $f \in L$ *and* $t \geq 0$*. Then we have that* $\int_0^t S(s) f ds \in \mathcal{D}(A)$ *and*

$$S(t)f - f = A \int_0^t S(s) f ds, \tag{1.19}$$

(b) *If* $f \in \mathcal{D}(A)$ *and* $t \geq 0$*, then we have* $S(t)f \in \mathcal{D}(A)$ *and moreover*

$$\frac{d}{dt} S(t) f = AS(t)f = S(t)Af. \tag{1.20}$$

As a consequence of this proposition we have the following result

Corollary 1.17. *Suppose that* $S(t)$ *is a strongly continuous semigroup on* L *and that* A *is the generator of* $S(t)$*. Thus we have* $\mathcal{D}(A)$ *is dense in* L *and* A *is closed.*

Throughout this work S_t will represent the semigroup defined as

$$(S_t h)(x) = \int_{\mathbb{R}^d} \rho_\alpha(t, x-y) h(y) dy \tag{1.21}$$

where $h \in B(\mathbf{R}^d)$ and ρ_α is the alpha - stable density defined as the fundamental solution of the Cauchy equation $\frac{\partial}{\partial t} u(t) = \Delta_\alpha u(t)$. Some additional properties of our semigroup used in this work are the following.

Proposition 1.18. *Let* $\psi \in L^\infty(\mathbb{R}^d)$ *and* $\phi \in L^1(\mathbb{R}^d)$*.*

(i) *For each* $t > 0$*, it follows that* $\|S_t \psi\|_\infty \leq \|\psi\|_\infty$ *and* $\|S_t \phi\|_1 \leq \|\phi\|_1$*.*

(ii) $\lim_{t\to\infty} t^{d/\alpha} (S_t \phi)(x) = p_1(0) \|\phi\|_1$ *uniformly in* $x \in \mathbb{R}^d$*. In particular,* $\lim_{t\to\infty} (S_t \phi)(x) = 0$ *uniformly in* $x \in \mathbb{R}^d$*.*

(iii) $\limsup_{\|x\|\to\infty} |\phi(x)| = 0$*. In particular,* $\limsup_{\|x\|\to\infty} |(S_t \phi)(x)| = 0$ *uniformly in* $t > 0$*.*

(iv) *If* $d > \alpha$ *then*

$$\limsup_{\|x\|\to\infty} \limsup_{t\to\infty} \int_0^t (S_s \phi)(x) ds = 0. \tag{1.22}$$

Proof. Property (i) readily follows from the definition of the norm (see [9]). Meanwhile, property (ii) is a consequence of the scale property of α-stable densities and the dominated convergence theorem (here we use the unimodal property of α-stable densities).

(iii) If $\limsup_{\|x\|\to\infty} |\phi(x)| = l > 0$, there is $M > 0$ sufficiently large so that $\inf\{|\phi(x)| : \|x\| \geq M\} > l/2$. As a consequence,

$$\|\phi\|_1 \geq \int_{\|x\|\geq M} |\phi(x)| dx \geq \int_{\|x\|\geq M} \frac{l}{2} dx = \infty. \tag{1.23}$$

The second assertion of (iii) readily follows now from property (i).

(*iv*) Let $\varepsilon > 0$, and take $M > 0$ such that $M^{1-d/\alpha}((d/\alpha)-1)^{-1} < \varepsilon/2$. The scale and unimodal properties of α-stable densities imply

$$\begin{aligned}\limsup_{t\to\infty}\int_0^t (S_s\phi)(x)ds &\leq \int_0^M (S_s\phi)(x)ds + \limsup_{t\to\infty}\int_M^t s^{-d/\alpha}p_1(0)\|\phi\|_1 ds \\ &\leq \int_0^M (S_s\phi)(x)ds + p_1(0)\|\phi\|_1\frac{\varepsilon}{2}, \quad \text{for all } x\in\mathbb{R}^d.\end{aligned} \tag{1.24}$$

The digression (*iii*) and the dominated convergence theorem yield

$$\limsup_{\|x\|\to\infty}\limsup_{t\to\infty}\int_0^t S_s\phi(x)ds \leq p_1(0)\|\phi\|_1\frac{\varepsilon}{2}. \tag{1.25}$$

Finally, we reach the conclusion when we let $\varepsilon \to 0$. □

1.5 The α-stable density

We define $p_\alpha(t,\cdot)$ as real-valued functions defined on $\mathbb{R}^d$, with Fourier transforms given by

$$\int_{\mathbb{R}^d} e^{z\cdot xi}p_\alpha(t,x)dx = e^{-t\|z\|^\alpha}, \quad \text{for all } t>0,\ z\in\mathbb{R}^d. \tag{1.26}$$

With $\cdot$ and $\|\cdot\|$ defined as the inner product and the Euclidean norm in $\mathbb{R}^d$, respectively. α-stable densities satisfy the following useful properties:

Proposition 1.19. *Let $p_\alpha(t,\cdot)$ be any α-stable density.*

(*i*) *For each $t>0$,*

$$\int_{\mathbb{R}^d} p_\alpha(t,y)dy = 1, \tag{1.27}$$

and $p_\alpha(t,x) > 0$, for all $x\in\mathbb{R}^d$ (density property).

(*ii*) *For each $t,s>0$ and $x\in\mathbb{R}^d$, $p_\alpha(ts,x) = t^{-d/\alpha}p_\alpha(s,t^{-1/\alpha}x)$* (scale property). *In particular, it follows that $p_\alpha(t,x)\leq t^{-d/\alpha}p_\alpha(1,0)$* (unimodal property).

Proposition 1.20. *Let $p_\alpha(t,\cdot)$ be any α-stable density. Thus it stands*

$$p_\alpha(t,x) = \int_0^\infty f_\alpha(t,\lambda)p_2(\lambda,x)d\lambda, \tag{1.28}$$

where p_2 is the Gaussian density. This is known as the subordination formula.

Lemma 1.21. *The function $(t,x)\mapsto p_\alpha(t,x)$ is in $C^\infty((0,\infty)\times\mathbb{R}^d)$.*

Proposition 1.22. *For each $t>0$,*

$$\lim_{x\to 0}\int_0^t \|p_\alpha(s,x+\cdot)-p_\alpha(s,\cdot)\|_1 ds = 0. \tag{1.29}$$

The proofs of these results can be found in [20, 4]

1.6 The fractional Laplacian

The fractional laplacian $(-\Delta)^\alpha$ is a operator which gives the standard laplacian when we take $\alpha = 1$. This operator can be defined in several equivalent forms:

Definition 1.23 (Using Fourier transform)**.** We define the fractional laplacian as the operator with symbol $|\xi|^{2\alpha}$. This means the following formula holds:

$$(-\hat{\Delta})^\alpha f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi) \tag{1.30}$$

for any f for which the right hand side is valid.

Definition 1.24 (Using functional calculus)**.** We know that the operator $-\Delta$ is a self-adjoint positive definite operator given a dense subset D of $L^2(\mathbf{R}^n)$. We can define $F(-\Delta)$ for any continuous function $F : \mathbf{R}^+ \to \mathbf{R}$. In this way we define $(-\Delta)^\alpha$.

Definition 1.25 (Using the heat semigroup)**.** We part from the formula

$$\lambda^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (\exp(-t\lambda) - 1) \frac{dt}{t^{1+\alpha}}. \tag{1.31}$$

with $\lambda \geq 0$ and $0 < \alpha < 1$. Suppose $\lambda = |\xi|^2$ and by using our first definition we get

$$(-\Delta)^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (\exp(t\Delta) f(x) - f(x)) \frac{dt}{t^{1+\alpha}}. \tag{1.32}$$

Definition 1.26 (Infinitesimal generator of a Levy process)**.** Suppose X_t is the α-stable Lévy process starting at 0 and suppose that f is a smooth function. Then we have

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{E}[f(x) - f(x + X_h)]. \tag{1.33}$$

In this particular work, the corresponding infinitesimal generator of the semigroup $\{S_t : t \geq 0\}$ is the fractional Laplacian Δ_α, whose domain is denoted by $D(\Delta_\alpha)$.

Proposition 1.27. *Let Δ_α represent the fractional Laplacian.*

(i) If $x \in \mathbb{R}^d$ is a global minimum of $\phi \in D(\Delta_\alpha)$ then $\Delta_\alpha \phi(x) \geq 0$.

(ii) Δ_α is a closed linear operator.

Proof. The proof of (i) can be found in [4]. Meanwhile, proposition (ii) is Corollary 1.2.5 in [18]. □

Definition 1.28. Let $B(\mathbb{R}^d)$ denote the space of bounded measurable real-valued functions defined on $\mathbb{R}^d$, and let $\{S_t : t \geq 0\}$ be the semigroup corresponding to the fractional Laplacian. By a *mild solution* we mean a continuous curve $u : [0, \infty) \to B(\mathbb{R}^d)$ satisfying

$$u(t) = S_t(\psi) + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds, \quad t \geq 0. \tag{1.34}$$

A *classical solution* of a partial differential equation of order k is a solution of the partial differential equation that is at least k times continuously differentiable.

2. Existence, regularity and positivity of solutions

In this section we will prove the existence of a solution for our problem of interest and we will prove several properties of said solution. Amongst these properties we have temporal regularity, positivity and global existence.

2.1 Local existence

In this section we will assume the following conditions

(i) $\psi, \varphi \in L^1(\mathbb{R}^d)$,

(ii) $\psi \in D(\Delta_\alpha) \cap B(\mathbb{R}^d)$ and $\psi \geq 0$,

(iii) $\varphi, g \in C(\mathbb{R}^d) \cap B(\mathbb{R}^d)$ and $\varphi, g \geq 0$, and

(iv) $f : [0, \infty) \to \mathbb{R}$ is convex with $f(0) = 0$.

The following theorem is one of our main results.

Theorem 2.1 (Existence). *The initial-value problem*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta_\alpha u(t,x) - g(x) f(u(t,x)) + \varphi(x), \\ u(0,x) = \psi(x), \end{cases} \tag{2.1}$$

has a classical solution.

To prove this theorem, we will need some technical lemmas. In them, we convey that E_T denotes the Banach space $C([0,T] : B(\mathbb{R}^d))$ for each $T > 0$ with the norm

$$|||u|||_T = \sup\{\|u(t)\|_\infty : t \in [0,T]\}. \tag{2.2}$$

Lemma 2.2. *There exist $R, T \in \mathbb{R}^+$ such that the function $F : E_T \to E_T$ given by*

$$(Fu)(t) = S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds, \tag{2.3}$$

is bounded in $B(0,R) \subseteq E_T$.

Proof. Beforehand, notice that

$$\begin{aligned}\left\|S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds\right\|_\infty &\le \|S_t\psi\|_\infty + \int_0^t \|S_{t-s}(\varphi - gf(u(s)))\|_\infty ds \\ &\le \|\psi\|_\infty + \int_0^t \left(\|\varphi\|_\infty + \|g\|_\infty \|f\|_\infty\right) ds.\end{aligned} \tag{2.4}$$

Suppose now that $u \in B(0,R) \subseteq E$, so that there exists $R > 0$ such that $u(s) \in [-R,R]$. As a consequence,

$$\begin{aligned}\|(Fu)(t)\| &\le \|\psi\|_\infty + \int_0^t \|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,R]} ds \\ &\le \|\psi\|_\infty + T\left(\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,R]}\right).\end{aligned} \tag{2.5}$$

Letting $R = \|\psi\|_\infty + 1$ and $T \le (\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,R]})^{-1}$, we obtain that $\|(Fu)(t)\| \le R$ for all $t \in [0,T]$. $\square$

Lemma 2.3. *The function $F : B(0,R) \to B(0,R)$ defined in Lemma 2.2 is continuous.*

Proof. Let $t > r$, and note that

$$\|S_t\psi - S_r\psi\| = \|S_r(S_{t-r}\psi - \psi)\| \le \|S_r\| \, \|S_{t-r}\psi - \psi\| \le \|S_{t-r}\psi - \psi\| \to 0 \tag{2.6}$$

when $r \uparrow t$. Using this result we have

$$\begin{aligned}&\left\|S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds - S_r\psi - \int_0^r S_{r-s}(\varphi - gf(u(s)))ds\right\| \\ &\quad\le \|S_t\psi - S_r\psi\| + \left\|\int_0^t S_{t-s}(\varphi - gf(u(s)))ds - \int_0^r S_{r-s}(\varphi - gf(u(s)))ds\right\| \\ &\quad\le \left\|\int_0^t S_{t-s}(\varphi - gf(u(s)))ds - \int_0^r S_{r-s}(\varphi - gf(u(s)))ds\right\| \\ &\quad\le \left\|\int_0^r S_{r-s}(S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s))))ds\right\| + \int_r^t \|S_{t-s}(\varphi - gf(u(s)))\| \, ds \\ &\quad\le \int_0^r \|S_{r-s}(S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s))))\| \, ds + \int_r^t \|S_{t-s}(\varphi - gf(u(s)))\| \, ds \\ &\quad\le \int_0^r \|S_{r-s}\| \, \|(S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s))))\| \, ds + \int_r^t \|S_{t-s}(\varphi - gf(u(s)))\| \, ds.\end{aligned} \tag{2.7}$$

The second term of the right-hand side of these inequalities is bounded by

$$\int_r^t \|S_{t-s}(\varphi - gf(u(s)))\| ds \le (t-r)(\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,R]}). \tag{2.8}$$

In turn, the right-hand side of (2.8) goes to 0 when r tends to t, so we only have to study the behavior first term. Note that

$$\begin{aligned}&\int_0^r \|S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s)))\| ds \\ &\quad= \int_0^T 1_{[0,r]}(s) \|S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s)))\|.\end{aligned} \tag{2.9}$$

The function $s \to 1_{[0,r]}(s)\|S_{t-r}(\varphi - gf(u(s)) - (\varphi - gf(u(s))\|$ is measurable and bounded, one bound being $2(\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,R]})$. By the uniform convergence theorem,

$$\lim_{r\to t^-} \int_0^r \|S_{t-r}(\varphi - gf(u(s))) - (\varphi - gf(u(s)))\| = 0. \tag{2.10}$$

We conclude that F is continuous. $\square$

Lemma 2.4. *Let $R = \|\psi\|_\infty + 1$. Then the mapping F is a contraction if*

$$T < \min\left\{ \frac{R}{\|g\|_\infty f(R)}, \frac{1}{\|\varphi\|_\infty + \|g\|_\infty \|f\|_{\ [0,R]}} \right\}. \tag{2.11}$$

Proof. Firstly, notice that Lemma 2.3 and the inequality (2.11) imply that the function F is continuous. On the other hand, for each $u, v \in B(0, R)$ it is readily checked that

$$\begin{aligned}
\|(Fu)(t) - (Fv)(t)\| &= \left\| S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds - S_t\psi - \int_0^t S_{t-s}(\varphi - gf(v(s)))ds \right\| \\
&= \left\| \int_0^t S_{t-s}(\varphi - gf(u(s))) - S_{t-s}(\varphi - gf(v(s)) \right\| \\
&= \left\| \int_0^t S_{t-s}(-gf(u(s)) + gf(v(s)))ds \right\| \\
&= \left\| \int_0^t S_{t-s}(g(f(u(s)) - f(v(s))))ds \right\| \\
&\le \int_0^t \|S_{t-s}(g(f(u(s)) - f(v(s))))\| \, ds \\
&\le \int_0^t \|g(f(u(s)) - f(v(s)))\| \, ds \\
&\le \int_0^t \|g\| \, \|f(u(s)) - f(v(s))\| \, ds \\
&\le \|g\|_\infty \int_0^t \|f(u(s)) - f(v(s))\| ds.
\end{aligned} \tag{2.12}$$

As a consequence, $u(s), v(s) \in [-R, R]$. Using the convexity of f, it follows that

$$\frac{f(u(s)) - f(v(s))}{u(s) - v(s)} \le \frac{f(R) - f(0)}{R}. \tag{2.13}$$

This inequality and (2.16) yield

$$\|(Fu)(t) - (Fv)(t)\| \le \|g\|_\infty \frac{Tf(R)}{R} \|u - v\|. \tag{2.14}$$

Using (2.11) again, we see that the coefficient multiplying $\|u - v\|$ at the right-hand side of (2.14) is less than 1. We conclude that F is a contraction. $\square$

Using now Lemmas 2.2 through 2.4 together with Banach's fixed-point theorem, there exists a unique continuous function $u \in B(0, R)$ that satisfies

$$u(t) = S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds, \quad \forall t \in [0, T]. \tag{2.15}$$

In such way, we have established the existence of a mild solution for problem (2.1). Finally we want to see that the mapping $u_0 \to u$ from $B(\mathbb{R}^d)$ to $C([0, T] : B(\mathbb{R}^d))$ is continuous. In other words, we wish to prove that the variation in the initial condition is continuous. To see this, let u and v be the solutions corresponding to the initial conditions u_0 and v_0 respectively. Thus we have

$$\begin{aligned}
\|u(t) - v(t)\|_\infty &= \left\| S_t u_0 + \int_0^t S_{t-s}(\varphi - gf(u(s)))ds - S_t v_0 - \int_0^t S_{t-s}(\varphi - gf(v(s)))ds \right\| \\
&\leq \|S_t u_0 - S_t v_0\| + \left\| \int_0^t S_{t-s}(g(f(v(s)) - f(u(s))))ds \right\| \\
&\leq \|S_t(u_0 - v_0)\| + \int_0^t \|S_{t-s}(g(f(v(s)) - f(u(s))))\| \, ds \\
&\leq \|u_0 - v_0\| + \int_0^t \|g(f(v(s)) - f(u(s)))\| \, ds \\
&\leq \|u_0 - v_0\| + \int_0^t \|g\|_\infty \|f(u(s)) - f(v(s))\| \, ds \\
&= \|u_0 - v_0\| + \|g\|_\infty \int_0^t \|f(u(s)) - f(v(s))\| \, ds \\
&\leq \|u_0 - v_0\| + \|g\|_\infty \frac{Tf(R)}{R} \|u - v\|.
\end{aligned} \tag{2.16}$$

Thus we have

$$\|u - v\| \left(1 - \|g\|_\infty \frac{f(R)T}{R}\right) \leq \|u_0 - v_0\|. \tag{2.17}$$

Since we have taken $T > 0$, it yields

$$\|u - v\| \leq \frac{R}{R - \|g\|_\infty f(R)T} \|u_0 - v_0\|. \tag{2.18}$$

From this previous inequality, the continuity in the initial parameter follows.

Corollary 2.5. *If $g \in B(\mathbb{R}^d)$, then the equation*

$$w(t) = S_t w(0) - \int_0^t S_{t-s}(gw(s))ds, w(0) \in B(\mathbb{R}^d), \tag{2.19}$$

has a unique solution $w \in C([0, T] : B(\mathbb{R}^d))$.

2.2 Temporal regularity

Theorem 2.6 (Temporal regularity)**.** *The solution of* (2.1) *is differentiable in time.*

Proof. Let u be the unique solution of (2.1), and consider the equation

$$v(t) = S_t v(0) - \int_0^t S_{t-s}(gf'(u(s))v(s))ds, \tag{2.20}$$

which has the unique solution $v(0) = \Delta_\alpha \psi - gf(\psi) + \varphi$. We wish to prove that $v(t)$ is the derivative of $u(t)$. To that end, let

$$u_h = \frac{u(t+h) - u(t)}{h}, \tag{2.21}$$

and observe that

$$\begin{aligned}\|u_h(t) - v(t)\| = \Bigg\| \frac{1}{h}\Bigg(S_{t+h}\psi + \int_0^{t+h} S_{t+h-s}(\varphi - gf(u(s)))ds - S_t\psi \\ - \int_0^t S_{t-s}(\varphi - gf(u(s)))ds \Bigg) - S_t v(0) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \Bigg\|.\end{aligned} \tag{2.22}$$

To simplify the right hand term, we will use the following the following equation

$$\begin{aligned}\int_0^{t+h} S_{t+h-s}(\varphi - gf(u(s)))ds &= \int_0^{h} S_{t+h-s}(\varphi - gf(u(s)))ds + \int_h^{t+h} S_{t+h-s}(\varphi - gf(u(s)))ds \\ &= S_t \int_0^h S_{h-s}(\varphi - gf(u(s)))ds + \int_0^t S_{t-s}(\varphi - gf(u(s+h)))ds.\end{aligned} \tag{2.23}$$

Using this equation, we have as a result

$$\begin{aligned}\|u_h(t) - v(t)\| &= \Bigg\| \frac{1}{h}\Bigg(S_{t+h}\psi + \int_0^{t+h} S_{t+h-s}(\varphi - gf(u(s)))ds - S_t\psi \\ &\quad - \int_0^t S_{t-s}(\varphi - gf(u(s)))ds \Bigg) - S_t v(0) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \Bigg\| \\ &= \Bigg\| \frac{1}{h}\Bigg(S_t S_h\psi + S_t\int_0^h S_{h-s}(\varphi - gf(u(s)))ds + \int_0^t S_{t-s}(\varphi - gf(u(s+h)))ds \Bigg) \\ &\quad - S_t\psi - \int_0^t S_{t-s}(\varphi - gf(u(s)))ds - S_t v(0)\Bigg) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \Bigg\| \\ &= \Bigg\| S_t \left(\frac{S_h\psi + \int_0^h S_{h-s}(\varphi - gf(u(s)))ds}{h} \right) \\ &\quad + \frac{1}{h}\left(-S_t\psi + \int_0^t S_{t-s}(\varphi - gf(u(s+h)))ds - \int_0^t S_{t-s}(\varphi - gf(u(s)))ds \right) \\ &\quad - S_t v(0) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \Bigg\| \\ &= \Bigg\| S_t \frac{u(h)}{h} + \frac{1}{h}(\int_0^t S_{t-s}(\varphi - gf(u(s+h)))ds) - \frac{S_t u(0)}{h} \\ &\quad - \frac{1}{h}\int_0^t S_{t-s}(\varphi - gf(u(s)))ds - S_t v(0) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \Bigg\|\end{aligned} \tag{2.24}$$

so that

$$
\begin{aligned}
\|u_h(t)-v(t)\| = & \left\| S_t(\frac{u(h)-u(0)}{h} - v(0)) + \frac{1}{h}\int_0^t S_{t-s}(\varphi - gf(u(s+h)))ds \right. \\
& \left. -\frac{1}{h}\int_0^t S_{t-s}(\varphi - gf(u(s)))ds + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \right\| \\
= & \left\| S_t(\frac{u(h)-u(0)}{h} - v(0)) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \right. \\
& \left. \frac{1}{h}\int_0^t S_{t-s}((\varphi - gf(u(s+h))) - (\varphi - gf(u(s))))ds \right\| \\
= & \left\| S_t(\frac{u(h)-u(0)}{h} - v(0)) + \int_0^t S_{t-s}(gf'(u(s))v(s))ds \right. \\
& \left. -\int_0^t S_{t-s}(g(\frac{f(u(s+h))-f(u(s))}{h}))ds \right\| .
\end{aligned}
\tag{2.25}
$$

Using the triangle inequality and Proposition 1.18 yields

$$
\begin{aligned}
\|u_h(t)-v(t)\| \leq & \left\| S_t(\frac{u(h)-u(0)}{h} - v(0)) \right\| + \left\| \int_0^t S_{t-s}(g(\frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)})(v-u_h(s))ds \right\| \\
& + \left\| \int_0^t S_{t-s}(g(f'(u(s)) - \frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)}v)ds \right\| \\
= & \|S_t\| \, \|u_h(0)-v(0)\|_\infty + \int_0^t \left\| S_{t-s}(g(\frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)})(v-u_h(s))) \right\| ds \\
& + \int_0^t \left\| S_{t-s}(g(f'(u(s)) - \frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)})v(s)) \right\| ds \\
\leq & \|u_h(0)-v(0)\| + \int_0^t \|g\| \left\| \frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)} \right\| \|u_h(s)-v(s)\| ds \\
& + \int_0^t \|g\| \| f'(u(s)) - \frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)} \| \|v(s)\|_\infty ds.
\end{aligned}
\tag{2.26}
$$

Let $0 < |h| < T$ for some $0 \leq s \leq T$. It is easy to see that $u(s+h) = u(0)$ if $s+h \leq 0$. Meanwhile, if $T \leq s+h$ then $u(s+h) = u(T)$. Note also that if $u(s+h) = u(s)$, then

$$
\frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)} = 0. \tag{2.27}
$$

This fact and the mean value theorem imply that there exists $\xi(s,h)$ between $u(s+h)$ and $u(s)$ such that

$$
\frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)} = f(\xi(s,h)). \tag{2.28}
$$

But $u \in B(0,R)$, so $|(\xi(s,h))| \leq R$. As a result,

$$
\left| \frac{f(u(s+h))-f(u(s))}{u(s+h)-u(s)} \right| \leq \|f'\|_{L[0,R]}. \tag{2.29}
$$

On the other hand, taking norm on both sides of (2.20), using the triangle inequality and properties of semigroups, we obtain the inequality

$$\|v(t)\| \leq \|v(0)\| + \int_0^t \|g\|\|f'\|_{L[0,R]}\|v(s)\|ds. \tag{2.30}$$

Moreover, as a consequence of Gronwall's inequality, it follows that $\|v(t)\| \leq \|v(0)\|e^{\|g\|\|f'\|_{L[0,R]}T}$. Substituting this inequality into (2.25) yields

$$\begin{aligned}\|u_h(t) - v(t)\| &\leq \|u_h(0) - v(0)\| + \|g\|\|v\|_{[0,T]} \int_0^T \left\| f'(u(s)) - \frac{f(u(s+h)) - f(u(s))}{u(s+h) - u(s)} \right\| ds \\ &\quad + \|g\|\|f'\|_{L[0,R]} \int_0^t \|u_h(s) - v(s)\|ds \\ &= \epsilon(h) + M \int_0^t \|u_h(s) - v(s)\|ds,\end{aligned} \tag{2.31}$$

where

$$M = \|g\|\|f'\|_{L[0,R]}, \tag{2.32}$$

$$\epsilon(h) = \|u_n(0) - v(0)\| + \|g\|\|v\|_{[0,T]} \int_0^T \left\| f'(u(s)) - \frac{f(u(s+h)) - f(u(s))}{u(s+h) - u(s)} \right\| ds. \tag{2.33}$$

By the dominated convergence theorem, the second term at the right-hand side of (2.31) tends to zero. On the other hand,

$$\lim_{h \to 0^+} \frac{u(h) - u(0)}{h} = v(0). \tag{2.34}$$

This implies that $\epsilon(h)$ tends to zero as $h \to 0$. Gronwall's inequality yields $\|u_n(t) - v(t)\| \leq \epsilon(h)e^{TM}$. As a consequence, $u(t)$ is differentiable and its derivative is equal to $v(t)$. $\square$

Finally, we will verify that the mild solution found in this section is indeed a classical solution of (2.1). To be able to prove that our solution is indeed a classical solution, we must verify that our solution u is in $D(\Delta)$. Yet this follows immediately from Proposition 1.12. Let

$$v(t) = S_t u_0 + \int_0^t T(t-s)(\varphi - gf(u(s)))ds. \tag{2.35}$$

Since the function $\varphi - gf(u(s))$ is continuously differentiable then

$$\begin{aligned}\frac{d}{dt}v(t) &= \frac{d}{dt}S_t u_0 + \frac{d}{dt}\int_0^t T(t-s)(\varphi - gf(u(s)))ds \\ &= \Delta_\alpha S_t u_0 + \frac{d}{dt}\int_0^t T(s)(\varphi - gf(u(t-s)))ds.\end{aligned} \tag{2.36}$$

Define now

$$\tilde{v}(t) = \int_0^t T(s)f(t-s)ds, \tag{2.37}$$

where $f(t-s) = \varphi - gf(u(t-s))$. We wish to study the behavior of

$$\begin{aligned}\lim_{h\to 0}\frac{\tilde{v}(t+h)-\tilde{v}(t)}{h} &= \lim_{h\to 0}\frac{\int_0^{t+h}T(s)f(t+h-s)ds - \int_0^{t}T(s)f(t-s)ds}{h}\\ &= \lim_{h\to 0}\frac{\int_0^{t+h}T(s)f(t+h-s)ds - \int_0^{t+h}T(s)f(t-s)ds + \int_t^{t+h}T(s)f(t-s)ds}{h}\\ &= \lim_{h\to 0}\frac{\int_0^{t+h}T(s)(f(t+h-s)-f(t-s))ds + \int_t^{t+h}T(s)f(t-s)ds}{h}\\ &= \lim_{h\to 0}\left\{\int_0^{t+h}T(s)(\frac{f(t-s+h)-f(t-s)}{h})ds + \frac{1}{h}\int_t^{t+h}T(s)f(t-s)ds\right\}.\end{aligned} \tag{2.38}$$

It is easy to verify that the second limit on the right-hand side of (2.38) is equal to $T(t)f(0)$. By taking the change of variable $r = s-t$ we have

$$\lim_{h\to 0}\frac{\int_t^{t+h}T(s)f(t-s)ds}{h} = \lim_{h\to 0}\frac{\int_0^{h}T(s+t)f(-s)ds}{h} = T(t)f(0). \tag{2.39}$$

On the other hand the first limit equals $\displaystyle\int_0^t T(s)f'(t-s)ds$ because, if we define

$$\zeta = \lim_{h\to 0}\int_0^{t+h}T(s)(\frac{f(t-s+h)-f(t-s)}{h})ds. \tag{2.40}$$

then we have

$$\begin{aligned}\zeta &= \lim_{h\to 0}\int_0^{t}T(s)(\frac{f(t-s+h)-f(t-s)}{h})ds\\ &\quad+\lim_{h\to 0}\int_t^{t+h}T(s)(\frac{f(t-s+h)-f(t-s)}{h})ds\\ &= \int_0^t T(s)f'(t-s)ds + \lim_{h\to 0}\int_t^{t+h}T(s)(\frac{f(t-s+h)-f(t-s)}{h})ds\\ &= \int_0^t T(s)f'(t-s)ds + \lim_{h\to 0}1_{[t,t+h]}(s)T(s)\frac{f(t-s+h)-f(t-s)}{h}\\ &= \int_0^t T(s)f'(t-s)ds + (T(s)f'(t-s))(0)\\ &= \int_0^t T(s)f'(t-s)ds.\end{aligned} \tag{2.41}$$

As a consequence,

$$\frac{d}{dt}\tilde{v} = T(t)f(0) + \int_0^t T(s)f'(t-s)ds. \tag{2.42}$$

In this way,

$$\frac{d}{dt}v(t) = \Delta_\alpha T(t)u_0 + T(t)(\varphi - gf(u(0))) + \int_0^t T(s)(-gf'(u(t-s))u'(t-s)ds. \tag{2.43}$$

On the other hand, we know that $\tilde{v}'(t) = \Delta_\alpha \tilde{v}(t) + f(t)$, since

$$\begin{aligned} \tilde{v}'(t) &= \lim_{h\to 0} \frac{S_h - I}{h} \tilde{v}(t) \\ &= \lim_{h\to 0} \frac{\tilde{v}(t+h) - \tilde{v}(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t-s+h) f(s) ds \\ &= \tilde{v}'(t) - f(t). \end{aligned} \tag{2.44}$$

Using this fact and Theorem 2.6, it follows that

$$\begin{aligned} \frac{d}{dt} v(t) &= \Delta_\alpha S_t u_0 + \Delta_\alpha \int_0^t T(t-s)(\varphi - gf(u(s))) ds + \varphi - gf(u(t)) \\ &= \Delta_\alpha v(t) - gf(u(t)) + \varphi. \end{aligned} \tag{2.45}$$

This implies that v is a classical solution of

$$\begin{cases} \dfrac{d}{dt} v(t) = \Delta_\alpha v(t) - gf(u(t)) + \varphi, \\ v(0) = u(0), \end{cases} \tag{2.46}$$

But u is also a mild solution of (2.46). Since the mild solution of (2.46) is unique, we conclude that $u = v$. As a result, u is a classical solution for (2.1).

2.3 Positivity

In this section we wish to prove the positivity of the solution of (2.1). But first we will start by proving the uniform continuity of u in x.

Lemma 2.7. *u is uniformly continuous in $\mathbb{R}^d$.*

Proof. Let $\tilde{t} \in [0,T]$ arbitrary fixed. Remember

$$u(t,x) = S_t \psi(x) + \int_0^t S_{t-s}(\varphi - gf(u(s)))(x) ds. \tag{2.47}$$

For the first term we observe

$$\begin{aligned} |(\rho_t * \psi)(x) - (\rho_{\tilde{t}} * \psi)(x)| &= \left| \int_{\mathbb{R}^d} \rho_t(y)\psi(x-y)dy - \int_{\mathbb{R}^d} \rho_{\tilde{t}}(y)\psi(x-y)dy \right| \\ &= \left| \int_{\mathbb{R}^d} (\rho_t(y) - \rho_{\tilde{t}}(y))\psi(x-y)dy \right| \\ &\le \|\psi\|_\infty \|\rho_t - \rho_{\tilde{t}}\|_1 . \end{aligned} \tag{2.48}$$

To study the second term, we define

$$h(x) = \left| \int_0^{\tilde{t}+\nu} \rho_{\tilde{t}+\nu-s} * (\varphi - gf(u(s)))(x) ds - \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x) ds \right| \tag{2.49}$$

and we note that

$$\begin{aligned}
h(x) &\leq \left|\int_0^{\tilde{t}+\nu} \rho_{\tilde{t}+\nu-s} * (\varphi - gf(u(s)))(x)ds - \int_0^{\tilde{t}+\nu} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds\right| \\
&\quad + \left|\int_0^{\tilde{t}+\nu} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds - \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds\right| \\
&= \left|\int_0^{\tilde{t}+\nu} (\rho_{\tilde{t}+\nu-s} - \rho_{\tilde{t}-s}) * (\varphi - gf(u(s)))(x)ds\right| \\
&\quad + \left|\int_{\tilde{t}}^{\tilde{t}+\nu} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds\right| \qquad (2.50) \\
&\leq \int_0^{\tilde{t}+\nu} \left\|\rho_{\tilde{t}+\nu-s} - \rho_{\tilde{t}-s}\right\| ds(\|\varphi\|_\infty + \|g\|_\infty \|f\|_{L_1^\infty[0,R]}) \\
&\quad + \int_{\tilde{t}}^{\tilde{t}+\nu} \left\|\rho_{\tilde{t}-s}\right\| ds(\|\varphi\|_\infty + \|g\|_\infty \|f\|_{L_1^\infty[0,R]}) \\
&\leq (\|\varphi\|_\infty + \|g\|_\infty \|f\|_{L_1^\infty[0,R]})(\int_0^{\tilde{t}+\nu} \left\|\rho_{\tilde{t}+\nu-s} - \rho_{\tilde{t}-s}\right\|_1 ds + |\nu|).
\end{aligned}$$

Since $\left\|\rho_{\tilde{t}+\nu-s} - \rho_{\tilde{t}-s}\right\|_1 \to 0$ when $\nu \to 0$ and moreover

$$\left\|\rho_{\tilde{t}+\nu-s} - \rho_{\tilde{t}-s}\right\|_1 \leq \left\|\rho_{\tilde{t}+\nu-s}\right\|_1 + \left\|\rho_{\tilde{t}-s}\right\|_1 = 2. \qquad (2.51)$$

the result holds as a consequence of the dominated convergence theorem. □

Theorem 2.8. *Let $(\tilde{t}, \tilde{x}) \in [0,T] \times \mathbb{R}^d$ arbitrary fixed. The function $x \to S_{\tilde{t}}\psi(x) = (\rho_{\tilde{t}} * \psi)(x)$ is continuous.*

Proof. We will begin by proving the continuity of

$$x \to \int_0^{\tilde{t}} S_{\tilde{t}-s}(\varphi - gf(u(s)))(x)ds = \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds. \qquad (2.52)$$

Note that if $x_n \to x$, then

$$\rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x_n) \to \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x), fors \in [0, \tilde{t}]. \qquad (2.53)$$

Additionally

$$\left|\rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)\right| \leq \|g\|_\infty \|f\|_{L_1^\infty[0,R]}. \qquad (2.54)$$

The continuity of (2.52) follows from the dominated convergence theorem. Using this previous result we have

$$
\begin{aligned}
\left|u(t,x)-u(\tilde{t},\tilde{x})\right| \quad &\leq \quad |(\rho_t * \psi)(x) - (\rho_{\tilde{t}} * \psi)(\tilde{x})| \\
&\quad + \left| \int_0^t \rho_{t-s} * (\varphi - gf(u(s)))(x)ds - \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(\tilde{x})ds \right| \\
&\leq \quad |(\rho_t * \psi)(x) - (\rho_{\tilde{t}} * \psi)(x)| + |(\rho_{\tilde{t}} * \psi)(x) - (\rho_{\tilde{t}} * \psi)(\tilde{x})| \\
&\quad + \left| \int_0^t \rho_{t-s} * (\varphi - gf(u(s)))(x)ds - \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds \right| \\
&\quad + \left| \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(x)ds - \int_0^{\tilde{t}} \rho_{\tilde{t}-s} * (\varphi - gf(u(s)))(\tilde{x})ds \right|.
\end{aligned}
\tag{2.55}
$$

By the uniform continuity of t in x, each of the terms on the right hand side of the inequality go to 0 when $x \to \tilde{x}$, yielding the result. □

Now that we have proven the uniform continuity of u we can start giving the proof of the positivity of u. We will begin by defining a function h such that $h(t) = \inf u(t,x)$ for $x \in \mathbb{R}^d$ and $t \in [0,T]$. We will also make use of the following lemmas.

Lemma 2.9. *The function h defined above is continuous in $[0,T]$.*

Proof. We wish to see that h is continuous in $\tilde{t} \in [0,T]$. By continuity of u we know that given $\epsilon > 0$ there exists $\delta > 0$ such that $\left|t - \tilde{t}\right| < \delta \Rightarrow \left|u(t,x) - u(\tilde{t},x)\right| < \epsilon$, for every $x \in \mathbb{R}^d$. Using this and properties of absolute value,

$$-\epsilon + u(\tilde{t},x) < u(t,x) < u(\tilde{t},x) + \epsilon. \tag{2.56}$$

Using properties of infimum we have

$$-\epsilon + \inf u(\tilde{t},x) \leq \inf u(t,x) \leq \inf u(\tilde{t},x) + \epsilon, \tag{2.57}$$

using properties of absolute value we conclude that $\left|\inf_{x\in\mathbb{R}^d} u(t,x) - \inf_{x\in\mathbb{R}^d} u(\tilde{t},x)\right| \leq \epsilon$. □

Now let $\tilde{t} \in [0,T]$ such that $h(\tilde{t}) = \inf_{t\in[0,T]} h(t)$. It is easy to verify the following lemma.

Lemma 2.10. *For $\tilde{t}$ we have that $\lim_{|x|\to\infty} \left|u(\tilde{t},x)\right| = 0$.*

Proof. Using the integral representation of u yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} |u(t,x)|\, dx \quad &\leq \quad \|\psi\|_1 + \int_0^t \|\varphi\|_1\, dt + \int_0^t c\, \|g\|_\infty \int_{\mathbb{R}^d} |u(s,x)|\, dxds \\
&\leq \quad (\|\psi\|_1 + T\, \|\varphi\|_1) + c\, \|g\|_\infty \int_0^t \int_{\mathbb{R}^d} |u(s,x)|\, dxds.
\end{aligned}
\tag{2.58}
$$

By Gronwall's inequality

$$\int_{\mathbb{R}^d} |u(t,x)|\, dx \leq (\|\psi\|_\infty + T\, \|\varphi\|_\infty) \exp\{c\, \|g\|_\infty\, T\}. \tag{2.59}$$

Now we note that

$$\limsup_{|x|\to\infty} (S_t\psi)(x) = \limsup_{|x|\to\infty} (\rho_t * \psi)(x). \tag{2.60}$$

This follows from the fact that
$$\int(\rho_t * \psi)(x)dx = \|\psi\|_1 < \infty. \tag{2.61}$$
implies that $\limsup_{|x|\to\infty}(\rho_t * \psi)(x) = 0$. This is seen by assuming
$$\limsup_{|x|\to\infty}(\rho_t * \psi)(x) = L > 0. \tag{2.62}$$
By definition, this implies for M sufficiently large, $\inf_{|x|\geq M}(\rho_t * \psi)(x) > \frac{1}{2}$. As a result,
$$\infty > \|\psi\| = \int_{\mathbb{R}^d}(\rho_t * \psi)(x)dx \geq \int_{|x|\geq M}(\rho_t * \psi)(x)dx \geq \infty. \tag{2.63}$$
Moreover, $\|\tilde{f}\|_1 \leq c\,\|u(t)\|_1 \leq C$, with C constant, $\forall t \in [0,T]$. As a consequence,
$$\limsup_{|x|\to\infty} f(u(t,x-y)) = \limsup_{|x|\to\infty} f(u(t,x)) = 0. \tag{2.64}$$
Finally, using the inequality
$$\rho(\tilde{t}-s,y)g(x-y)\tilde{f}(u(s,x-y)) \leq \|g\|_\infty \tilde{f}(\|u\|)\rho(\tilde{t}-s,y). \tag{2.65}$$
and using the dominated convergence theorem we have
$$\limsup_{|x|\to\infty}\int_0^{\tilde{t}}\int_{\mathbb{R}^d} g(x-y)\tilde{f}(u(s,x-y))\rho(\tilde{t}-s,y)dyds = 0. \tag{2.66}$$
Hence we can conclude $\lim_{|x|\to\infty}\left|u(\tilde{t},x)\right| = 0$. □

We will now suppose that $h(\tilde{t}) < 0$. By definition of h we can find a $M > 0$ such that $\left|u(\tilde{t},x)\right| < -\frac{h(\tilde{t})}{2}$, for every $|x| > M$. This implies $u(\tilde{t},x) > \frac{h(\tilde{t})}{2}$, for every $|x| > M$. From this we can define
$$\inf_{x\in\mathbb{R}^d} u(\tilde{t},x) = \inf_{|x|\leq M} u(\tilde{t},x) = u(\tilde{t},\tilde{x}). \tag{2.67}$$
In this way $u(\tilde{t},\tilde{x}) = h(\tilde{t}) = \inf_{(t,x)\in[0,T]X\mathbb{R}^d} u(t,x)$. Define $\xi = u(\tilde{t},\tilde{x}) < 0$. If $\tilde{t} = 0$ then $\psi(\tilde{x}) = u(0,\tilde{x}) = \xi < 0$. Thus $\tilde{t} \in (0,T]$. Now we will define the function $\tilde{h}$ such that $\tilde{h}(t) = \inf_{x\in\mathbb{R}^d} V(x,t)$, with $V(t,x) = u(t,x) - K(\tilde{t}-t)$ and $K = -\frac{\xi}{6\tilde{t}} > 0$, for $t \in [0,\tilde{t}]$. The continuity of u on t uniform in x implies $\tilde{h}$ is continuous in $[0,\tilde{t}]$. As a consequence there exists $\hat{t} \in [0,\tilde{t}]$ such that
$$\tilde{h}(\hat{t}) = \inf_{t\in[0,\tilde{t}]}\tilde{h}(t) = \inf_{t\in[0,\tilde{t}]}\inf_{x\in\mathbb{R}^d} V(t,x). \tag{2.68}$$

By 2.10 we have that $\lim_{|x|\to\infty} V(\hat{t},x) = -K(\tilde{t}-\hat{t})$. As a consequence there exists $M > 0$ such that $\left|V(\hat{t},x) + K(\tilde{t}-\hat{t})\right| < -\frac{\xi}{6}$. As a result we have $V(\hat{t},x) > \frac{\xi}{6} - K(\tilde{t}-\hat{t})$, $\forall\,|x| > M$. On the other hand, we know $\lim_{t\to\tilde{t}} u(t,\tilde{x}) = \xi$. Hence there exists $\delta > 0$ such that for $0 < t-\tilde{t} < \delta$ we have $\left|u(t,\tilde{x}) - u(\tilde{t},\tilde{x})\right| < -\frac{\xi}{2}$. It follows that $u(t,\tilde{x}) < \frac{\xi}{2}$ when $0 < t-\tilde{t} < \delta$. Thus if $t \in (\tilde{t}-\delta,\tilde{t})$ we have
$$u(t,\tilde{x}) < \frac{\xi}{2} < -K\tilde{t} + \frac{\xi}{6} = \frac{\xi}{3}. \tag{2.69}$$

Using the fact that $-K\tilde{t} < K(\hat{t}-t)$ and 2.69 yields

$$u(t,\tilde{x}) < -K\tilde{t} + \frac{\xi}{6} < K(\hat{t}-t) + \frac{\xi}{6}. \tag{2.70}$$

Using 2.70 and replacing $K(\hat{t}-t)$ with $-K(t+\tilde{t}-\tilde{t}-\hat{t})$ we have

$$u(t,\tilde{x}) < -K(t+\tilde{t}+-\tilde{t}+\hat{t}) + \frac{\xi}{6}. \tag{2.71}$$

Separating this term, passing $-K(t-\tilde{t})$ to the other side and using the definition of $V(t,\tilde{x})$ we have $V(t,\tilde{x}) < \frac{\xi}{6} - K(\tilde{t}-\hat{t})$. Applying definition of $\hat{t}$ and definition of $V(t,\tilde{x})$ we have

$$\tilde{h}(\hat{t}) \leq \tilde{h}(t) \leq V(t,\tilde{x}) < \frac{\xi}{6} - K(\tilde{t}-\hat{t}). \tag{2.72}$$

It follows that $\inf_{x\in\mathbb{R}^d} V(\hat{t},x) < \frac{\xi}{6} - K(\tilde{t}-\hat{t})$. Because of this there must exist $z \in \mathbb{R}^d$ such that $V(\hat{t},z) < \frac{\xi}{6} - K(\tilde{t}-\hat{t})$. This would imply that $|z| \leq M$ and that

$$h(\hat{t}) = \inf_{x\in\mathbb{R}^d} V(\hat{t},x) = \inf_{|x|\leq M} V(\hat{t},x) = V(\hat{t},\hat{x}) = \inf_{t\in[0,\tilde{t}]} \inf_{x\in\mathbb{R}^d} V(t,x). \tag{2.73}$$

On the other hand since $V(\hat{t},\hat{x}) \leq u(\tilde{t},\tilde{x})$ we have that $u(\hat{t},\hat{x}) - K(\tilde{t}-\hat{t}) \leq \xi$. Passing the second term to the other side and using the fact that K and $\hat{t}$ are greater than 0 we have $u(\hat{t},\hat{x}) < \xi + K\tilde{t}$. By definition we have that

$$\xi + K\tilde{t} = (\frac{1}{3\tilde{t}}(-\frac{\xi}{2}))\tilde{t} + \xi = \frac{5}{6}\xi \tag{2.74}$$

Since $\xi < 0$ this tells us that $u(\hat{t},\hat{x}) < 0$. If we take $\hat{t} = 0$ we reach $V(0,\hat{x}) < \frac{\xi}{6} - K(\tilde{t})$. Subtracting $K(\tilde{t}-0)$ on the left hand side, simplifying the right hand side and using the definition of $\psi(\tilde{x})$ we reach $\psi(\tilde{x}) < \frac{\xi}{6} < 0$. Thus $\hat{t} \in (0,\tilde{t}]$. This implies that

$$\frac{\partial}{\partial t}V(\hat{t},\hat{x}) = \lim_{h\downarrow 0} \frac{V(\hat{t},\hat{x}) - V(\hat{t}-h,\hat{x})}{h} \leq 0. \tag{2.75}$$

Using 2.75, the fact that $\tilde{f}(u(\hat{t},\hat{x})) = 0$ and

$$\Delta_\alpha V(\hat{t},\hat{x}) = \Delta_\alpha u(\hat{t},\hat{x}) - K\Delta_\alpha(\tilde{t}-\hat{t}) = \Delta_\alpha u(\hat{t},\hat{x}). \tag{2.76}$$

We have

$$0 \leq \Delta_\alpha u(\hat{t},\hat{x}) - g(\hat{x})\tilde{f}(u(\hat{t},\hat{x})) + \varphi(\hat{x}) = \frac{\partial}{\partial t}u(\hat{t},\hat{x}) \leq -K < 0 \tag{2.77}$$

since $\hat{x}$ is global infimum of V. This means that the point $(\hat{t},\hat{x}) \in [0,T] \times \mathbb{R}^d$ does not satisfy the equation. Hence $\xi \geq 0$. As a result we have that our solution is non-negative.

2.4 Global existence

Up to this point, we have proven that our model has a local positive solution. We wish to prove that our local solution is in fact, a global solution. In other words, we wish to prove that the maximum time of existance t_{max} of is not finite. For this purpose, we will assume that t_{max} is finite. Then,

$$\lim_{t \to t_{max}} \|u(t)\| = \infty. \tag{2.78}$$

Since otherwise there is a sequence $t_n \uparrow t_{max}$ such that $\|u(t_n)\| \leq C$ for all n, for some fixed C. We now define

$$V(t) = u(t_n + t), t \geq 0. \tag{2.79}$$

From this definition we have

$$\begin{aligned} V(t) &= S_{t_n+t}\varphi + \int_0^{t_n+t} S_{t_n+t-s}(\varphi - gf(u(s)))ds \\ &= S_t u(t_n) + \int_0^t S_{t-s}(\varphi - gf(u(t_n+s)))ds \\ &= S_t V(0) + \int_0^t S_{t-s}(\varphi - gf(V(s)))ds. \end{aligned} \tag{2.80}$$

But we also know that taking

$$\xi = S_t[S_{t_n}\psi + \int_0^{t_n} S_{t_n-s}(\varphi - gf(u(s)))ds] + \int_0^t S_{t-s}(\varphi - gf(u(t+s)))ds \tag{2.81}$$

we have

$$\begin{aligned} \xi &= S_{t+tn}\psi + \int_0^{t_n} S_{t+t_n-s}(\varphi - gf(u(s)))ds \\ &\quad + \int_0^t S_{t-s}(\varphi - gf(u(t_n+s))ds \\ &= S_{t+tn}\psi + \int_0^{t_n} S_{t+t_n-s}(\varphi - gf(u(s)))ds \\ &\quad + \int_{t_n}^{t_n+s} S_{t_n+t-s}(\varphi - gf(u(s))ds. \end{aligned} \tag{2.82}$$

By Banach's fixed point principle there exists

$$\tilde{T} = \min\{\frac{\tilde{R}}{\|g\|_\infty f(\tilde{R})}, \frac{1}{\|\varphi\|_\infty + \|g\|_\infty \|f\|_{[0,\tilde{R}]}}\}. \tag{2.83}$$

Such that

$$1 + \|u(t_n)\| \leq 1 + C = \tilde{R}. \tag{2.84}$$

In this way, $\tilde{T}$ is independant of n and we have that $V(t)$ is a unique solution in the interval $[0, \tilde{T}]$. Now let us take $\epsilon = \frac{\tilde{T}}{2} > 0$. Since $t_n \uparrow t_{max}$ there exists n_0 positive integer such that $t_{n_0} \in (t_{max} - \tilde{T}, t_{max})$. This implies that $t_{max} < t_{n_0} + \tilde{T}$. Finally, we define the function:

$$w(t) = u(t), t \leq t_{n_0}, \tag{2.85}$$

$$w(t) = V(t - t_{n_0}), t \geq t_{n_0}, \tag{2.86}$$

for $t \in [0, t_{n_0} + \tilde{T}]$. We know that w is a mild solution for

$$\tilde{w}(t) = S_T \tilde{w}(0) + \int_0^t S_{t-s}(\varphi - gf(\tilde{u}(s)))ds \tag{2.87}$$

for any $t \in [0, t_{n_0} + \tilde{T}]$. As a result

$$\lim_{t \to t_{max}} \|u(t)\| = \infty \tag{2.88}$$

but that is a contradiction due to the fact that our solution u is bounded. As a result, we have that t_{max} is not finite.

3. Integrability and asymptotic behavior

We will start this section using the time-monotonicity of the solution u of (1) in order to obtain a solution v of (2).

Theorem 3.1. *Let $d > \alpha$ and suppose that*

(i) $\varphi \in C(\mathbb{R}^d) \cap B(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *with* $\varphi \geq 0$, *and*

(ii) $f : [0, \infty) \to \mathbb{R}$ *is convex and satisfies* $f(0) = 0$.

Then there is a solution $u_\infty \in \overline{B}_{\|\psi\|_\infty+1}(0)$ *of (2) satisfying the boundary condition* $\lim_{\|x\|\to\infty} u_\infty(x) = 0$. *If* $\alpha = 2$ *and* $d \in \{1, 2\}$ *we require that* $\varphi \in C_c(\mathbb{R}^d)$, $\liminf_{\|x\|\to\infty} g(x) > 0$, *and* $g(x) > 0$ *for all* $x \in \mathbb{R}^d$.

Proof.

- **Case 1:** $d > \alpha$. Consider (1) with $\psi \equiv 0$, and let v be the solution of

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) &= \Delta_\alpha v(t,x) - g(x)(\tilde{f})'(u(t,x))v(t,x), \\ v(0,x) &= \varphi(x), \end{cases} \tag{3.1}$$

where $\tilde{f}(x) = f(\max\{0, x\})$, for each $x \in \mathbb{R}$. We readily see that $\partial_t u(t,x) = v(t,x) \geq 0$. We start by bounding the following term

$$\begin{aligned} \left\| S_t V(0) - \int_0^t S_{t-s}(gf'(u(s))V(s)ds \right\|_\infty &\leq \|S_t V(0)\| + \int_0^t \|S_{t-s}(gf'(u(s))V(s))\| \, ds \\ &\leq \|V(s)\|_\infty + \int_0^t \|gf'(u(s))V(s)\| \, ds \\ &\leq \|\varphi\|_\infty + \int_0^t \|g\|_\infty \|f'(u(s))\| \, \|V(s)\| \, ds \\ &\leq \|\varphi\|_\infty + \|g\|_\infty \int_0^t \|f'\|_{[0,R]} \, R ds \\ &\leq \|\varphi\|_\infty + T \|g\|_\infty \|f'\|_{[0,R]} \, R. \end{aligned} \tag{3.2}$$

By taking $T = \frac{1}{\|g\|_\infty \|f'\|_{[0,R]} R}$, $R = \|\varphi\|_\infty + 1$ we have $\|(FV)(t)\| \leq R$. By using Banach's contraction principle we obtain T' time of existence of a solution V for our new equation. To prove the positivity of our new solution, we begin by defining

$$h(t) = \inf_{x \in \mathbb{R}^d} V(t, x). \tag{3.3}$$

From this definition we derive the following lemmas,

Lemma 3.2. *The function h defined above is continuous in $[0, T']$.*

Lemma 3.3. $\lim_{|x| \to \infty} \left| V(\tilde{t}, x) \right| = 0.$

The proofs of these lemma are similar to those in section 2.3. We will now suppose $h(\tilde{t}) < 0$. This implies there exists $M > 0$ such that

$$\left| V(\tilde{t}, x) \right| < -\frac{h(\tilde{t}}{2}. \tag{3.4}$$

for $|x| > M$. From this we can define

$$V(\tilde{t}, \tilde{x}) = \inf_{x \in \mathbb{R}^d} V(\tilde{t}, x) = h(\tilde{t}). \tag{3.5}$$

Let $\xi = V(\tilde{t}, \tilde{x}) < 0$. If $\tilde{t} = 0$ this would imply $\varphi(\tilde{x}) = V(0, \tilde{x}) = \xi < 0$. Thus we must take $\tilde{t}$ such that $\tilde{t} \in (0, T']$. Now we will define a new function $\tilde{h}(t) = \inf_{x \in \mathbb{R}^d} V'(t, x)$ with V' defined as

$$V'(t, x) = V(t, x) - K(\tilde{t} - t). \tag{3.6}$$

for $t \in [0, \tilde{t}]$ and $K = -\frac{\xi}{6\tilde{t}}$. By the continuity of V the continuity of V' follows immediately. Since V' is continuous in $[0, \tilde{t}]$ we can find $\hat{t} \in [0, \tilde{t}]$ such that

$$\tilde{h}(\hat{t}) = \inf_{t \in [0, \tilde{t}]} V'(t). \tag{3.7}$$

By 3.3 we can see that $\lim_{|x| \to \infty} V'(\tilde{t}, x) = -K(\tilde{t} - \hat{t})$. From this we can find $M > 0$ such that

$$V'(\hat{t}, x) > \frac{\xi}{6} - K(\tilde{t} - \hat{t}). \tag{3.8}$$

On the other hand we know that $\lim_{t \to \tilde{t}} V(t, \tilde{x}) = \xi$. Proceeding as before, this implies that $V(t, \tilde{x}) < \frac{\xi}{2}$ in $0 < t - \tilde{t} < \delta$ for some $\delta > 0$. This result leads to the following inequality

$$V(t, \tilde{x}) < \frac{\xi}{2} < -K\tilde{t} - (\frac{1}{3}(-\frac{\xi}{2})) = \frac{\xi}{3}. \tag{3.9}$$

Rearranging terms and using the definition of V' yield

$$V'(t, \tilde{x}) < \frac{\xi}{6} - K(\tilde{t} - \hat{t}). \tag{3.10}$$

Applying the definitions of $\hat{t}$ and of V' we have the following inequality

$$\tilde{h}(\hat{t}) \leq \tilde{h}(t) \leq V'(t, \tilde{x}) < \frac{\xi}{6} - K(\tilde{t} - \hat{t}). \tag{3.11}$$

As an direct consequence we have that $\inf_{x\in\mathbb{R}^d} V'(\hat{t},x) < \frac{\xi}{6} - K(\tilde{t}-\hat{t})$, this implies there exists $z \in \mathbb{R}^d$ such that $V'(\hat{t},z) < \frac{\xi}{6} - K(\tilde{t}-\hat{t})$. Now we note, if $\hat{t}=0$ this implies $V'(0,\hat{x}) < \frac{\xi}{6} < 0$, thus $\hat{t} \in (0,\tilde{t}]$. As a result we have

$$\frac{\partial}{\partial t}V'(\hat{t},\hat{x}) = \frac{\partial}{\partial t}V(\hat{t},\hat{x}) + K \leq 0. \tag{3.12}$$

Finally, this implies

$$0 > -K \geq \frac{\partial}{\partial t}V(\hat{t},\hat{x}) = \Delta_\alpha V(\hat{t},\hat{x}) - g(\hat{x})\tilde{f}(u(\hat{t},\hat{x})) + \varphi(\hat{x}) \geq 0. \tag{3.13}$$

Since $\hat{x}$ is global infimum of V'. As a result we have that our solution is non-negative. Now, considering the corresponding mild equation of (3.1), we deduce that

$$\frac{\partial}{\partial t}u(t,x) \leq S_t\varphi(x). \tag{3.14}$$

From Proposition 1.18 (ii), it follows that there exists $\delta > 1$ such that $S_t\varphi(x) \leq ct^{-d/\alpha}$, for all $x \in \mathbb{R}^d$ and $t \geq \delta$. If we take $t > s \geq \delta$ then (3.14) results in

$$u(t,x) - u(s,x) \leq \frac{\alpha c}{\alpha - d}\left(t^{1-d/\alpha} - s^{1-d/\alpha}\right). \tag{3.15}$$

Using now (1.34) and $d > \alpha$, we reach that $u(t,x) \leq c(\delta + \delta^{1-d/\alpha}$ holds for all $x \in \mathbb{R}^d$ and $t \geq \delta$. This implies that $\lim_{t\to\infty} u(t,x) =: u_\infty(x)$ exists and, moreover, one can easily prove that the convergence is uniform in x. On the other hand, using (3.14) and Proposition 1.18(ii) we see that $\lim_{t\to\infty}\partial_t u(t,x) = 0$ holds uniformly in x. Thus, $\lim_{t\to\infty}\Delta_\alpha u(t,x) = g(x)f(u_\infty(x)) - \varphi(x)$ is uniform in x. Since Δ_α is a closed operator by Proposition 1.27(ii), then $u_\infty \in D(\Delta_\alpha)$ is a solution of (2). Moreover $u_\infty \neq 0$ if $\varphi \neq 0$. The positivity of $u(t,x)$ implies then that

$$\limsup_{\|x\|\to\infty} u_\infty(x) \leq \limsup_{\|x\|\to\infty}\limsup_{t\to\infty}\int_0^t S_s\phi(x)ds. \tag{3.16}$$

Proposition 1.18(iv) implies that $\lim_{\|x\|\to\infty} u_\infty(x) = 0$.

- **Case 2:** $\alpha = 2$ and $d \in \{1,2\}$. We show firstly that

$$\lim_{\|x\|\to\infty} u(t,x) = 0 \text{ uniformly in } t > 0. \tag{3.17}$$

By hypothesis, $L = \liminf_{\|x\|\to\infty} g(x) > 0$. Recall that $R > 0$ is such that $\mathrm{supp}(\varphi) \subseteq \overline{B}(0,R)$ and $\inf\{g(x) : \|x\| > R\} \geq L/2$. Set $v(x) = \chi(\|x\|)$, where $\chi : [R,\infty) \to \mathbb{R}$ is the corresponding solution of (1.13). Using (1), for all $\|x\| > R$ and $t > 0$ the following holds:

$$\begin{aligned} \Delta(u(t,x) - v(x)) &\geq \frac{L}{2}f(u(t,x)) - \frac{L}{2}f(v(x)) - \frac{d-1}{\|x\|}\chi'(\|x\|) \\ &> \frac{L}{2}\left(f(u(t,x)) - f(v(x))\right). \end{aligned} \tag{3.18}$$

If there exists $(t_0,x_0) \in (0,\infty) \times (\mathbb{R}^d \backslash \overline{B}(0,R))$ such that $u(t_0,x_0) > v(x_0)$, then there is $\tilde{x}_0 \in$

$\overline{B}(x_0, \tilde{R})$ with $\tilde{R} = (\|x_0\| - R)/2$, for which

$$u(t_0, \tilde{x}_0) - v(\tilde{x}_0) = \max\{u(t_0, x) - v(x) : x \in \overline{B}(x_0, \tilde{R})\}. \tag{3.19}$$

As a consequence,

$$0 = \Delta(u(t_0, \tilde{x}_0) - v(\tilde{x}_0)) > \frac{L}{2}\left(f(u(t_0, \tilde{x}_0)) - f(v(\tilde{x}_0))\right), \tag{3.20}$$

and the strict monotonicity of f yields $0 > u(t_0, \tilde{x}_0) - v(\tilde{x}_0) \geq u(t_0, x_0) - v(x_0) > 0$. It follows that

$$u(t, x) \leq v(x), \text{ for all } t > 0 \text{ and } \|x\| > R. \tag{3.21}$$

The desired uniform limits follow then from Proposition 1.15. Take now $M > R$ and $t > 0$, and let $x_t \in \overline{B}(0, M)$ such that $u(t, x_t) = \max\{u(t, x) : \|x\| \leq M\}$. If $\|x_t\| = M$ then (3.21) implies that

$$u(t, x) \leq \max\{v(x) : \|x\| = M\}, \text{ for all } \|x\| \leq M. \tag{3.22}$$

In the case $\|x_t\| < M$, we readily note that $0 \leq \partial_t u(t, x_t) = \Delta u(t, x_t) - g(x_t) f(u(t, x)) + \varphi(x_t)$. Moreover, notice that

$$u(t, x) \leq f^{-1}\left(\frac{\|\varphi\|_\infty}{\inf\{g(x) : x \in \mathbb{R}^d\}}\right), \text{ for all } t > 0 \text{ and } \|x\| < M. \tag{3.23}$$

The strict positivity of the infimum follows now from the fact that $\inf\{g(x) : \|x\| > R\} \geq L/2$, and that $g(x) > 0$ for all $x \in \mathbb{R}^d$. On the other hand, the identity $\lim_{\|x\| \to \infty} v(x) = 0$ and the expressions (3.21), (3.22) and (3.23) imply that $\sup\{u(t, x) : t \geq 0, x \in \mathbb{R}^d\} < \infty$. Hence, $\lim_{t \to \infty} u(t, x) := u_\infty(x)$ exists in this case. Moreover, we claim that the convergence is uniform on $\|x\| \leq M$, with $M > R$. To prove that, we check firstly that $\{u(m, \cdot)\}_{m=1}^{\infty}$ is equicontinuous on $\overline{B}(0, M)$. Indeed, let $m \in \mathbb{N}$ be arbitrary but fixed, and let $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned} |u(m, x) - u(m, y)| &\leq \int_0^m \int_{\mathbb{R}^d} |\varphi(z) - g(z) f(u(s, z))| |p_\alpha(t - s, x - z) - p_\alpha(t - s, y - z)| dz ds \\ &\leq (\|\varphi\|_\infty + \|g\|_\infty f(c)) \int_0^m \int_{\mathbb{R}^d} |p_\alpha(s, x - y + z) - p_\alpha(s, z)| dz ds; \end{aligned} \tag{3.24}$$

the equicontinuity of $\{u(m, \cdot)\}_{m=1}^{\infty}$ follows now from Proposition 1.22. Since $\lim_{m \to \infty} u(m, x) = u_\infty(x)$, Theorem 7.5.6 in [3] implies that the convergence is uniform on $\overline{B}(0, M)$, so u is continuous on $\mathbb{R}^d$. Dini's theorem implies next that the convergence $\lim_{t \to \infty} u(t, x) = u_\infty(x)$ is uniform on $\overline{B}(0, M)$, for each $M > R$. Now, we will check that the convergence is uniform in $\mathbb{R}^d$. To that end, let $\varepsilon > 0$. By (3.17), there exits $M_\varepsilon > R$ such that

$$|u(t, x)| < \frac{\varepsilon}{2}, \text{ for all } t > 0 \text{ and } \|x\| > M_\varepsilon. \tag{3.25}$$

Moreover, the uniform convergence on $\overline{B}(0, M)$ guarantees that there exists $t_\varepsilon > 0$ for which

$$|u(t, x) - u_\infty(x)| < \frac{\varepsilon}{2}, \text{ for all } t > t_\varepsilon \text{ and } \|x\| \leq M_\varepsilon. \tag{3.26}$$

Propositions (3.25) and (3.26) yield $|u(t,x)-u_\infty(x)|\le\varepsilon$, for all $t>t_\varepsilon$ and $x\in\mathbb{R}^d$. To complete the proof, we proceed now as in Case 1. $\square$

Let $R>0$ be such that $\operatorname{supp}(\varphi)\subseteq\overline{B}(0,R)$ and $\inf\{g(x):\|x\|>R\}\ge L/2$. Define the function

$$\boldsymbol{F}(t)=\int_t^{\chi(R)}\frac{ds}{(F(s))^{1/2}},\quad 0<t\le\chi(R), \tag{3.27}$$

where χ is the solution of (1.13) and

$$F(t)=\int_0^t f(s)ds,\quad t\ge 0. \tag{3.28}$$

From (3.21), we readily check that $u_\infty(x)\le\chi(\|x\|)$, for all $\|x\|\ge R$. The following result provides other explicit bounds.

Theorem 3.4. *Suppose that*

(i) $\psi,\varphi\in L^1(\mathbb{R}^d)$,

(ii) $\psi\in D(\Delta_\alpha)\cap B(\mathbb{R}^d)$ *and* $\psi\ge 0$,

(iii) $\varphi,g\in C(\mathbb{R}^d)\cap B(\mathbb{R}^d)$ *and* $\varphi,g\ge 0$, *and*

(iv) $f:[0,\infty)\to\mathbb{R}$ *is convex with* $f(0)=0$.

and let $\alpha=2$ *and* $d\in\{1,2\}$. *If* $f'(0+)>0$ *then*

$$u(t,x)\le\exp\left(-\frac{1}{2}\|x\|\sqrt{f'(0+)}\right),\ \textit{for all } \|x\|\ge R \textit{ and } t>0. \tag{3.29}$$

If $f'(0+)=0$ *then*

$$u(t,x)\le\boldsymbol{F}^{-1}(\sqrt{3}\|x\|),\ \textit{for all } \|x\|\ge R \textit{ and } t>0. \tag{3.30}$$

As a consequence, if $f'(0+)>0$ *then* u_∞ *is always integrable. However, if* $f'(0+)=0$ *then* u_∞ *is integrable when* $\int_R^\infty\boldsymbol{F}^{-1}(r)r^{d-1}dr<\infty$.

Proof. An application of L'Hôpital's rule yields

$$\lim_{r\to\infty}\left(\frac{\chi'(r)}{\chi(r)}\right)^2=\lim_{r\to\infty}\frac{f(\chi(r))}{\chi(r)}. \tag{3.31}$$

On the other hand, the convexity of f implies that $\lim_{x\downarrow 0}f(x)/x=f'(0+)\ge 0$.

- **Case 1:** $f'(0+)>0$. Since $\lim_{r\to\infty}\chi'(r)/\chi(r)=-\sqrt{f'(0+)}$, there is $M>0$ large enough such that
$$\exp\left(-\frac{3}{2}r\sqrt{f'(0+)}\right)<\chi(r)<\exp\left(-\frac{1}{2}r\sqrt{f'(0+)}\right),\ \text{for all } r>M. \tag{3.32}$$
The integrability of u readily follows now.

- **Case 2:** $f'(0+)=0$. By L'Hôspital's rule, $\lim_{r\to\infty}(\chi'(r))^2/F(\chi(r))=2$. So, for $M>0$ large enough,
$$-\sqrt{3}\,(F(\chi(r)))^{1/2}<\chi'(r)<-(F(\chi(r)))^{1/2},\ \text{for all } r\ge M. \tag{3.33}$$

The comparison lemma for ordinary differential equations yields $y(r) \le \chi(r) \le z(r)$, for all $r \ge M$. Here,

$$\begin{aligned} y'(r) &= -\left(F(y(r))\right)^{1/2}, \quad y(M) = \chi(M), && (3.34) \\ z'(r) &= -\sqrt{3}\left(F(z(r))\right)^{1/2}, \quad z(M) = \chi(M). && (3.35) \end{aligned}$$

Moreover, the solutions are given by $y(r) = \boldsymbol{F}^{-1}(r)$ and $z(r) = \boldsymbol{F}^{-1}(\sqrt{3}r)$. We may use now L'Hôspital's rule to see that $\lim_{t\downarrow 0} \boldsymbol{F}(t) = \infty$, so that y and z are well defined for all $r \ge M$. To conclude the proof, it suffices to observe that (3.21) implies that $u(t,x) \le v(x) = \chi(\|x\|)$, for all $t > 0$ and $\|x\| > R$. □

Conclusions and discussions

In this work, we studied the temporal monotonicity of the positive solutions of the parabolic semilinear partial differential equation (1). Moreover, we used our results to establish the existence of solutions for the elliptic equation (2). In particular, we observed that the source term f is fundamental in order to guarantee the integrability of the solutions. In fact, the ordinary differential equation (1.13) only involves such term. That fact was noted in one of the examples of this manuscript, in which the integrability of u_∞ only requires to impose conditions on β. Ultimately, all the goals established at the beginning of this work were met and, more importantly, a new and innovative approach was given in several of the arguments presented throughout this work. This work represents a contribution in the study of partial differential equations with fractional diffusion. Currently, there has been an increase in interest in the study and applications of partial differential equations with this operator. As one of the problems which remain open after the conclusion of this work, it is still pending to check if the integrability conditions for u_∞ are necessary. Additionally, the case when $d = 1$ and $\alpha \in [1, 2)$ still needs to be studied, and new techniques must be developed to tackle that case. This is an interesting problem since the solution can be used to see how much time an $(\alpha, 1, \beta)$-superprocess spends in bounded Borel subset of $\mathbb{R}^d$, see [14].

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