UNIVERSIDAD AUTONOMA DE AGUASCALIENTES

CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL DIFFUSION: NUMERICAL METHODS AND APPLICATIONS TO IMAGE DENOISING

PRESENTA

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PARA OPTAR POR EL GRADO DE MAESTRO EN CIENCIAS EN MATEMÁTICAS APLICADAS

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FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
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PRESENTE

Por medio de la presente, en mi calidad de tutor designado del estudiante **JOEL ALBA PÉREZ** con ID 128308 quien realizó la tesis titulada: **PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL DIFFUSION: NUMERICAL METHODS AND APPLICATIONS TO IMAGE DENOISING**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

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JOEL ALBA PEREZ
MAESTRIA EN CIENCIAS CON OPCION A
COMPUTACION, MATEMATICAS APLICADAS

Estimado alumno:

Por medio de este conducto me permito comunicar a Usted que habiendo recibido los votos aprobatorios de los revisores de su trabajo de tesis y/o caso práctico titulado: **"PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL DIFFUSION: NUMERICAL METHODS AND APPLICATIONS TO IMAGE DENOISING"** hago de su conocimiento que puede imprimir dicho documento y continuar con los trámites para la presentación de su examen de grado.

Sin otro particular me permito saludarle muy afectuosamente.

ATENTAMENTE
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Resumen

En este manuscrito, trabajamos con las ecuaciones de Burgers–Fisher y Burgers–Huxley en multiples dimensiones, las cuales son ecuaciones diferenciales parabólicas. En estas ecuaciones tomamos el término de difusión y advección como fraccionario de tipo Riesz, y el término de reacción como no lineal. Consideramos condiciones iniciales y de frontera como positivas y acotadas. Existen soluciones analíticas de estas ecuaciones que son del tipo onda viajera, positivas y acotadas. Proponemos dos métodos basados en diferencias finitas para aproximar las soluciones de estas ecuaciones. El primer método es un método implícito el cual está basado en la técnica de Crank–Nikolson. El segundo método es un método explicito el cual está basado en la técnica de Bhattacharya. Ambos métodos se basan en el uso de las diferencias centradas fraccionarias, ya que éstas permiten aproximar la derivada fraccionaria de Riesz. Para cada método se estudian sus propiedades estructurales (existencia, unicidad, positividad y acotación) como sus propiedades numéricas (consistencia, estabilidad y convergencia). Por último, para cada método se hacen simulaciones, con el objetivo de ilustrar las aproximaciones a las soluciones analíticas y, además, mostrar que los métodos son capaces de preservar sus propiedades estructurales y numéricas.

Abstract

In this manuscript, we work with the well–known Burgers–Fisher and Burgers–Huxley equations in multiple dimensions, which are parabolic differential equations. In these equations, the diffusion and advection terms are fractional of Riesz type, and the reaction term is nonlinear. We consider the initial–boundary conditions as positive and bounded. We know that some analytical solutions of these equations are traveling–wave solutions, positive and bounded. We propose two methods based on finite differences to approximate the solutions of these equations. The first method is an implicit method which is based on the Crank–Nicolson technique. The second method is an explicit method which is based on the Bhattacharya approach. Both methods are based on the use of fractional centered differences, which help us to approximate the Riesz fractional derivatives. For each method, we study the structural properties (existence, uniqueness, positivity and boundedness) and the numerical properties (consistency, stability, and convergence). Finally, for each method, we perform some simulations to depict the numerical approximations. Moreover, we show that all methods are capable to preserve the structural properties.

Introduction

Aims and scope

Partial differential equations have been a fundamental piece in the scientific and technological development in the world. They allow us, in between other things, to study the structure and the behavior of diverse physical phenomena in the nature. In this thesis, we work with Burgers' equation who was proposed as a model of turbulent fluid motion by J. M. Burgers in several articles [1]. We can obtained the Burgers' equation as result of combining nonlinear wave motion with linear diffusion and is the simplest model for analyzing combined effect of nonlinear advection and diffusion [2]. In recent years, there has been an interest in the Burgers' equation, since, this equatuion is found naturally in a number of diverse contexts [3]. Is for this reason that this equation is used to study the gas dynamics [4] , traffic flow [5], acoustic [6], heat conduction [7], among others physical phenomena.

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order [8]. Recently, considerable interest in fractional calculus has been stimulated by the applications found in numerical analysis and different areas of physics and engineering [8]. Notable contributions to the fractional calculus were made successively by Laplace, Fourier, Abel, Liouville, Riemann, Heaviside and – in the present century – by Bateman, Hardy, Weyl, Riesz, and Courant, as well as by many pure and applied mathematicians of lesser reknown [9]. Due to their contributions, we count with the Riemann–Liouville, Marchaud–Hadamard, Weyl, Riesz, and Grünwald–Letnikov fractional derivatives in space who are defined in [10] and the Caputo fractional derivative in time who is defined in [11]. In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function, called fractional differential equations [12].

The study of physical phenomena with long-range interactions has been an interesting topic in last years due to their several applications. We can find applications in optical solitons [13], molecular dynamics [14], among others. We follow the work made by Vasily E. Tarasov [15]. We considered a discrete interacting particle system in one dimension, that is modeled by a discrete motion equation. The objective is to obtain the continuous motion equation, where we can find the Riesz fractional derivative. In this thesis, we use the Riesz fractional derivative due to his nature, moreover, this derivative allows us to preserve the properties and features of the physical phenomena.

Image processing is an important and wide topic of investigation in computer science due to several different methods to restore the information on a digital picture. Moreover, the work of Perona and Malik in anisotropic diffusion was the cornerstone for more development of this topic [16]. In this thesis, we are interested in methods based on partial differential equations to image denoising since are methods that preserve edges and features of the image. However, sometimes the use of these methods presents drawbacks due to the selection of parameters in the partial differential equation, in the discretization and the numerical scheme that we will use [17].

The purpose of this thesis is to apply a method based on Burgers–Fisher and Burger–Huxley equations in multiple dimensions where the diffusion, convection, and advection terms are fractionals of Riesz type to image denoising. Moreover, we consider positive and bounded initial–boundary conditions, and the rectangular domain is closed and bounded. We work with two discrete methods, one based in Crank–Nikolson implicit technique and other in Bhattacharya explicit technique, both, in finite differences. For each technique, we show the structural properties (numerical solutions exists and are unique, positive and bounded). Also, we show the numerical properties. The method with Crank–Nikolson technique has quadratic consistency, is stable and has quadratic convergence. The method with Bhattacharya technique has linear consistency, is stable ans has linear convergence. Finally, we implement the methods in different images to appreciate the obtained results.

Work Organization

This thesis is sectioned as follows.

- Chapter 1 provides a way to obtain the Riesz fractional derivative in space. Indeed, we work with a discrete system of interacting particles, where the distance between particles is uniform and the particles are in fact oscillators. We can model the dynamic of this system using a discrete equation of motion. Subsequently, we apply a serie of operations to discrete equation motion to obtain a continuous equation motion, where we can find the Riesz fractional derivative in space.

- Chapter 2 presents two partial differential equations, the well know Burgers–Fisher and Burgers–Huxley equations in two dimensions, where the diffusion and advection terms are fractionals of Riesz type, and the initial–boundary conditions are positive and bounded. We use a linear three–steps Crank–Nikolson technique with fractional centered differences to get approximations to analytical solutions [18] which are positive and bounded. It is worth mentioning that the finite–difference method preserves the positivity and the boundedness of the approximations. We show that our technique has quadratic consistency, is stable and has quadratic convergence order.

- Chapter 3 shows a parabolic equation where the diffusion and advection terms are fractionals of Riesz type. We work with the generalization of the Burgers–Fisher and Burgers–Huxley models. We want to approximate the solutions of this models using a variable–step Bhattacharya–type finite–difference scheme with the fractional centered differences. This technique is explicit since we obtain the solutions in an easy way and preserves the positivity, boundedness and the monotonicity of the approximations as the analytical solutions. Also, this technique is consistency, stable and convergent.

- This thesis closes with a section of conclusions for each chapter and a list of relevant references.

1. Preliminaries

1.1 Introduction

Derivatives are one of the most important tools in mathematics, which were developed by Isaac Newton and Gottfried Leibniz. Derivatives allow us study easy physical models as complicate physical models using partial differential equations. Also, there exists models based in partial differential equations to image restoration, as example we have the diffusion equation [19], [20].

Equations which involve derivatives or integrals of noninteger order are very successful in describing anomalous kinetics and transport and continuous time random walks. Usually, the fractional equations for dynamics or kinetics apper as some phenomenological models. Recently, a method to obtain fractional analogues of equations of motion was considered for sets of coupled particles with long-range interaction. Examples of systems with interacting oscillators, spin or waves are used for many applications in physics [21] , chemistry [22] and biology [23].

In this chapter we consider a physical discrete system of interacting particles, where the system is modeled by a discrete motion equation. The aim is to obtain the continuous motion equation of this physical system where we can find the Riesz fractional derivative in space, using the continuous limit process. The continuous limit process is a transform that involves the Fourier series transform, the limit when the distance between particle tends to zero and the inverse Fourier transform. We define the concept of α-interactions, since the α-interaction give us the order in the Riesz fractional derivative. Also, we generalize the physical discrete system in the three-dimension case to obtain the continuous analogous.

1.2 Transform Operation

In this subsection we define the discrete model which describes the dynamic of the oscillators. Also, we follow the concept of Fourier series transform and the inverse Fourier transform to define a transform operation, this transform operation will help us to get the continuous model analogous to discrete model.

Let $t > 0$, we consider a system of interacting particles called oscillators whose dynamics is described by the equations of motion

$$\frac{du_n}{dt}(t) = I_n(u(t)) + F(u_n(t)) \tag{1.1}$$

for all $n \in \mathbb{Z}$, where u_n are displacements from the equilibrium. The term $F(u_n)$ which represent an interaction of the oscillators u_n with an external force, the term $I_n(u)$ for linear long-range interaction is defined by

$$I_n(u) \equiv \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} J(n,m)[u_n - u_m] \tag{1.2}$$

where $J \in L^2(\mathbb{Z})$ describes the dynamical between particles.

We consider a wide class of interactions (1.2) that create a possibility of presenting the continuous medium equations with fractional derivatives. We need to define the operation which transforms the discrete model (1.1) for $u_n(t)$ into continuous medium equation for $u(x,t)$. We assume that $u_n(t)$ are Fourier coefficients of some function $\hat{u}(k,t)$. Then we define the field $\hat{u}(k,t)$ on $[-K/2, K/2]$ as

$$\hat{u}(k,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{-ikx_n} = \mathcal{F}_{\Delta}\{u_n(t)\}, \tag{1.3}$$

where $x_n = n\Delta x$, $\Delta x = 2\pi/K$ is the distance between oscillators and

$$u_n(t) = \frac{1}{K}\int_{-K/2}^{K/2} \hat{u}(k,t) e^{ikx_n} dk = \mathcal{F}_{\Delta}^{-1}\{\hat{u}(k,t)\}. \tag{1.4}$$

These equations are the basis for the Fourier transform, which is obtained by transforming from a discrete variable to a continuous one in the limit $\Delta x \longrightarrow 0$ $(K \longrightarrow \infty)$. The Fourier transform can be derived from (1.3) and (1.4) in the limit as $\Delta x \longrightarrow 0$. Replace the discrete $u_n(t) = (2\pi/K)u(x_n,t)$ with continuous $u(x,t)$ while letting $x_n = n\Delta x = 2\pi n/K \longrightarrow x$. Then change the sum to an integral and equations (1.3), (1.4) become

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx = \mathcal{F}\{u(x,t)\}, \tag{1.5}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx} dk = \mathcal{F}^{-1}\{\tilde{u}(k,t)\}. \tag{1.6}$$

Here,

$$\tilde{u}(k,t) = \mathcal{L}\tilde{u}(k,t) \tag{1.7}$$

and $\mathcal{L}$ denotes the passage to the limit $\Delta x \longrightarrow 0$ $(K \longrightarrow \infty)$. Note that $\tilde{u}(k,t)$ is a Fourier transform of the field $u(x,t)$ and $\hat{u}(k,t)$ is a Fourier series transform of $u_n(t)$, where we can use $u_n(t) = (2\pi/K)u(n\Delta x, t)$. The function $\tilde{u}(k,t)$ can be derived from $\hat{u}(k,t)$ in the limit $\Delta x \longrightarrow 0$.

The map of a discrete model into the continuous one can be defined by the transform operation.

Definition 1.1. We define the transform operation $\hat{T}$ as a combination of the following operations:

1. The Fourier series transform:

$$\mathcal{F}_{\Delta} : u_n(t) \longrightarrow \hat{u}(k,t) \tag{1.8}$$

2. The passage to the limit $\Delta x \longrightarrow 0$:

$$\mathcal{L} : \hat{u}(k,t) \longrightarrow \tilde{u}(k,t) \tag{1.9}$$

3. The inverse Fourier transform:

$$\mathcal{F}^{-1} : \tilde{u}(k,t) \longrightarrow u(x,t) \tag{1.10}$$

Then, the operation

$$\hat{T} = \mathcal{F}^{-1}\mathcal{L}\mathcal{F}_{\Delta} \tag{1.11}$$

is called a transform operation, since it performs a transform of a discrete model of coupled oscillators into the continuous medium model.

1.3 From Discrete to Continuous Equation

The main aim in this subsection is to define the concept of α-interaction to obtain the continuous equation with Riesz fractional derivative from discrete equation (1.1), using the transform operator (1.11) previously defined. Let us consider the interparticle interaction that is described by (1.2), then for $J \in L^2(\mathbb{Z})$ and for all $m,n \in \mathbb{Z}$, the term $J(n,m)$ satisfies the condition

$$J(n,m) = J(n-m) = J(m-n), \qquad \sum_{k=-\infty}^{\infty} |J(k)|^2 < \infty \tag{1.12}$$

and note that $J(-k) = J(k)$ for all $k \in \mathbb{Z}$.

Definition 1.2. The interaction terms (1.2) and (1.12) in the equation of motion (1.1) are called α-interaction if the function

$$\hat{J}_\alpha(k) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-ikn} J(n) = 2\sum_{n=1}^{\infty} J(n)\cos(kn) \tag{1.13}$$

satisfies the condition

$$\lim_{k\longrightarrow 0} \frac{|\hat{J}_\alpha(k) - \hat{J}_\alpha(0)|}{|k|^\alpha} = A_\alpha \tag{1.14}$$

where $\alpha > 0$ and $0 < |A_\alpha| < \infty$.

Condition (1.14) means that $\hat{J}_\alpha(k) - \hat{J}_\alpha(0) = O(|k|^\alpha)$, i.e.

$$\hat{J}_\alpha(k) - \hat{J}_\alpha(0) = A_\alpha |k|^\alpha + R_\alpha(k), \tag{1.15}$$

for $k \longrightarrow 0$, where

$$\lim_{k\longrightarrow 0} \frac{R_\alpha(k)}{|k|^\alpha} = 0 \tag{1.16}$$

Proposition 1.1. *The transform operation $\hat{T}$ maps the discrete equations of motion*

$$\frac{du_n(t)}{dt} = \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} J(n,m)[u_n(t) - u_m(t)] + F(u_n(t)) \tag{1.17}$$

with noninteger α-interaction into the fractional continuous medium equations:

$$\frac{\partial}{\partial t}u(x,t) - G_\alpha A_\alpha \frac{\partial^\alpha}{\partial|x|^\alpha}u(x,t) - F(u(x,t)) = 0 \tag{1.18}$$

where $\partial^\alpha/\partial|x|^\alpha$ *is the Riesz fractional derivative and* $G_\alpha = |\Delta x|^\alpha$ *is a finite parameter.*

Proof. To derive the equation for the field $\hat{u}(k,t)$, we multiply equation (1.17) by $e^{-ikn\Delta x}$, and summing over n from $-\infty$ to ∞. Then,

$$\sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}\frac{du_n(t)}{dt} = \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} e^{-ikn\Delta x}J(n,m)[u_n - u_m] + \sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}F(u_n) \tag{1.19}$$

The left-hand side of (1.19) gives

$$\sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}\frac{du_n(t)}{dt} = \frac{d}{dt}\sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}u_n(t) = \frac{d\hat{u}(k,t)}{dt} \tag{1.20}$$

where $\hat{u}(k,t)$ is defined by (1.3). The second term on the right-hand side of (1.19) is

$$\sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}F(u_n) = \mathcal{F}_\Delta\{F(u_n)\}. \tag{1.21}$$

The first term on the right-hand side of (1.19) is

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} e^{-ikn\Delta x}J(n,m)[u_n - u_m] &= \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} e^{-ikn\Delta x}J(n,m)u_n \\ &\quad - \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} e^{-ikn\Delta x}J(n,m)u_m.\end{aligned} \tag{1.22}$$

The first term on the right-hand side of (1.22) gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left[\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} e^{-ikn\Delta x}J(n,m)u_n\right] &= \sum_{n=-\infty}^{\infty}\left[e^{-ikn\Delta x}u_n\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty} J(n,m)\right] \\ &= \sum_{n=-\infty}^{\infty}\left[e^{-ikn\Delta x}u_n\sum_{\substack{m'=-\infty\\ m'\neq 0}}^{\infty} J(m')\right] \\ &= \sum_{n=-\infty}^{\infty} e^{-ikn\Delta x}u_n\hat{J}_\alpha(0) \\ &= \hat{u}(k,t)\hat{J}_\alpha(0),\end{aligned} \tag{1.23}$$

since $m,n\in\mathbb{Z}$, we can take $m'\in\mathbb{Z}$ such that $m = m'+n$, and where we use (1.12) with $J(n,m'+n) =$

$J(m')$, the transform (1.3) and

$$\hat{J}_\alpha(k\Delta x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-ikn\Delta x} J(n) = \mathcal{F}_\Delta\{J(n)\}. \tag{1.24}$$

Similarly, the second term on the right-hand side of (1.22) gives

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\left[\sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} e^{-ikn\Delta x} J(n,m) u_m\right] &= \sum_{m=-\infty}^{\infty}\left[\sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} e^{-ikn\Delta x} J(n,m) u_m\right] \\
&= \sum_{m=-\infty}^{\infty} u_m \left[\sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} e^{-ikn\Delta x} J(n,m)\right] \\
&= \sum_{m=-\infty}^{\infty} u_m \left[\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{\infty} e^{-ik(n'+m)\Delta x} J(n')\right] \\
&= \sum_{m=-\infty}^{\infty} u_m \left[\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{\infty} e^{-ikn'\Delta x} e^{-ikm\Delta x} J(n')\right] \\
&= \sum_{m=-\infty}^{\infty} u_m e^{-ikm\Delta x} \left[\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{\infty} e^{-ikn'\Delta x} J(n')\right] \\
&= \sum_{m=-\infty}^{\infty} u_m e^{-ikm\Delta x} \hat{J}_\alpha(k\Delta x) \\
&= \hat{u}(k,t)\hat{J}_\alpha(k\Delta x)
\end{aligned} \tag{1.25}$$

since $m, n \in \mathbb{Z}$, we can take $n' \in \mathbb{Z}$ such that $n = n' + m$, and where we use (1.12) with $J(n'+m, m) = J(n')$, the transform (1.3) and (1.24).
As a result, equation (1.19) has the form

$$\frac{d\hat{u}(k,t)}{dt} = [\hat{J}_\alpha(0) - \hat{J}_\alpha(k\Delta x)]\hat{u}(k,t) + \mathcal{F}_\Delta\{F(u_n)\}. \tag{1.26}$$

where $\mathcal{F}_\Delta\{F(u_n)\}$ is an operator notation for the Fourier series transform of $F(u_n)$. □

The Fourier series transform $\mathcal{F}_\Delta$ of (1.17) gives (1.26). We will be interested in the limit $\Delta x \longrightarrow 0$. Using (1.1), equation (1.26) can be written as

$$\frac{d\hat{u}(k,t)}{dt} - [-A_\alpha |k\Delta x|^\alpha - R_\alpha(k\Delta x)]\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n)\} = 0$$

then

$$\frac{d\hat{u}(k,t)}{dt} - |\Delta x|^\alpha [-A_\alpha |k|^\alpha - R_\alpha(k\Delta x)|\Delta x|^{-\alpha}]\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n)\} = 0$$

Finally, we have

$$\frac{d\hat{u}(k,t)}{dt} - G_\alpha \hat{\mathcal{T}}_{\alpha,\Delta}(k)\hat{u}(k,t) - \mathcal{F}_\Delta\{F(u_n)\} = 0 \tag{1.27}$$

where we use finite parameter $G_\alpha = |\Delta x|^\alpha$ and

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha - R_\alpha(k\Delta x)|\Delta x|^{-\alpha}. \tag{1.28}$$

Note that R_α satisfies the condition

$$\lim_{\Delta x \longrightarrow 0} \frac{R_\alpha(k\Delta x)}{|\Delta x|^\alpha} = 0.$$

The expression for $\hat{\mathcal{T}}_{\alpha,\Delta}(k)$ can be considered as a Fourier transform of the operator (1.2).

The passage to the limit $\Delta x \longrightarrow 0$ for the third term of (1.27) gives

$$\mathcal{L} : \mathcal{F}_\Delta F(u_n) \longrightarrow \mathcal{L}\mathcal{F}_\Delta F(u_n). \tag{1.29}$$

Then,

$$\mathcal{L}\mathcal{F}_\Delta\{F(u_n)\} = \mathcal{F}\{\mathcal{L}F(u_n)\} = \mathcal{F}\{F(\mathcal{L}u_n)\} = \mathcal{F}\{F(u(x,t))\} \tag{1.30}$$

where we use $\mathcal{L}\mathcal{F}_\Delta = \mathcal{F}\mathcal{L}$.

As a result, equation (1.27) in the limit $\Delta x \longrightarrow 0$ obtains

$$\frac{\partial \tilde{u}(k,t)}{\partial t} - G_\alpha \hat{\mathcal{T}}_\alpha(k)\tilde{u}(k,t) - \mathcal{F}\{F(u(x,t))\} = 0, \tag{1.31}$$

where

$$\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t), \qquad \hat{\mathcal{T}}_\alpha(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -A_\alpha |k|^\alpha.$$

The inverse Fourier transform of (1.31) gives

$$\frac{\partial u(x,t)}{\partial t} - G_\alpha \mathcal{T}_\alpha(x) u(x,t) - F(u(x,t)) = 0, \tag{1.32}$$

where $\mathcal{T}_\alpha(x)$ is an operator,

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = A_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}. \tag{1.33}$$

In (1.33), we used the connection [24] between the Riesz fractional derivative and its Fourier transform $|k|^\alpha \longleftrightarrow -\partial^\alpha/\partial|x|^\alpha$. As a result, we obtain continuous medium equations (1.18).

Example 1.1. In this example, we are interested to obtain the continuous motion equation where we can find the Riesz fractional derivative considering the interaction term coefficient as $J(n) = \frac{1}{n^2}$, then the α-interaction is defined as follows

$$J_\alpha(k) = 2\sum_{n=1}^{\infty} \frac{\cos(kn)}{n^2} = \frac{1}{6}(3k^2 - 6\pi k + 2\pi^2), \quad 0 \le k \le 2\pi$$

thus

$$A_\alpha = \lim_{k\longrightarrow 0} \frac{|J_\alpha(k) - J_\alpha(0)|}{|k|^\alpha} = \lim_{k \longrightarrow 0} \frac{k^2/2 - \pi k}{k^\alpha} = -\pi, \quad if \;\; \alpha = 1$$

and

$$\frac{\partial}{\partial t} u(x,t) + \pi G_1 \frac{\partial}{\partial |x|} u(x,t) - F(u(x,t)) = 0$$

therefore, we have the Riesz fractional derivative of $u(x,t)$ of order one with respect x.

$J(n)$	$\mathcal{T}_\alpha(x)$
$\left(\frac{(-1)^n\pi^{\alpha+1}}{\alpha+1}\right)-\frac{(-1)^n\pi^{1/2}}{(\alpha+1)\lvert n\rvert^{\alpha+1/2}}L_1(\alpha+3/2,1/2,\pi n)$	$-\partial^\alpha/\partial\lvert x\rvert^\alpha$
$\frac{(-1)^n}{n^2}$	$-(1/2)\partial^2/\partial x^2$
$\frac{1}{n^2}$	$-i\pi\partial/\partial x$
$\lvert n\rvert^{-(\beta+1)}$, $(0<\beta<2,\beta\neq 1)$	$-2\Gamma(-\beta)\cos(\pi\beta/2)\partial^\beta/\partial\lvert x\rvert^\beta$
$\lvert n\rvert^{-(\beta+1)}$, $(\beta>2,\beta\neq 3,4,\ldots)$	$\zeta(\beta-1)\partial^2/\partial x^2$
$\frac{(-1)^n}{\Gamma(1+\alpha/2+n)\Gamma(1+\alpha/2-n)}$, $(\alpha>-1/2)$	$-\frac{1}{\Gamma(\alpha+1)}\partial^\alpha/\partial\lvert x\rvert^\alpha$
$\frac{(-1)^n}{a^2-n^2}$	$-\frac{a\pi}{2\sin(a\pi)}\partial^2/\partial x^2$
$\frac{1}{n!}$	$-2e\partial^2/\partial x^2$

Table 1.1: Examples of Riesz fractional derivatives for some interaction terms coefficients. The first column is the interaction term coefficient and the second column is the coefficient and the Riesz fractional derivative.

Example 1.2. In this example, we are interested to obtain the continuous motion equation where we can find the Riesz fractional derivative considering the interaction term coefficient as $J(n)=\frac{1}{n!}$, then the α-interaction is defined as follows

$$J_\alpha(k)=2\sum_{n=1}^{\infty}\frac{\cos(kn)}{n!}=2\left[\sum_{n=0}^{\infty}\frac{\cos(kn)}{n!}-1\right]=2\left[e^{\cos(k)}\cos(\sin(k))-1\right]$$

thus

$$A_\alpha=\lim_{k\longrightarrow 0}\frac{|J_\alpha(k)-J_\alpha(0)|}{|k|^\alpha}=\lim_{k\longrightarrow 0}\frac{2e^{\cos(k)}\cos(\sin(k))-2e}{k^2}=-2e, \quad if \quad \alpha=2$$

and

$$\frac{\partial}{\partial t}u(x,t)+2eG_2\frac{\partial^2}{\partial|x|^2}u(x,t)-F(u(x,t))=0$$

therefore, we have the Riesz fractional derivative of $u(x,t)$ of order two with respect x.

In table 1.1, we can find more examples about the Riesz fractional derivative. We are considering more interaction term coefficients with its corresponding Riesz fractonal derivative.

1.4 Fractional three-dimensional lattice equation

In this part we consider the generalization of our system of interacting particles. We work the three-dimensional case obtaining the continuous model with Riesz fractional derivative in each variable. Then the dynamic of the particles is described by equations of motion

$$\frac{du_{\mathbf{n}}}{dt}(t)=\sum_{\substack{\mathbf{m}=-\infty\\ \mathbf{m}\neq\mathbf{n}}}^{\infty}J(\mathbf{n},\mathbf{m})[u_{\mathbf{n}}-u_{\mathbf{m}}](t)+F(u_{\mathbf{n}}(t)) \tag{1.34}$$

for all $t > 0$ and $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, and $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$ where the coefficient satisfies $J(\mathbf{n}, \mathbf{m}) = J(\mathbf{n} - \mathbf{m}) = J(\mathbf{m} - \mathbf{n})$. Analogously, we suppose that $u_{\mathbf{n}}(t)$ are Fourier coefficients of the function $\hat{u}(\mathbf{k}, t)$:

$$\hat{u}(\mathbf{k}, t) = \sum_{\mathbf{n}=-\infty}^{\infty} u_{\mathbf{n}}(t) e^{-i\mathbf{k}\mathbf{x}_{\mathbf{n}}} = \mathcal{F}_{\Delta}\{u_{\mathbf{n}}(t)\}, \tag{1.35}$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and

$$\mathbf{r}_{\mathbf{n}} = \sum_{i=1}^{3} n_i \mathbf{a}_i. \tag{1.36}$$

Here, $\mathbf{a}_i$ are the translational vectors of the lattice. The continuous medium model can be derived in the limit $|\mathbf{a}_i| \longrightarrow 0$.

To derive the equation for $\hat{u}(\mathbf{k}, t)$, we multiply (1.34) by $e^{-i\mathbf{k}\mathbf{r}_{\mathbf{n}}}$, and summing over $\mathbf{n}$. Then, we obtain

$$\frac{d\hat{u}(\mathbf{k}, t)}{dt} = [\hat{J}_{\alpha}(0) - \hat{J}_{\alpha}(\mathbf{k}\mathbf{a})]\hat{u}(\mathbf{k}, t) + \mathcal{F}_{\Delta}\{F(u_{\mathbf{n}})\}, \tag{1.37}$$

where $\mathcal{F}_{\Delta}\{F(u_{\mathbf{n}})\}$ is an operator notation for the Fourier series transform of $F(u_{\mathbf{n}})$ and

$$\hat{J}_{\alpha}(\mathbf{k}\mathbf{a}) = \sum_{\mathbf{n}=-\infty}^{\infty} e^{-i\mathbf{k}\mathbf{r}_{\mathbf{n}}} J(\mathbf{n}). \tag{1.38}$$

Definition 1.3. For the three-dimensional lattice, we say that (1.38) is a α-interaction with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, if satisfies the conditions

$$\lim_{k_i \longrightarrow 0} \frac{|\hat{J}_{\alpha}(\mathbf{k}) - \hat{J}_{\alpha}(0)|}{|k_i|^{\alpha_i}} = A_{\alpha_i}, \quad i = 1, 2, 3, \tag{1.39}$$

where $0 < |A_{\alpha_i}| < \infty$.

Condition (1.39) means that

$$\hat{J}_{\alpha}(\mathbf{k}) - \hat{J}_{\alpha}(0) = \sum_{i=1}^{3} A_{\alpha_i} |k_i|^{\alpha_i} + \sum_{i=1}^{3} R_{\alpha_i}(\mathbf{k}) \tag{1.40}$$

where

$$\lim_{k_i \longrightarrow 0} \frac{R_{\alpha_i}(\mathbf{k})}{|k_i|^{\alpha_i}} = 0. \tag{1.41}$$

Proposition 1.2. *The transform operation $\hat{T}$ defined in the previous section maps the discrete equations of motion*

$$\frac{du_{\boldsymbol{n}}}{dt}(t) = \sum_{\substack{\boldsymbol{m}=-\infty \\ \boldsymbol{m}\neq\boldsymbol{n}}}^{\infty} J(\boldsymbol{n}, \boldsymbol{m})[u_{\boldsymbol{n}} - u_{\boldsymbol{m}}](t) + F(u_{\boldsymbol{n}}(t)) \tag{1.42}$$

with noninteger α-interaction into the fractional continuous medium equations:

$$\frac{\partial u(\boldsymbol{r}, t)}{\partial t} = -\sum_{i=1}^{3} A_{\alpha_i} \frac{\partial^{\alpha_i} u(\boldsymbol{r}, t)}{\partial |x|^{\alpha_i}} + F(u(\boldsymbol{r}, t)). \tag{1.43}$$

where we obtain the Riesz fractional derivative in each variable $\partial^{\alpha_1}/\partial x^{\alpha_1}$, $\partial^{\alpha_2}/\partial y^{\alpha_2}$ and $\partial^{\alpha_3}/\partial z^{\alpha_3}$.

Proof. The proof of this proposition is analogous to proof of proposition 1.1. □

1.5 Conclusion

In this chapter, we worked with a discrete system of particles which was modeled by a discrete motion equation. We defined a transform operation which help us to transform the discrete motion equation to a continuos motion equation analogous. We defined the concept of $\alpha - intercation$. In the continuos motion equation we can find the Riesz fractional derivative of order α in space. We extended the discrete system of particles to the three–dimensional case. This chapter is important since we obtained the Riesz fractional derivative naturally, moreover, this derivative allows to keep the features and structure of the physical phenomena that we are working.

2. A method for anomalously convective and diffusive problems

This chapter is motivated by a generalization of the well-known Burgers–Fisher and Burgers–Huxley equations in multiple dimensions, considering Riesz fractional diffusion and convection. Initial-boundary conditions, which are positive and bounded, are imposed on a closed and bounded rectangular domain. In this chapter we propose a finite-difference method to approximate the positive and bounded solutions of the fractional model. The methodology is a linear three-steps Crank–Nicolson technique which is based on the use of fractional centered differences. The properties of fractional centered differences are employed to establish the existence and the uniqueness of solutions of the finite-difference method, as well as the capability of the technique to preserve the positivity and the boundedness of the approximations. We show in this chapter that the method is capable of preserving some of the constant solutions of the continuous model. Additionally, we prove that our technique is a second-order consistent, stable and quadratically convergent scheme. Suitable bounds for the numerical solutions are also derived in this work. Finally, some illustrative simulations show that the method is able to preserve the positivity and the boundedness of the numerical approximations, in agreement with the analytic results proved in this chapter.

2.1 Introduction

Considering the before chapter, the use of Riesz fractional derivatives in the modeling of physical problems through partial differential equations is justified mathematically in the continuous limit of certain particle systems. Moreover, various fractional models from science and engineering are also capable of preserving some physical quantities. As examples, we may consider some gradient and Hamiltonian extensions of the Hemlholtz conditions for phase space and some fractional equivalents of the Fokker–Planck equation for fractal media [25], continuous-limit approximations of systems of coupled oscillators with power-law interactions [26] and mathematical models with fractional dynamics resulting in optimal control theory [27]. It is important to point out that some of these quantities are fractional forms of Hamiltonians [25], whence a natural direction of investigation in scientific computing is the design of new computational techniques that preserve the relevant quantities of a physical system described by fractional partial differential equations. It is worth pointing out that this task has been accomplished recently for fractional hyperbolic partial differential equations that extend the well known sine-Gordon and nonlinear Klein–Gordon models from relativistic quantum mechanics, which are models for which a Hamiltonian function exists [28].

The literature also has reports of methods for fractional partial differential equations that do not necessarily preserve the structure of the solutions, but most of the methods proposed are numerically efficient techniques. For example, some highly accurate numerical schemes have been proposed for multi-dimensional space variable-order fractional Schrödinger equations [29] and some techniques have been used to approximate the solutions of Riesz fractional advection-dispersion equations [30]. Other approximation methods based on Legendre polynomials have been designed to solve the fractional two-dimensional heat conduction equation [31], to approximate the solutions of the multi-term time-fractional wave-diffusion equation [32], to solve numerically the two-dimensional variable-order fractional percolation equation in non-homogeneous porous media [33], to estimate the solutions of $(3+1)$-dimensional generalized fractional KdV–Zakharov–Kuznetsov equations through an improved fractional sub-equation method [34] and to solve fractional sub-diffusion equations with variable coefficients [35]. As a conclusion, many reports show that the development of numerical techniques to solve fractional partial differential equations has been a fruitful avenue of research, but few reports have striven to design structure-preserving techniques for those systems.

In this work, the notion of 'structure preservation' not only refers to the capability of numerical methods to preserve analogues of physical quantities. More generally, these concepts also refer to the capacity of a computational technique to preserve mathematical features of the relevant solutions of continuous systems. Such features may naturally arise from the physical context of the problem. A typical example is the condition of positivity (or non-negativity) of the solutions, which is a natural requirement for problems in which the variables of interest are measured in absolute scales [36]. Other characteristics include the boundedness [37], the monotonicity [38] and the convexity of approximations [39]. In the present work, we will consider an initial-boundary-value problem governed by a multidimensional parabolic equation with Riesz fractional diffusion and convection. The problem is a generalization of various equations from mathematical physics, including the well known Burgers–Fisher and the Burgers–Huxley models, which are equations for which there exist positive and bounded solutions under suitable conditions. In this manuscript, we will propose a structure-preserving and numerically efficient technique to approximate the solutions of that model using fractional centered differences.

2.2 Preliminaries

2.2.1 Mathematical model

Throughout this work, we suppose that $a, b, c, d \in \mathbb{R}$ satisfy $a < b$ and $c < d$, and we assume that $T > 0$. Assume that $B = (a, b) \times (c, d)$ and define $\Omega = B \times (0, T)$. We will employ $\overline{B}$ and $\overline{\Omega}$ to represent respectively the closures of B and Ω under the standard topology of $\mathbb{R}^3$, and we will use ∂B to denote the boundary of B. In this manuscript, $u : \overline{\Omega} \to \mathbb{R}$ will represent a function, and let $x = (x_1, x_2)$.

Definition 2.1. Let $\alpha > -1$ and suppose that n is a nonnegative integer such that $n - 1 < \alpha \leq n$. The *Riesz fractional derivatives* of u of order α with respect to x_1 and with respect to x_2 at the point (x, t)

are defined respectively by

$$\frac{\partial^\alpha u}{\partial|x_1|^\alpha}(x,t) = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)}\frac{\partial^n}{\partial x_1^n}\int_a^b \frac{u(\xi,x_2,t)d\xi}{|x_1-\xi|^{\alpha+1-n}}, \quad \forall(x,t)\in\Omega, \tag{2.1}$$

$$\frac{\partial^\alpha u}{\partial|x_2|^\alpha}(x,t) = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)}\frac{\partial^n}{\partial x_2^n}\int_c^d \frac{u(x_1,\xi,t)d\xi}{|x_2-\xi|^{\alpha+1-n}}, \quad \forall(x,t)\in\Omega. \tag{2.2}$$

Here Γ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds, \quad \forall z>0. \tag{2.3}$$

For the remainder of this work, we will use α_i, β_i and λ_i to represent nonnegative real numbers such that $1<\alpha_i\leq 2$, $0<\beta_i<1$ for each $i\in\{1,2\}$. Let $p\geq 0$, and let $\phi:\overline{B}\to\mathbb{R}$ and $\psi:\partial B\times[0,T]\to\mathbb{R}$ be functions whose ranges are subsets of some closed and bounded interval $I\subseteq\mathbb{R}$. Assume additionally that the compatibility condition $\phi(x)=\psi(x,0)$ holds for each $x\in\partial B$. With these conventions, the problem under consideration in this work is the nonlinear initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) = \sum_{i=1}^{2}\frac{\partial^{\alpha_i}u}{\partial|x_i|^{\alpha_i}}(x,t) + u^p(x,t)\sum_{i=1}^{2}\lambda_i\frac{\partial^{\beta_i}u}{\partial|x_i|^{\beta_i}}(x,t) + u(x,t)f(u(x,t)), \quad \forall(x,t)\in\Omega, \\ \text{such that } \begin{cases} u(x,0)=\phi(x), & \forall x\in\overline{B}, \\ u(x,t)=\psi(t), & \forall(x,t)\in\partial B\times[0,T]. \end{cases} \end{aligned} \tag{2.4}$$

Here, f is in general a real-valued function defined on some open subinterval of $[0,\rho]$, for some $\rho>0$. For practical purposes, we may assume that $u(x,t)\in[0,\rho]$ for each $(x,t)\in\overline{\Omega}$, and that f has the form

$$f(u(x,t)) = 1-u^p(x,t), \quad \forall(x,t)\in\overline{\Omega}, \tag{2.5}$$

or

$$f(u(x,t)) = (1-u^p(x,t))(u^p(x,t)-\gamma), \quad \forall(x,t)\in\overline{\Omega}. \tag{2.6}$$

The partial differential equation (2.4) is a convection-diffusion-reaction model that generalizes many particular equations from mathematical physics. For instance, the convection term is a generalized form of the corresponding term in the classical Burgers' equation [40]. Meanwhile, the reaction factor (2.5) was studied independently and simultaneously by R. A. Fisher [41] and A. N. Kolmogorov, I. G. Petrovskii and N. Piskunov [42] in 1937 in the context of population dynamics, and the reaction law (2.6) is a form of the Hodgkin–Huxley regime appearing in some studies on the electric activity of nerves [43]. In view of these remarks, the model (2.4) with reaction factor f given by (2.5) is a fractional generalized Burgers–Fisher equation, while the equation (2.4) with reaction (2.6) is a fractional generalized Burgers–Huxley equation.

The following examples provide exact traveling-wave solutions of the one-dimensional Burgers–Huxley and the Burgers–Fisher equations when $p=1$. It is worth noting that these functions are positive and bounded.

Example 2.1 (Burgers–Fisher equation). The one-dimensional partial differential equation of (2.4) with reaction (2.5), $\alpha_1=2$, $\beta_1=1$, $\lambda=\lambda_1$ and $p=1$ has traveling-wave solutions which are bounded within the interval $(0,1)$, one example being the function

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\lambda}{4}\left(x - \left(\frac{\lambda}{2} + \frac{2}{\lambda}\right)t\right)\right], \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{2.7}$$

Clearly, this function is a traveling-wave front that connects asymptotically the stationary solutions $u = 0$ and $u = 1$ of our model. Moreover, for every fixed $x_0 \in \mathbb{R}$, the function $u(x_0, t)$ is monotone in time. Likewise, for every fixed $t_0 \in \mathbb{R}^+$, $u(x, t_0)$ is a monotone function in the variable x. □

Example 2.2 (Burgers–Huxley equation). A one-dimensional form of our Burgers–Huxley model has been investigated in the literature [18] when $\alpha_1 = 2$, $\beta_1 = 1$, $\lambda = \lambda_1$ and $p = 1$. In that case, the Burgers–Huxley model has the traveling-wave solutions

$$u_\pm(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left[c_\pm(x - v_1^\pm t)\right], \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.8}$$

and

$$u_\pm(x,t) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[c_\pm \gamma(x - v_2^\pm t)\right], \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.9}$$

where

$$c_\pm = \frac{\lambda \pm \sqrt{\lambda^2 + 8}}{8}, \tag{2.10}$$

$$v_1^\pm = -\frac{\lambda}{2} + \frac{(1 - 2\gamma)(\lambda \pm \sqrt{\lambda^2 + 8})}{4}, \tag{2.11}$$

$$v_2^\pm = -\frac{\gamma\lambda}{2} - \frac{(2 - \gamma)(\lambda \pm \sqrt{\lambda^2 + 8})}{4} \tag{2.12}$$

(see [44]). These two solutions are monotone fronts that are bounded within $(0,1)$ or within $(0,\gamma)$, respectively. □

2.2.2 Fractional centered differences

In this work, we follow a finite-difference approach to approximate the solutions of (2.4), and use fractional centered differences to approximate Riesz space-fractional derivatives. In the present section, we recall the definition of fractional centered differences and record their most important properties [45].

Definition 2.2. For any function $f : \mathbb{R} \to \mathbb{R}$, and any $h > 0$ and $\alpha > -1$, the *fractional centered difference* of order α of f at the point x is defined as

$$\Delta_h^{(\alpha)} f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{2.13}$$

where

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{Z}. \tag{2.14}$$

For computational purposes, it is convenient to possess an iterative formula to calculate the coefficients of the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$. Using induction one may readily check that

$$g_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2+1)^2}, \tag{2.15}$$

$$g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha+1}{\alpha/2+k+1}\right) g_k, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{2.16}$$

Lemma 2.1 (Wang *et al.* [46])**.** *If* $0 < \alpha \leq 2$ *and* $\alpha \neq 1$ *then the coefficients* $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ *satisfy:*

(a) $g_0^{(\alpha)} > 0$,

(b) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ *for all* $k \neq 0$, *and*

(c) $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$. *As a consequence, it follows that* $g_0^{(\alpha)} = -\displaystyle\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} g_k^{(\alpha)}$.

Lemma 2.2 (Wang *et al.* [46])**.** *Let* $f \in \mathcal{C}^5(\mathbb{R})$, *and assume that all its derivatives up to order five are integrable. If* $0 < \alpha \leq 2$ *and* $\alpha \neq 1$ *then, for almost all* x,

$$-\frac{\Delta_h^{\alpha} f(x)}{h^{\alpha}} = \frac{\partial^{\alpha} f(x)}{\partial |x|^{\alpha}} + \mathcal{O}(h^2). \tag{2.17}$$

2.3 Numerical method

2.3.1 Finite-difference scheme

Let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$, for each $q \in \mathbb{N}$. In this work, we will follow a finite-difference approach to solve the system (2.4). Let $K, M, N \in \mathbb{N}$, and define the spatial partition norms $h_{x_1} = (b-a)/M$ and $h_{x_2} = (d-c)/N$ in the x_1 and the x_2 directions, respectively. We will consider uniform partitions of the intervals $[a,b]$ and $[c,d]$, respectively, of the forms

$$a = x_{1,0} < x_{1,1} < \ldots < x_{1,m} < \ldots < x_{1,M} = b, \quad \forall m \in \overline{I}_M, \tag{2.18}$$

$$c = x_{2,0} < x_{2,1} < \ldots < x_{2,n} < \ldots < x_{2,N} = d, \quad \forall n \in \overline{I}_N. \tag{2.19}$$

Let $J = I_{M-1} \times I_{N-1}$, $\overline{J} = \overline{I}_M \times \overline{I}_N$ and $\partial J = \overline{J} \cap \partial B$. We will also fix a non-necessarily uniform partition of $[0, T]$ consisting of K subintervals, namely,

$$0 = t_0 < t_1 < \ldots < t_k < \ldots < t_K = T, \quad \forall k \in \overline{I}_K. \tag{2.20}$$

Define $\tau_k = t_{k+1} - t_k$, for each $k \in \overline{I}_{K-1}$. For each $n \in \overline{I}_N$ and $m \in \overline{I}_M$, let $v_{m,n}^k$ represent an approximation to the exact value of the solution u of (2.4) at $(x_{1,m}, x_{2,n}, t_k)$, and define $x_{m,n} = (x_{1,m}, x_{2,n})$ and $\phi_{m,n}^i = \phi^i(x_{m,n})$ for $i = 0, 1$. Here ϕ^i will represent the exact solution of (2.4) at the time t_i. Let $\psi_{m,n}^k = \psi(x_{m,n}, t_k)$ for each $(m,n) \in \partial J$ and $k \in \overline{I}_K$.

In the present work, we will employ the following discrete operators for each $(m,n) \in J$ and $k \in$

I_{K-1}:

$$\delta_t^{(1)} v_{m,n}^k = \frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{2\tau_k}, \tag{2.21}$$

$$\mu_t^{(1)} v_{m,n}^k = \frac{v_{m,n}^{k+1} + v_{m,n}^{k-1}}{2}. \tag{2.22}$$

Obviously, these operators provide second-order approximations of the partial derivative of u with respect to t at $(x_{m,n}, t_k)$ and the value of u at that point. Without loss of generality in the next proofs we consider a uniform partition of $[0, T]$.

Lemma 2.3. *Let $v \in \mathcal{C}^3(\overline{\Omega})$ where $\Omega = B \times (0, T)$ with $B = (a, b) \times (c, d)$, then:*

$$\delta_t^{(1)} v_{m,n}^k = \frac{\partial v(x_{1,n}, x_{2,m}, t_k)}{\partial t} + \mathcal{O}(\tau^2) \tag{2.23}$$

Proof. Expanding $v_{m,n}^{k+1}$ and $v_{m,n}^{k-1}$ in Taylor's series with center in t_k

$$\begin{aligned} v_{m,n}^{k+1} = {} & v(x_{1,m}, x_{2,n}, t_k) + \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}(t_{k+1} - t_k) + \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{(t_{k+1} - t_k)^2}{2!} \\ & + \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^*)}{\partial t^3}\frac{(t_{k+1} - t_k)^3}{3!} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} v_{m,n}^{k-1} = {} & v(x_{1,m}, x_{2,n}, t_k) + \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}(t_{k-1} - t_k) + \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{(t_{k-1} - t_k)^2}{2!} \\ & + \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^{**})}{\partial t^3}\frac{(t_{k-1} - t_k)^3}{3!} \end{aligned} \tag{2.25}$$

where t^* and t^{**} are in (t_k, t_{k+1}) and (t_{k-1}, t_k), respectively. Also, as $\tau = t_{k+1} - t_k$ and $\tau = t_k - t_{k-1}$ then in (2.24) and (2.25) we have:

$$\begin{aligned} v_{m,n}^{k+1} = {} & v(x_{1,m}, x_{2,n}, t_k) + \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}\tau + \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{\tau^2}{2!} \\ & + \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^*)}{\partial t^3}\frac{\tau^3}{3!} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} v_{m,n}^{k-1} = {} & v(x_{1,m}, x_{2,n}, t_k) - \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}\tau + \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{\tau^2}{2!} \\ & - \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^{**})}{\partial t^3}\frac{\tau^3}{3!} \end{aligned} \tag{2.27}$$

Subtracting (2.27) from (2.26) and dividing it by 2τ, we have:

$$\begin{aligned} \frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{2\tau} = {} & \frac{v(x_{1,m}, x_{2,n}, t_k)}{2\tau} + \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}\frac{\tau}{2\tau} + \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{\tau^2}{4\tau} \\ & + \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^*)}{\partial t^3}\frac{\tau^3}{12\tau} - \frac{v(x_{1,m}, x_{2,n}, t_k)}{2\tau} + \frac{\partial v(x_{1,m}, x_{2,n}, t_k)}{\partial t}\frac{\tau}{2\tau} \\ & - \frac{\partial^2 v(x_{1,m}, x_{2,n}, t_k)}{\partial t^2}\frac{\tau^2}{4\tau} + \frac{\partial^3 v(x_{1,m}, x_{2,n}, t^{**})}{\partial t^3}\frac{\tau^3}{12\tau} \end{aligned} \tag{2.28}$$

removing some terms, we have:

$$\frac{v_{m,n}^{k+1}-v_{m,n}^{k-1}}{2\tau}=\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}+\frac{\partial^3 v(x_{1,m},x_{2,n},t^*)}{\partial t^3}\frac{\tau^2}{12}+\frac{\partial^3 v(x_{1,m},x_{2,n},t^{**})}{\partial t^3}\frac{\tau^2}{12} \tag{2.29}$$

thus

$$\begin{aligned}\left|\frac{v_{m,n}^{k+1}-v_{m,n}^{k-1}}{2\tau}-\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\right| &= \left|\frac{\partial^3 v(x_{1,m},x_{2,n},t^*)}{\partial t^3}+\frac{\partial^3 v(x_{1,m},x_{2,n},t^{**})}{\partial t^3}\right|\frac{\tau^2}{12}\\ &\leq \left[\left|\frac{\partial^3 v(x_{1,m},x_{2,n},t^*)}{\partial t^3}\right|+\left|\frac{\partial^3 v(x_{1,m},x_{2,n},t^{**})}{\partial t^3}\right|\right]\frac{\tau^2}{12}\end{aligned} \tag{2.30}$$

since $\overline{\Omega}\subset\mathbb{R}^3$ is compact, then there exists a constant $\kappa>0$ such that $\left|\frac{\partial^3 v(x_1,x_2,t)}{\partial t^3}\right|\leq\kappa$ for all (x_1,x_2,t) in Ω then

$$\left|\frac{v_{m,n}^{k+1}-v_{m,n}^{k-1}}{2\tau}-\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\right|\leq(\kappa+\kappa)\frac{\tau^2}{12}=\frac{\kappa}{6}\tau^2 \tag{2.31}$$

therefore $\delta_t^{(1)}v_{m,n}^k$ is a second-order consistency operator.

□

Lemma 2.4. *Let $v\in\mathcal{C}^3(\overline{\Omega})$ where $\Omega=B\times(0,T)$ with $B=(a,b)\times(c,d)$, then:*

$$\mu_t^{(1)}v_{m,n}^k=v(x_{1,m},x_{2,n},t_k)+\mathcal{O}(\tau^2) \tag{2.32}$$

Proof. Expanding $v_{m,n}^{k+1}$ and $v_{m,n}^{k-1}$ in Taylor's series with center in t_k

$$v_{m,n}^{k+1}=v(x_{1,m},x_{2,n},t_k)+\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}(t_{k+1}-t_k)+\frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2}\frac{(t_{k+1}-t_k)^2}{2!} \tag{2.33}$$

and

$$v_{m,n}^{k-1}=v(x_{1,m},x_{2,n},t_k)+\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}(t_{k-1}-t_k)+\frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\frac{(t_{k-1}-t_k)^2}{2!} \tag{2.34}$$

where t^* and t^{**} are in (t_k,t_{k+1}) and (t_{k-1},t_k) respectively. Also, as $\tau=t_{k+1}-t_k$ and $\tau=t_k-t_{k-1}$ then in (2.33) and (2.34) we have:

$$v_{m,n}^{k+1}=v(x_{1,m},x_{2,n},t_k)+\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\tau+\frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2}\frac{\tau^2}{2!} \tag{2.35}$$

and

$$v_{m,n}^{k-1}=v(x_{1,m},x_{2,n},t_k)-\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\tau+\frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\frac{\tau^2}{2!} \tag{2.36}$$

if we do the sum of (2.35) and (2.36), divided by 2, we have:

$$\begin{aligned}\frac{v_{m,n}^{k+1}+v_{m,n}^{k-1}}{2} &= \frac{v(x_{1,m},x_{2,n},t_k)}{2}+\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\frac{\tau}{2}+\frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2}\frac{\tau^2}{4}\\ &\quad+\frac{v(x_{1,m},x_{2,n},t_k)}{2}-\frac{\partial v(x_{1,m},x_{2,n},t_k)}{\partial t}\frac{\tau}{2}+\frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\frac{\tau^2}{4}\end{aligned} \tag{2.37}$$

removing some terms, we have:

$$\frac{v_{m,n}^{k+1}+v_{m,n}^{k-1}}{2} = v(x_{1,m},x_{2,n},t_k) + \frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2}\frac{\tau^2}{4} + \frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\frac{\tau^2}{4} \tag{2.38}$$

thus

$$\begin{aligned}\left|\frac{v_{m,n}^{k+1}+v_{m,n}^{k-1}}{2} - v(x_{1,m},x_{2,n},t_k)\right| &= \left|\frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2} + \frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\right|\frac{\tau^2}{4} \\ &\leq \left[\left|\frac{\partial^2 v(x_{1,m},x_{2,n},t^*)}{\partial t^2}\right| + \left|\frac{\partial^2 v(x_{1,m},x_{2,n},t^{**})}{\partial t^2}\right|\right]\frac{\tau^2}{4}\end{aligned} \tag{2.39}$$

since $\overline{\Omega} \subset \mathbb{R}^3$ is compact, then there exists a constant $\kappa > 0$ such that $\left|\frac{\partial^2 v(x_1,x_2,t)}{\partial t^2}\right| \leq \kappa$ for all (x_1,x_2,t) in Ω then

$$\left|\frac{v_{m,n}^{k+1}+v_{m,n}^{k-1}}{2} - v(x_{1,m},x_{2,n},t_k)\right| \leq (\kappa+\kappa)\frac{\tau^2}{4} = \frac{\kappa}{2}\tau^2 \tag{2.40}$$

therefore $\mu_t^{(1)} v_{m,n}^k$ is a second-order consistency operator.

□

Let $0 < \alpha \leq 2$ with $\alpha \neq 1$. For each $(m,n) \in J$ and each $k \in \overline{I}_K$ we define the linear operators

$$\delta_{x_1}^{(\alpha)} v_{m,n}^k = -\frac{1}{h_{x_1}^\alpha}\sum_{j=0}^{M} g_{m-j}^{(\alpha)} v_{j,n}^k, \tag{2.41}$$

$$\delta_{x_2}^{(\alpha)} v_{m,n}^k = -\frac{1}{h_{x_2}^\alpha}\sum_{j=0}^{N} g_{n-j}^{(\alpha)} v_{m,j}^k. \tag{2.42}$$

In light of Lemma 2.2, these operators yield second-order approximations of the fractional derivatives of u of order α with respect to x_1 and x_2, respectively, at the point $(x_{m,n}, t_k)$. With this nomenclature, the finite-difference method to approximate the solutions of (2.4) is given by

$$\delta_t^{(1)} v_{m,n}^k = \sum_{i=1}^{2} \mu_t^{(1)} \delta_{x_i}^{(\alpha_i)} v_{m,n}^k + (v_{m,n}^k)^p \sum_{i=1}^{2} \lambda_i \mu_t^{(1)} \delta_{x_i}^{(\beta_i)} v_{m,n}^k + f(v_{m,n}^k)\mu_t^{(1)} v_{m,n}^k, \quad \forall (m,n,k) \in J \times I_{K-1},$$
$$\text{such that} \begin{cases} v_{m,n}^0 = \phi_{m,n}^0, & \forall (m,n) \in \overline{J}, \\ v_{m,n}^1 = \phi_{m,n}^1, & \forall (m,n) \in \overline{J}, \\ v_{m,n}^k = \psi_{m,n}^k, & \forall (m,n,k) \in \partial J \times \overline{I}_K, \end{cases} \tag{2.43}$$

2.3.2 Equivalent representations

The purpose of this section is to provide alternative representations of the finite-difference scheme (2.43). To that end, we will require additional nomenclature. Throughout this work we will convey that

$$R_{x_i}^{(\alpha)} = \tau_k h_{x_i}^{-\alpha}, \tag{2.44}$$

for each $i = 1,2$ and each $\alpha \in (0,1) \cup (1,2]$. Here, we are dropping the dependence of $R_{x_1}^{(\alpha)}$ and $R_{x_2}^{(\alpha)}$ on k for the sake of briefness. To build the alternative representation, first, we substitute the discrete operators (2.21) and (2.22) in (2.43), then:

$$\begin{aligned}\frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{2\tau_k} &= \mu_t^{(1)}\left[\frac{-1}{h_{x_1}^{\alpha_1}}\sum_{i=0}^{M} g_{m-i}^{(\alpha_1)} v_{i,n}^{k}\right] + \mu_t^{(1)}\left[\frac{-1}{h_{x_2}^{\alpha_2}}\sum_{i=0}^{N} g_{n-i}^{(\alpha_2)} v_{m,i}^{k}\right] + \lambda_1\left(v_{m,n}^{k}\right)^p \mu_t^{(1)}\left[\frac{-1}{h_{x_1}^{\beta_1}}\sum_{i=0}^{M} g_{m-i}^{(\beta_1)} v_{i,n}^{k}\right] \\ &\quad + \lambda_2\left(v_{m,n}^{k}\right)^p \mu_t^{(1)}\left[\frac{-1}{h_{x_2}^{\beta_2}}\sum_{i=0}^{N} g_{n-i}^{(\beta_2)} v_{m,i}^{k}\right] + f\left(v_{m,n}^{k}\right)\mu_t^{(1)} v_{m,n}^{k}\end{aligned} \tag{2.45}$$

reordering, we have:

$$\begin{aligned}\frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{2\tau_k} &= \frac{-1}{h_{x_1}^{\alpha_1}}\sum_{i=0}^{M} g_{m-i}^{(\alpha_1)}\left[\mu_t^{(1)} v_{i,n}^{k}\right] + \frac{-1}{h_{x_2}^{\alpha_2}}\sum_{i=0}^{N} g_{n-i}^{(\alpha_2)}\left[\mu_t^{(1)} v_{m,i}^{k}\right] + \lambda_1\left(v_{m,n}^{k}\right)^p \frac{-1}{h_{x_1}^{\beta_1}}\sum_{i=0}^{M} g_{m-i}^{(\beta_1)}\left[\mu_t^{(1)} v_{i,n}^{k}\right] \\ &\quad + \lambda_2\left(v_{m,n}^{k}\right)^p \frac{-1}{h_{x_2}^{\beta_2}}\sum_{i=0}^{N} g_{n-i}^{(\beta_2)}\left[\mu_t^{(1)} v_{m,i}^{k}\right] + f\left(v_{m,n}^{k}\right)\left[\mu_t^{(1)} v_{m,n}^{k}\right]\end{aligned} \tag{2.46}$$

applying the discrete operator $\mu_t^{(1)}$, we have:

$$\begin{aligned}\frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{2\tau_k} &= \frac{-1}{h_{x_1}^{\alpha_1}}\sum_{i=0}^{M} g_{m-i}^{(\alpha_1)}\left[\frac{v_{i,n}^{k+1} + v_{i,n}^{k-1}}{2}\right] - \frac{1}{h_{x_2}^{\alpha_2}}\sum_{i=0}^{N} g_{n-i}^{(\alpha_2)}\left[\frac{v_{m,i}^{k+1} + v_{m,i}^{k-1}}{2}\right] \\ &\quad - \lambda_1\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_1}^{\beta_1}}\sum_{i=0}^{M} g_{m-i}^{(\beta_1)}\left[\frac{v_{i,n}^{k+1} + v_{i,n}^{k-1}}{2}\right] - \lambda_2\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_2}^{\beta_2}}\sum_{i=0}^{N} g_{n-i}^{(\beta_2)}\left[\frac{v_{m,i}^{k+1} + v_{m,i}^{k-1}}{2}\right] \\ &\quad + f\left(v_{m,n}^{k}\right)\left[\frac{v_{m,n}^{k+1} + v_{m,n}^{k-1}}{2}\right]\end{aligned} \tag{2.47}$$

distributing some terms, we have:

$$\begin{aligned}\frac{v_{m,n}^{k+1} - v_{m,n}^{k-1}}{\tau_k} &= \frac{-1}{h_{x_1}^{\alpha_1}}\sum_{i=0}^{M} g_{m-i}^{(\alpha_1)} v_{i,n}^{k+1} - \frac{1}{h_{x_1}^{\alpha_1}}\sum_{i=0}^{M} g_{m-i}^{(\alpha_1)} v_{i,n}^{k-1} - \frac{1}{h_{x_2}^{\alpha_2}}\sum_{i=0}^{N} g_{n-i}^{(\alpha_2)} v_{m,i}^{k+1} - \frac{1}{h_{x_2}^{\alpha_2}}\sum_{i=0}^{N} g_{n-i}^{(\alpha_2)} v_{m,i}^{k-1} \\ &\quad - \lambda_1\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_1}^{\beta_1}}\sum_{i=0}^{M} g_{m-i}^{(\beta_1)} v_{i,n}^{k+1} - \lambda_1\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_1}^{\beta_1}}\sum_{i=0}^{M} g_{m-i}^{(\beta_1)} v_{i,n}^{k-1} \\ &\quad - \lambda_2\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_2}^{\beta_2}}\sum_{i=0}^{N} g_{n-i}^{(\beta_2)} v_{m,i}^{k+1}\, \lambda_2\left(v_{m,n}^{k}\right)^p \frac{1}{h_{x_2}^{\beta_2}}\sum_{i=0}^{N} g_{n-i}^{(\beta_2)} v_{m,i}^{k-1} \\ &\quad + f\left(v_{m,n}^{k}\right) v_{m,n}^{k+1} + f\left(v_{m,n}^{k}\right) v_{m,n}^{k-1}\end{aligned} \tag{2.48}$$

defining as $R_{x_i}^{(\alpha)} = \tau_k h_{x_i}^{-\alpha}$, we have:

$$
\begin{aligned}
v_{m,n}^{k+1} - v_{m,n}^{k-1} \;=\; & -R_{x_1}^{(\alpha_1)} \sum_{i=0}^{M} g_{m-i}^{(\alpha_1)} v_{i,n}^{k+1} - R_{x_1}^{(\alpha_1)} \sum_{i=0}^{M} g_{m-i}^{(\alpha_1)} v_{i,n}^{k-1} - R_{x_2}^{(\alpha_2)} \sum_{i=0}^{N} g_{n-i}^{(\alpha_2)} v_{m,i}^{k+1} - R_{x_2}^{(\alpha_2)} \sum_{i=0}^{N} g_{n-i}^{(\alpha_2)} v_{m,i}^{k-1} \\
& -\lambda_1 \left(v_{m,n}^{k}\right)^p R_{x_1}^{(\beta_1)} \sum_{i=0}^{M} g_{m-i}^{(\beta_1)} v_{i,n}^{k+1} - \lambda_1 \left(v_{m,n}^{k}\right)^p R_{x_1}^{(\beta_1)} \sum_{i=0}^{M} g_{m-i}^{(\beta_1)} v_{i,n}^{k-1} \\
& -\lambda_2 \left(v_{m,n}^{k}\right)^p R_{x_2}^{(\beta_2)} \sum_{i=0}^{N} g_{n-i}^{(\beta_2)} v_{m,i}^{k+1} - \lambda_2 \left(v_{m,n}^{k}\right)^p R_{x_2}^{(\beta_2)} \sum_{i=0}^{N} g_{n-i}^{(\beta_2)} v_{m,i}^{k-1} \\
& +\tau_k f\left(v_{m,n}^{k}\right) v_{m,n}^{k+1} + \tau_k f\left(v_{m,n}^{k}\right) v_{m,n}^{k-1}
\end{aligned}
\tag{2.49}
$$

reording the terms, in the left-side the term with $v_{m,n}^{k+1}$ and in the right-side the term with $v_{m,n}^{k-1}$ in each sum, thus:

$$
\begin{aligned}
& \left[1 - \tau_k f\left(v_{m,n}^{k}\right) + \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} + \left(v_{m,n}^{k}\right)^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)}\right] v_{m,n}^{k+1} \\
+ & \sum_{i=0, i\neq m}^{M} \left[\lambda_1 \left(v_{m,n}^{k}\right)^p R_{x_1}^{(\beta_1)} g_{m-i}^{(\beta_1)} + R_{x_1}^{(\alpha_1)} g_{m-i}^{(\alpha_1)}\right] v_{i,n}^{k+1} + \sum_{i=0, i\neq n}^{N} \left[\lambda_2 \left(v_{m,n}^{k}\right)^p R_{x_2}^{(\beta_2)} g_{n-i}^{(\beta_2)} + R_{x_2}^{(\alpha_2)} g_{n-i}^{(\alpha_2)}\right] v_{m,i}^{k+1} \\
= & \left[1 + \tau_k f\left(v_{m,n}^{k}\right) - \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} - \left(v_{m,n}^{k}\right)^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)}\right] v_{m,n}^{k-1} \\
- & \sum_{i=0, i\neq m}^{M} \left[\lambda_1 \left(v_{m,n}^{k}\right)^p R_{x_1}^{(\beta_1)} g_{m-i}^{(\beta_1)} + R_{x_1}^{(\alpha_1)} g_{m-i}^{(\alpha_1)}\right] v_{i,n}^{k-1} - \sum_{i=0, i\neq n}^{N} \left[\lambda_2 \left(v_{m,n}^{k}\right)^p R_{x_2}^{(\beta_2)} g_{n-i}^{(\beta_2)} + R_{x_2}^{(\alpha_2)} g_{n-i}^{(\alpha_2)}\right] v_{m,i}^{k-1}
\end{aligned}
\tag{2.50}
$$

now, for each $(m,n) \in J$ and $k \in I_{K-1}$, let us define

$$
\mathfrak{a}_{m,n}^{k} = 1 - \tau_k f(v_{m,n}^{k}) + \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} + (v_{m,n}^{k})^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)}, \tag{2.51}
$$

$$
\mathfrak{b}_{m,n,j}^{k,x_i} = \begin{cases} -\lambda_1 R_{x_1}^{(\beta_1)} g_{m-j}^{(\beta_1)} (v_{m,n}^{k})^p - R_{x_1}^{(\alpha_1)} g_{m-j}^{(\alpha_1)}, & \text{if } i = 1 \text{ and } j \in \overline{I}_M \\ -\lambda_2 R_{x_2}^{(\beta_2)} g_{n-j}^{(\beta_2)} (v_{m,n}^{k})^p - R_{x_2}^{(\alpha_2)} g_{n-j}^{(\alpha_2)}, & \text{if } i = 2 \text{ and } j \in \overline{I}_N, \end{cases} \tag{2.52}
$$

$$
\mathfrak{c}_{m,n}^{k} = 1 + \tau_k f(v_{m,n}^{k}) - \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} - (v_{m,n}^{k})^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)} = 2 - \mathfrak{a}_{m,n}^{k}. \tag{2.53}
$$

It is easy to see that for each $(m,n,k) \in J \times I_{K-1}$, the following hold:

$$
\mathfrak{a}_{m,n}^{k} = 1 - \tau_k f(v_{m,n}^{k}) - \mathfrak{b}_{m,n,m}^{k,x_1} - \mathfrak{b}_{m,n,n}^{k,x_2}, \tag{2.54}
$$

$$
\mathfrak{c}_{m,n}^{k} = 1 + \tau_k f(v_{m,n}^{k}) + \mathfrak{b}_{m,n,m}^{k,x_1} + \mathfrak{b}_{m,n,n}^{k,x_2}, \tag{2.55}
$$

$$
\mathfrak{c}_{m,n}^{k} = 2 - \mathfrak{a}_{m,n}^{k}. \tag{2.56}
$$

With this notation, the difference equations of (2.43) may be equivalently rewritten as the following Crank–Nicolson-type linear system, whose recursive equations are valid for each $(m,n,k) \in J \times I_{K-1}$:

$$\mathfrak{a}^k_{m,n} v^{k+1}_{m,n} - \sum_{\substack{j=0 \\ j\neq m}}^{M} \mathfrak{b}^{k,x_1}_{m,n,j} v^{k+1}_{j,n} - \sum_{\substack{j=0 \\ j\neq n}}^{N} \mathfrak{b}^{k,x_2}_{m,n,j} v^{k+1}_{m,j} = \mathfrak{c}^k_{m,n} v^{k-1}_{m,n} + \sum_{\substack{j=0 \\ j\neq m}}^{M} \mathfrak{b}^{k,x_1}_{m,n,j} v^{k-1}_{j,n} + \sum_{\substack{j=0 \\ j\neq n}}^{N} \mathfrak{b}^{k,x_2}_{m,n,j} v^{k-1}_{m,j},$$
$$\text{such that} \begin{cases} v^0_{m,n} = \phi^0_{m,n}, & \forall (m,n) \in \overline{J}, \\ v^1_{m,n} = \phi^1_{m,n}, & \forall (m,n) \in \overline{J}, \\ v^k_{m,n} = \psi^k_{m,n}, & \forall (m,n,k) \in \partial J \times \overline{I}_K, \end{cases} \tag{2.57}$$

Alternatively, a matrix representation of (2.57) is readily at hand. To that end, for each $k \in \overline{I}_K$ we order lexicographically the set $\{v^k_{m,n} : (m,n) \in \overline{J}\}$, along with the initial and the boundary conditions, into the $(M+1)\times(N+1)$-dimensional real vectors v^k, v_0, v_1 and ψ^k, respectively, for each $k \in \overline{I}_K$. More precisely, for each $j \in \overline{I}_M$ define the $(N+1)$-dimensional vectors

$$v^k_j = (v^k_{j,0}, v^k_{j,1}, \ldots, v^k_{j,N-1}, v^k_{j,N})^\top, \tag{2.58}$$

$$\phi^i_j = (\phi^i_{j,0}, \phi^i_{j,1}, \ldots, \phi'_{j,N-1} \phi^i_{j,N})^\top, \quad i = 0,1, \tag{2.59}$$

$$\psi^k_j = \begin{cases} (\psi^k_{j,0}, \psi^k_{j,1}, \ldots, \psi^k_{j,N-1}, \psi^k_{j,N})^\top, & \text{if } j \in \{0, M\}, \\ (\psi^k_{j,0}, 0, \ldots, 0, \psi^k_{j,N})^\top, & \text{if } j \in I_{M-1}. \end{cases} \tag{2.60}$$

Then

$$v^k = v^k_0 \oplus v^k_1 \oplus \cdots \oplus v^k_{M-1} \oplus v^k_M, \tag{2.61}$$

$$\phi_i = \phi^i_0 \oplus \phi^i_1 \oplus \cdots \oplus \phi^i_{M-1} \oplus \phi^i_M, \quad i = 0,1, \tag{2.62}$$

$$\psi^k = \psi^k_0 \oplus \psi^k_1 \oplus \cdots \oplus \psi^k_{M-1} \oplus \psi^k_M. \tag{2.63}$$

where $\oplus$ represents the vector operation of juxtaposition.

Let I represent the identity matrix of size $(N+1)\times(N+1)$, and define the following real matrices of sizes $(N+1)\times(N+1)$:

$$B^k_{m,j} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mathfrak{b}^{k,x_1}_{m,1,j} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{b}^{k,x_1}_{m,2,j} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathfrak{b}^{k,x_1}_{m,N-2,j} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathfrak{b}^{k,x_1}_{m,N-1,j} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \tag{2.64}$$

$$C_m^k = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\mathfrak{b}_{m,1,0}^{k,x_2} & \mathfrak{a}_{m,1}^{k} & -\mathfrak{b}_{m,1,2}^{k,x_2} & \cdots & -\mathfrak{b}_{m,1,N-1}^{k,x_2} & -\mathfrak{b}_{m,1,N}^{k,x_2} \\ -\mathfrak{b}_{m,2,0}^{k,x_2} & -\mathfrak{b}_{m,2,1}^{k,x_2} & \mathfrak{a}_{m,2}^{k} & \cdots & -\mathfrak{b}_{m,2,N-1}^{k,x_2} & -\mathfrak{b}_{m,2,N}^{k,x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathfrak{b}_{m,N-2,0}^{k,x_2} & -\mathfrak{b}_{m,N-2,1}^{k,x_2} & -\mathfrak{b}_{m,N-2,2}^{k,x_2} & \cdots & -\mathfrak{b}_{m,N-2,N-1}^{k,x_2} & -\mathfrak{b}_{m,N-2,N}^{k,x_2} \\ -\mathfrak{b}_{m,N-1,0}^{k,x_2} & -\mathfrak{b}_{m,N-1,1}^{k,x_2} & -\mathfrak{b}_{m,N-1,2}^{k,x_2} & \cdots & \mathfrak{a}_{m,N-1}^{k} & -\mathfrak{b}_{m,N-1,N}^{k,x_2} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.65}$$

Let A^k be the $(M+1)(N+1) \times (M+1)(N+1)$ block matrix defined by

$$A^k = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -B_{1,0}^k & C_1^k & -B_{1,2}^k & \cdots & -B_{1,M-1}^k & -B_{1,M}^k \\ -B_{2,0}^k & -B_{2,1}^k & C_2^k & \cdots & -B_{2,M-1}^k & -B_{2,M}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -B_{M-2,0}^k & -B_{M-2,1}^k & -B_{M-2,2}^k & \cdots & -B_{M-2,M-1}^k & -B_{M-2,M}^k \\ -B_{M-1,0}^k & -B_{M-1,1}^k & -B_{M-1,2}^k & \cdots & C_{M-1}^k & -B_{M-1,M}^k \\ 0 & 0 & 0 & \cdots & 0 & I \end{pmatrix}. \tag{2.66}$$

Here the zeros represent zero matrices of sizes $(N+1) \times (N+1)$.

On the other hand, let us define the matrix $D_{m,j}^k$ of size $(N+1) \times (N+1)$ by

$$D_{m,j}^k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathfrak{b}_{m,1,0}^{k,x_2} & \mathfrak{c}_{m,1}^{k} & \mathfrak{b}_{m,1,2}^{k,x_2} & \cdots & \mathfrak{b}_{m,1,N-2}^{k,x_2} & \mathfrak{b}_{m,1,N-1}^{k,x_2} & \mathfrak{b}_{m,1,N}^{k,x_2} \\ \mathfrak{b}_{m,2,0}^{k,x_2} & \mathfrak{b}_{m,2,1}^{k,x_2} & \mathfrak{c}_{m,2}^{k} & \cdots & \mathfrak{b}_{m,2,N-2}^{k,x_2} & \mathfrak{b}_{m,2,N-1}^{k,x_2} & \mathfrak{b}_{m,2,N}^{k,x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathfrak{b}_{m,N-2,0}^{k,x_2} & \mathfrak{b}_{m,N-2,1}^{k,x_2} & \mathfrak{b}_{m,N-2,2}^{k,x_2} & \cdots & \mathfrak{c}_{m,N-2}^{k} & \mathfrak{b}_{m,N-2,N-1}^{k,x_2} & \mathfrak{b}_{m,N-2,N}^{k,x_2} \\ \mathfrak{b}_{m,N-1,0}^{k,x_2} & \mathfrak{b}_{m,N-1,1}^{k,x_2} & \mathfrak{b}_{m,N-1,2}^{k,x_2} & \cdots & \mathfrak{b}_{m,N-1,N-2}^{k,x_2} & \mathfrak{c}_{m,N-1}^{k} & \mathfrak{b}_{m,N-1,N}^{k,x_2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{2.67}$$

In turn, we introduce the $(M+1)(N+1) \times (M+1)(N+1)$ block matrix

$$E^k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ B_{1,0}^k & D_1^k & B_{1,2}^k & \cdots & B_{1,M-1}^k & B_{1,M}^k \\ B_{2,0}^k & B_{2,1}^k & D_2^k & \cdots & B_{2,M-1}^k & B_{2,M}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{M-2,0}^k & B_{M-2,1}^k & B_{M-2,2}^k & \cdots & B_{M-2,M-1}^k & B_{M-2,M}^k \\ B_{M-1,0}^k & B_{M-1,1}^k & B_{M-1,2}^k & \cdots & D_{M-1}^k & B_{M-1,M}^k \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.68}$$

With this notation, the finite-difference method (2.43) can be in vector form as the recursive system

$$A^k v^{k+1} = E^k v^{k-1} + \psi^k, \quad \forall k \in I_{K-1}, \quad \text{such that} \begin{cases} v^0 = v_0, \\ v^1 = v_1. \end{cases} \tag{2.69}$$

In the following sections, we will establish the main properties of this technique. Among other features, we will establish the existence and the uniqueness of the numerical solutions under suitable conditions of the parameters. Structural and numerical properties of (2.69) will be proved also, and numerical simulations based on an implementation of (2.69) will illustrate the validity of our results.

2.4 Structural properties

In this stage of our work, we prove the most important structural properties of the finite-difference method (2.43). Concretely, we will show that the method has a unique solution under appropriate conditions, and that the method is capable of preserving the positivity and the boundedness of the approximations. The cornerstone of our discussion will be the concept of Minkowski matrices, which are defined next.

Definition 2.3. A square real matrix is a *Z-matrix* if all its off-diagonal entries are less than or equal to zero.

Definition 2.4. We say that a square real matrix A is a *Minkowski matrix* if the following three properties are satisfied:

(i) A is a Z-matrix,

(ii) all the diagonal entries of A are positive, and

(iii) there exists a diagonal matrix D with positive diagonal elements, such that AD is strictly diagonally dominant.

Definition 2.5. We say that a (not necessarily square) real matrix A is *nonnegative* if every entry of A is a nonnegative number; such fact will be denoted by $A \geq 0$. If ρ is any real number, we say that A is *bounded from above* by ρ if every entry of A is less than or equal to ρ, a fact that will be represented by $A \leq \rho$. If $\rho > 0$ then we use the notation $0 \leq A \leq \rho$ to represent that $A \geq 0$ and $A \leq \rho$.

Obviously, an n-dimensional real vector v satisfies $v \leq \rho$ if and only if $\rho e - v > 0$, where e is the n-dimensional vector all of whose components are equal to 1. In our investigation, Minkowski matrices will be important in view that they are nonsingular. Moreover, if A is a Minkowski matrix then $A^{-1} \geq 0$ (see [47] and references therein).

Lemma 2.5. *Let $k \in I_{K-1}$ and suppose that $v^k \geq 0$. If $\tau_k f(v^k_{m,n}) < 1$ for each $(m,n) \in J$ then the matrix in* (2.66) *is a Minkowski matrix.*

Proof. Note that the off-diagonal entries of A^k are equal to zero, or of the form $-\mathfrak{b}^{k,x_i}_{m,n,j}$ for suitable $(m,n) \in J$ with $j \in \overline{I}_M \setminus \{m\}$ if $i = 1$, and $j \in \overline{I}_N \setminus \{n\}$ if $i = 2$. Using Lemma 2.4(b), it readily follows that

$$-\mathfrak{b}^{k,x_1}_{m,n,j} = \lambda_1 R^{(\beta_1)}_{x_1} g^{(\beta_1)}_{m-j} (v^k_{m,n})^p + R^{(\alpha_1)}_{x_1} g^{(\alpha_1)}_{m-j} \leq 0, \quad \forall (m,n,j) \in J \times (\overline{I}_M \setminus \{m\}). \tag{2.70}$$

Similarly, $-\mathfrak{b}_{m,n,j}^{k,x_2}$ for each $(m,n,j) \in J \times (\overline{I}_N \setminus \{n\})$. This implies that A^k is a Z-matrix. On the other hand, the diagonal entries of that matrix are either equal to 1, or of the form $\mathfrak{a}_{m,n}^k$ for some $(m,n) \in J$. By Lemma 2.4(a) and the hypotheses, it follows that $\mathfrak{a}_{m,n}^k > 1 - \tau_k f(v_{m,n}^k) \geq 0$ for each $(m,n) \in J$, which means that the property (ii) of Definition 2.4 holds. Finally, note that some rows have entries all equal to zero except at the diagonal which is equal to 1, so the condition of strict diagonal dominance is satisfied for those rows. For the remaining rows, the diagonal entry is of the form $\mathfrak{a}_{m,n}^k$ for some $(m,n) \in J$, while the nonzero off-diagonal entries are equal to $-\mathfrak{b}_{m,n,j}^{k,x_1}$ for $j \in \overline{I}_M \setminus \{m\}$, and $-\mathfrak{b}_{m,n,j}^{k,x_2}$ for $j \in \overline{I}_N \setminus \{n\}$. Using properties (a), (b) and (c) of Lemma 2.4, it follows that

$$\begin{aligned}
\sum_{\substack{j=0\\ j\neq m}}^{M} |\mathfrak{b}_{m,n,j}^{k,x_1}| + \sum_{\substack{j=0\\ j\neq n}}^{N} |\mathfrak{b}_{m,n,j}^{k,x_2}| &= -\lambda_1 R_{x_1}^{(\beta_1)} (v_{m,n}^k)^p \sum_{\substack{j=0\\ j\neq m}}^{M} g_{m-j}^{(\beta_1)} - R_{x_1}^{(\alpha_1)} \sum_{\substack{j=0\\ j\neq m}}^{M} g_{m-j}^{(\alpha_1)} - \lambda_2 R_{x_2}^{(\beta_2)} (v_{m,n}^k)^p \sum_{\substack{j=0\\ j\neq n}}^{N} g_{n-j}^{(\beta_2)} \\
&\quad - R_{x_2}^{(\alpha_2)} \sum_{\substack{j=0\\ j\neq n}}^{N} g_{n-j}^{(\alpha_2)} \qquad (2.71) \\
&\leq \lambda_1 R_{x_1}^{(\beta_1)} (v_{m,n}^k)^p g_0^{(\beta_1)} + R_{x_1}^{(\alpha_1)} g_0^{(\alpha_1)} + \lambda_2 R_{x_2}^{(\beta_2)} (v_{m,n}^k)^p g_0^{(\beta_2)} + R_{x_2}^{(\alpha_1)} g_0^{(\alpha_2)} \\
&< \mathfrak{a}_{m,n}^k
\end{aligned}$$

for each $(m,n) \in J$. This means that A^k is strictly diagonally dominant. Since all the properties of Definition 2.4 are satisfied, we conclude that A^k is a Minkowski matrix. □

For the sake of simplicity, we will use $f(v^k)$ to represent the sequence $(f(v_{m,n}^k))_{(m,n)\in\overline{J}}$, for each $k \in \overline{I}_K$. The following result establishes conditions under which the system (2.43) yields solutions at each iteration. Its proof is a direct consequence of the previous lemma and the properties of Minkowski matrices.

Theorem 2.1 (Existence and uniqueness). *Let $k \in I_{K-1}$, and assume that $v^k \geq 0$. If $\tau_k f(v^k) < 1$ then the recursive equation of* (2.69) *has a unique solution.*

Proof. By Lemma 2.5, the matrix A^k is a Minkowski matrix, so it is nonsingular. It follows that the vector equation $A^k v^{k+1} = E^k v^{k-1} + \psi^k$ has a unique solution, as desired. □

The next result summarizes the most important structural properties of the finite-difference scheme (2.43).

Theorem 2.2 (Positivity and boundedness). *Let $k \in I_{K-1}$, $\rho > 0$ and $s = \sup_{[0,\rho]} |f|$. Suppose that $0 \leq v^k \leq \rho$,*

$$s\tau_k \leq R_{x_1}^{(\alpha_1)} \sum_{j=0}^{M} g_{m-j}^{(\alpha_1)} + R_{x_2}^{(\alpha_2)} \sum_{j=0}^{N} g_{n-j}^{(\alpha_2)}, \quad \forall (m,n) \in J, \tag{2.72}$$

and

$$s\tau_k + \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} + \rho^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)} < 1 \tag{2.73}$$

are satisfied. If $0 \leq \psi^k \leq \rho$ then $0 \leq v^{k+1} \leq \rho$.

Proof. Note that the inequality (2.73) implies that $\tau_k f(v^k) < 1$, so A^k is a Minkowski matrix and the method yields a unique solution v^{k+1}. To establish the positivity of the approximation at the time

t_{k+1}, note firstly that the off-diagonal entries of E^k are zero, or of the form $\mathfrak{b}^{k,x_i}_{m,n,j}$ for some $(m,n) \in J$ and $j \in \overline{I}_M \setminus \{m\}$ in the case that $i = 1$, and $j \in \overline{I}_N \setminus \{n\}$ if $i = 2$. The proof of Lemma 2.5 shows that the off-diagonal entries of E^k are nonnegative in any case. Meanwhile, the components in the diagonal of E^k are zero, or of the form $\mathfrak{c}^k_{m,n}$ for some $(m,n) \in J$. Clearly,

$$\mathfrak{c}^k_{m,n} \geq 1 - \tau_k |f(v^k_{m,n})| - \sum_{i=1}^{2} R^{(\alpha_i)}_{x_i} g^{(\alpha_i)}_0 - \rho^p \sum_{i=1}^{2} \lambda_i R^{(\beta_i)}_{x_i} g^{(\beta_i)}_0 > 0, \quad \forall (m,n) \in J. \tag{2.74}$$

This implies that $E^k \geq 0$. Moreover, the vector $E^k v^{k-1} + \psi^k \geq 0$, and it follows that

$$v^{k+1} = (A^k)^{-1}(E^k v^{k-1} + \psi^k) \geq 0. \tag{2.75}$$

In order to establish the boundedness, we let e be the vector of the same dimension as v^{k+1} all of whose entries are equal to 1, and let $u^{k+1} = \rho e - v^{k+1}$ Substituting into the recursive vector equation of (2.69), we readily obtain

$$A^k u^{k+1} = \rho A^k e - E^k v^{k-1} - \psi^k. \tag{2.76}$$

Let f be the vector on the right-hand side of (2.76). The components of f are of the form $\rho - \psi^k_{j,n}$ for $(j,n) \in \{0, M\} \times \overline{I}_N$ or $(j,n) \in I_{M-1} \times \{0, N\}$ (yielding nonnegative expressions), or of the form

$$\begin{aligned} f_{m,n} &= \rho \left(\mathfrak{a}^k_{m,n} - \sum_{\substack{j=0 \\ j \neq m}}^{M} \mathfrak{b}^{k,x_1}_{m,n,j} - \sum_{\substack{j=0 \\ j \neq n}}^{N} \mathfrak{b}^{k,x_2}_{m,n,j} \right) - \mathfrak{c}^k_{m,n} v^{k-1}_{m,n} - \sum_{\substack{j=0 \\ j \neq m}}^{M} \mathfrak{b}^{k,x_1}_{m,n,j} v^{k-1}_{j,n} - \sum_{\substack{j=0 \\ j \neq n}}^{N} \mathfrak{b}^{k,x_2}_{m,n,j} v^{k-1}_{m,j} \\ &\geq 2\rho \left(1 - \mathfrak{c}^k_{m,n} - \sum_{\substack{j=0 \\ j \neq m}}^{M} b^{k,x_1}_{m,n,j} - \sum_{\substack{j=0 \\ j \neq n}}^{N} b^{k,x_2}_{m,n,j} \right) = 2\rho \left(-\tau_k f(v^k_{m,n}) - \sum_{j=0}^{M} b^{k,x_1}_{m,n,j} - \sum_{j=0}^{N} b^{k,x_2}_{m,n,j} \right) \\ &\geq 2\rho \left(-s\tau_k + R^{(\alpha_1)}_{x_1} \sum_{j=0}^{M} g^{(\alpha_1)}_{m-j} + R^{(\alpha_2)}_{x_2} \sum_{j=0}^{N} g^{(\alpha_2)}_{n-j} \right) \geq 0, \end{aligned} \tag{2.77}$$

for suitable $(m,n) \in J$. Here, we have used Lemma 2.4 and the inequality (2.72). As a consequence, note that $f \geq 0$. This and the fact that A^k is a Minkowski matrix imply that $u^{k+1} \geq 0$ or, equivalently, that $v^{k+1} \leq \rho$, as desired. □

We would like to examine now the feasibility of the constraints in Theorem 2.2. In a first stage, note that the inequality (2.72) may be multiplied on both sides by τ_k^{-1} to obtain

$$s \leq h_{x_1}^{-\alpha_1} \sum_{j=0}^{M} g^{(\alpha_1)}_{m-j} + h_{x_2}^{-\alpha_2} \sum_{j=0}^{N} g^{(\alpha_2)}_{n-j}, \tag{2.78}$$

which is satisfied for sufficiently small values of h_{x_1} and h_{x_2}. Having chosen the spatial step-sizes that satisfy (2.72), note that the inequality (2.73) is equivalent to the condition

$$\tau_k \left(s + \sum_{i=1}^{2} g^{(\alpha_i)}_0 h_{x_i}^{-\alpha_i} + \rho^p \sum_{i=1}^{2} \lambda_i g^{(\beta_i)}_0 h_{x_i}^{-\beta_i} \right) < 1, \tag{2.79}$$

which holds for sufficiently small values of τ_k.

The following result establishes the existence of a constant solution for (2.43), which is also a constant solution of the continuous model (2.4). The proof is straightforward in light of the vector form of the scheme given by (2.69), Lemma 2.5 and mathematical induction.

Theorem 2.3 (Constant solution). *Consider the problem* (2.43) *with homogeneous initial and boundary conditions. Then the constant sequence of zero vectors is the unique solution of the finite-difference method.*

□

2.5 Numerical results

In this section, we prove the main numerical properties of our finite-difference scheme, namely, the second-order consistency, the stability and the quadratic convergence of the method. In the following, we will assume that the range of the solution u of (2.4) is a subset of $[0,\rho]$ and that f is a smooth function defined on $[0,\rho]$, where $\rho > 0$. Moreover, we define the following continuous and discrete functionals, respectively:

$$\begin{aligned} \mathcal{L}u(x,t) &= \frac{\partial u}{\partial t}(x,t) - \sum_{i=1}^{2} \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) - u^p(x,t)\sum_{i=1}^{2}\lambda_i \frac{\partial^{\beta_i} u}{\partial |x_i|^{\beta_i}}(x,t) - u(x,t)f(u(x,t)), && (2.80)\\ Lu^k_{m,n} &= \delta_t^{(1)} v^k_{m,n} - \sum_{i=1}^{2}\mu_t^{(1)}\delta_{x_i}^{(\alpha_i)} v^k_{m,n} - (v^k_{m,n})^p \sum_{i=1}^{2}\lambda_i \mu_t^{(1)}\delta_{x_i}^{(\beta_i)} v^k_{m,n} - f(v^k_{m,n})\mu_t^{(1)} v^k_{m,n}, && (2.81)\end{aligned}$$

for each $(x,t) \in \Omega$ and $(m,n,k) \in J \times I_{K-1}$. Moreover, we will employ h to represent the vector (h_{x_1}, h_{x_2}).

Theorem 2.4 (Consistency). *Let $u \in \mathcal{C}^5(\overline{\Omega})$, and suppose that the range of u is a subset of $[0,\rho]$. If $\tau < 1$ and $f \in \mathcal{C}^1([0,\rho])$ then there exists a constant $C > 0$ independent of h and τ such that for each $(m,n,k) \in J \times I_{K-1}$,*

$$\left| Lu^k_{m,n} - \mathcal{L}u(x_{m,n}, t_k) \right| \le C(\tau^2 + \|h\|^2). \qquad (2.82)$$

Proof. We employ here the usual argument with Taylor polynomials and the identity (2.17). Note that the condition on the continuous differentiability of u implies that there exist constants $C_1, C_2^{(\alpha_i)}, C_3^{(\beta_i)}, C_4 > 0$ for $i \in \{1,2\}$ such that

$$\begin{aligned} \left| \delta_t^{(1)} u^k_{m,n} - \frac{\partial u}{\partial t}(x_{m,n}, t_k)\right| &\le C_1 \tau^2, && (2.83)\\ \left| \mu_t^{(1)}\delta_{x_i}^{(\alpha_i)} u^k_{m,n} - \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x_{m,n}, t_k)\right| &\le C_2^{(\alpha_i)}(\tau^2 + h_{x_i}^2), \quad i = 1,2, && (2.84)\\ \left| (u^k_{m,n})^p \mu_t^{(1)}\delta_{x_i}^{(\beta_i)} u^k_{m,n} - u^p(x_{m,n},t_k)\frac{\partial^{\beta_i} u}{\partial |x_i|^{\beta_i}}(x_{m,n}, t_k)\right| &\le |u^k_{m,n}|^p \left| \mu_t^{(1)}\delta_{x_i}^{(\beta_i)} u^k_{m,n} - \frac{\partial^{\beta_i} u}{\partial |x_i|^{\beta_i}}(x_{m,n}, t_k)\right| \\ &\le C_3^{(\beta_i)}(\tau^2 + h_{x_i}^2), \quad i = 1,2, && (2.85)\\ \left| f(u^k_{m,n})\mu_t^{(1)} u^k_{m,n} - u(x_{m,n},t_k) f(u(x_{m,n},t_k))\right| &= |f(u^k_{m,n})||\mu_t^{(1)} u^k_{m,n} - u(x_{m,n},t_k)| \\ &\le C_4\tau^2, && (2.86)\end{aligned}$$

for all $(m,n,k) \in J \times I_{K-1}$. The conclusion of this theorem is readily reached using the triangle inequality and defining the constant $C = \max\{C_1, C_2^{(\alpha_1)}, C_2^{(\alpha_2)}, \lambda_1 C_3^{(\beta_1)}, \lambda_2 C_3^{(\beta_2)}, C_4\}$. □

The following lemma will be a useful tool to show that the method (2.43) is stable and quadratically convergent. In the next results, for each $v \in \mathbb{R}^m$ we let

$$\|v\|_\infty = \max\{|v_i| : i = 1, \ldots, m\}. \tag{2.87}$$

Lemma 2.6 (Chen *et al.* [48]). *Let A be a real matrix of size $m \times m$ that satisfies*

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \leq |a_{ii}| - 1, \quad \forall i \in \{1, \ldots, m\}. \tag{2.88}$$

Then $\|v\|_\infty \leq \|Av\|_\infty$ is satisfied for all $v \in \mathbb{R}^m$.

Lemma 2.7. *Let $k \in I_{K-1}$ and $\rho > 0$. Let $s = \sup_{[0,\rho]} |f|$ and suppose that $0 \leq v^k \leq \rho$. If (2.72) and (2.73) are satisfied, then $\|v\|_\infty \leq \|A^k v\|_\infty$ holds for any $v \in \mathbb{R}^{(M+1)(N+1)}$.*

Proof. In light of Lemma 2.6, we only need to show that the matrix A^k satisfies the inequality (2.88). As in the proof of Lemma 2.5, we note that some rows of the matrix A^k have all entries equal to zero except at the diagonal, which is equal to 1. In those cases, the inequality of Lemma 2.6 is trivially satisfied. For the remaining rows, the diagonal is of the form $\mathfrak{a}_{m,n}^k$ for some $(m,n) \in J$, while the nonzero off-diagonal components are equal to $-\mathfrak{b}_{m,n,j}^{k,x_1}$ for $j \in \overline{I}_M \setminus \{m\}$, and $-\mathfrak{b}_{m,n,j}^{k,x_2}$ for $j \in \overline{I}_N \setminus \{n\}$. Using the inequality (2.72), we obtain

$$\begin{aligned}
|\mathfrak{a}_{m,n}^k| - 1 &= -\tau_k f(v_{m,n}^k) + \sum_{i=1}^{2} R_{x_i}^{(\alpha_i)} g_0^{(\alpha_i)} + (v_{m,n}^k)^p \sum_{i=1}^{2} \lambda_i R_{x_i}^{(\beta_i)} g_0^{(\beta_i)} \\
&\geq \left(-s\tau_k + R_{x_1}^{(\alpha_1)} \sum_{j=0}^{M} g_{m-j}^{(\alpha_1)} + R_{x_2}^{(\alpha_2)} \sum_{j=0}^{N} g_{n-j}^{(\alpha_2)} \right) + \lambda_1 R_{x_1}^{(\beta_1)} (v_{m,n}^k)^p \sum_{j=0}^{M} g_{m-j}^{(\beta_1)} \\
&\quad + \lambda_2 R_{x_2}^{(\beta_2)} (v_{m,n}^k)^p \sum_{j=0}^{N} g_{n-j}^{(\beta_2)} + \sum_{\substack{j=0 \\ j \neq m}}^{M} \mathfrak{b}_{m,n,j}^{k,x_1} + \sum_{\substack{j=0 \\ j \neq n}}^{N} \mathfrak{b}_{m,n,j}^{k,x_2} \\
&\geq \sum_{\substack{j=0 \\ j \neq m}}^{M} |\mathfrak{b}_{m,n,j}^{k,x_1}| + \sum_{\substack{j=0 \\ j \neq n}}^{N} |\mathfrak{b}_{m,n,j}^{k,x_2}|.
\end{aligned} \tag{2.89}$$

Thus the inequality (2.88) holds for each row of A^k. The conclusion of the present lemma readily follows now. □

Lemma 2.7 will be employed next to calculate some *a priori* bounds for the numerical solutions of (2.43) and to establish the stability of the method in some particular scenarios. Recall that for each real matrix E of size $q \times q$, the infinity norm of E is given by

$$\|E\|_\infty = \sup\{\|Ev\|_\infty : v \in \mathbb{R}^q \text{ such that } \|v\|_\infty = 1\} = \max_{1 \leq i \leq q} \sum_{j=1}^{q} |e_{ij}|. \tag{2.90}$$

Lemma 2.8. *Let $k \in I_{K-1}$ and suppose that* (2.72) *and* (2.73) *are satisfied. Then $E^k \geq 0$ and $\|E^k\|_\infty < 1$.*

Proof. We had already established that $E^k \geq 0$ in the proof of Theorem 2.2. On the other hand, using (2.72) it is easy to check that

$$\mathfrak{c}^k_{m,n} + \sum_{\substack{j=0 \\ j\neq m}}^{M} \mathfrak{b}^{k,x_1}_{m,n,j} + \sum_{\substack{j=0 \\ j\neq n}}^{N} \mathfrak{b}^{k,x_2}_{m,n,j} \leq 1 + \tau_k f(v^k_{m,n}) - R^{(\alpha_1)}_{x_1} \sum_{j=0}^{M} g^{(\alpha_1)}_{m-j} - R^{(\alpha_2)}_{x_2} \sum_{j=0}^{N} g^{(\alpha_2)}_{n-j} < 1 \tag{2.91}$$

for each $(m,n) \in J$. We conclude that $\|E^k\|_\infty < 1$, as desired. □

Lemma 2.7 will be employed next to calculate some *a priori* bounds for the numerical solutions of (2.43) and to establish the stability of the method in some particular scenarios.

Theorem 2.5 (*A priori* bounds)**.** *Let $\rho > 0$, and let $s = \sup_{[0,\rho]} |f|$. Let $(v^k)_{k=0}^K$ be a solution of* (2.69) *which is bounded in $[0,\rho]$, and suppose that* (2.72) *and* (2.73) *are satisfied for each $k \in \overline{I}_{K-1}$. Then for each $k \in \mathbb{N}$ with $k < N/2$,*

$$\|v^{2k}\|_\infty \leq \left(\prod_{l=1}^{k} \|E^{2l-1}\|_\infty \right) \|v_0\|_\infty + \sum_{l=1}^{k} \left(\|\psi^{2l-1}\|_\infty \prod_{j=l+1}^{k} \|E^{2j-1}\|_\infty \right), \tag{2.92}$$

$$\|v^{2k+1}\|_\infty \leq \left(\prod_{l=1}^{k} \|E^{2l}\|_\infty \right) \|v_1\|_\infty + \sum_{l=1}^{k} \left(\|\psi^{2l}\|_\infty \prod_{j=l+1}^{k} \|E^{2j}\|_\infty \right). \tag{2.93}$$

Proof. Notice that the assumptions of Lemma 2.7 are satisfied for each $k \in I_{K-1}$. Using that lemma, we obtain that

$$\|v^{k+1}\|_\infty \leq \|A^k v^{k+1}\|_\infty = \|E^k v^{k-1} + \psi^k\|_\infty \leq \|E^k\|_\infty \|v^{k-1}\|_\infty + \|\psi^k\|_\infty, \quad \forall k \in I_{K-1}. \tag{2.94}$$

The conclusions of the theorem readily follow now using a recursive argument. □

Under the assumptions of Lemma 2.8 and Theorem 2.5, the conclusion of the last theorem can be substantially simplified. Indeed, if homogeneous Dirichlet conditions are considered at the boundary of the spatial domain, then $\|v^k\|_\infty \leq \max\{\|v_0\|_\infty, \|v_1\|_\infty\}$ for each $k \in \overline{I}_K$.

We establish next the properties of stability and convergence for our method in some particular scenarios.

Theorem 2.6 (Stability)**.** *Let $\rho > 0$ and $p = 0$, and assume that $|f| = s \in \mathbb{R}$. Let $(u^k)_{k=0}^K$ and $(v^k)_{k=0}^K$ be solutions of* (2.43) *bounded in $[0,\rho]$ for the initial-boundary data $(\phi^1_u, \phi^2_u, \psi)$ and $(\phi^1_v, \phi^2_v, \psi)$, respectively. If* (2.72) *and* (2.73) *hold for each $k \in \overline{I}_{K-1}$, then*

$$\|u^k - v^k\|_\infty \leq \max\{\|u_0 - v_0\|_\infty, \|u_1 - v_1\|_\infty\}, \quad \forall k \in \overline{I}_K. \tag{2.95}$$

Proof. The hypotheses guarantee that the matrices A^k and E^k are all identical to some constant matrices A and E, respectively, for each $k \in I_{K-1}$. Note that the inequality (2.95) is trivially satisfied when $k \in \{0,1\}$, so suppose that it holds for some $k \in \{1,\ldots,K-1\}$. Lemmas 2.7 and 2.8 yield then

$$\|u^{k+1} - v^{k+1}\|_\infty \leq \|A(u^{k+1} - v^{k+1})\|_\infty \leq \|E(u^{k-1} - v^{k-1})\|_\infty \leq \|u^{k-1} - v^{k-1}\|_\infty. \tag{2.96}$$

The conclusion of this result follows now by induction. □

Theorem 2.7 (Convergence). *Let $\rho > 0$ and $p = 0$, and suppose that $u \in \mathcal{C}^5(\overline{\Omega})$ is a solution of (2.4) which is bounded in $[0, \rho]$. Let $\tau < 1$ and suppose that $|f| = s \in \mathbb{R}$. Let $(v^k)_{k=0}^K$ be a solution of (2.43) which is bounded in $[0, \rho]$, and suppose that (2.72) and (2.73) hold for each $k \in \overline{I}_K$. Then there exists a constant $\kappa \in \mathbb{R}$ independent of τ and h such that*

$$\|u^k - v^k\|_\infty \leq \kappa(\tau^2 + \|h\|^2), \quad \forall k \in \overline{I}_K. \tag{2.97}$$

Proof. Let $\epsilon^k = u^k - v^k$ for each $k \in \overline{I}_K$. Beforehand, note that the exact and the numerical solutions coincide for the initial-boundary data, which means in particular that $\|\epsilon^0\|_\infty = \|\epsilon^1\|_\infty = 0$. Using Theorem 2.4 together with Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned} \|\epsilon^{k+1}\|_\infty &\leq \|A(u^{k+1} - v^{k+1})\|_\infty \leq \|E(u^{k-1} - v^{k-1})\|_\infty + \|Au^{k+1} - Eu^{k-1} - \psi^k\|_\infty \\ &= \|\epsilon^{k-1}\|_\infty + \tau\|Lu^k - \mathcal{L}u^k\|_\infty \leq \|\epsilon^{k-1}\|_\infty + \tau C(\tau^2 + \|h\|^2), \end{aligned} \tag{2.98}$$

which yields that $\|\epsilon^{k+1}\|_\infty - \|\epsilon^{k-1}\|_\infty \leq \tau C(\tau^2 + \|h\|^2)$ for each $k \in I_{K-1}$. The conclusion of this results readily follows from this inequality with $\kappa = TC$. □

Finally, we provide some computer simulations to show that the finite-difference method (2.43) is capable of preserving the main analytical features of the solutions of interest of (2.4). Concretely, we illustrate the capability of the method to preserve the positivity and the boundedness. The simulations were obtained using our own implementation of the method in ©Matlab 8.5.0.197613 (R2015a), on a ©Sony Vaio PCG-5L1P laptop computer with Kubuntu 16.04 as operating system. In terms of computational times, we are aware that better results may be obtained with more modern equipment and more modest Linux/Unix distributions.

Example 2.3. Let us consider the continuous model (2.4) with parameters $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = \lambda_2 = 0$, $p = 1$, $\gamma = 0.6$, and f is given by (2.6). We will consider the spatial domain $B = (-200, 200) \times (-200, 200) \subseteq \mathbb{R}^2$, and the computational constants $h_{x_1} = h_{x_2} = 4$ and $\tau = 0.05$. Let us fix homogeneous Dirichlet conditions on the boundary of B, and consider the initial profiles

$$\phi^1(x, y) = \phi^2(x, y) = \begin{cases} 0.2, & \text{if } (x, y) = (0, 0), \\ 0, & \text{otherwise.} \end{cases} \tag{2.99}$$

Notice that we consider a problem without convective effects and with partial derivatives of integer order. In such situation, the classical solution of the initial-boundary-value problem (2.4) is nonnegative and bounded from above by γ. Figure 2.1 shows snapshots of the approximate solution u as a function of x and y, for the times (a) $t = 5$, (b) $t = 10$, (c) $t = 15$, (d) $t = 20$, (e) $t = 25$ and (f) $t = 30$. The solutions suggest that the method is capable of preserving the positivity and the boundedness of the approximations, in agreement with Theorem 2.2. □

Example 2.4. Let us consider the same problem as in Example 2.3, using the constants $\alpha_1 = 1.9$, $\alpha_2 = 1.95$, $\beta_1 = 0.8$, $\beta_2 = 0.9$, $\lambda_1 = \lambda_2 = 1$, $p = 1$ and $\gamma = 0.6$, together with the same computational parameters and the same initial-boundary conditions as in the previous example. The results of our simulations are shown in Figure 2.2. In this case, anomalous diffusion and convection are considered

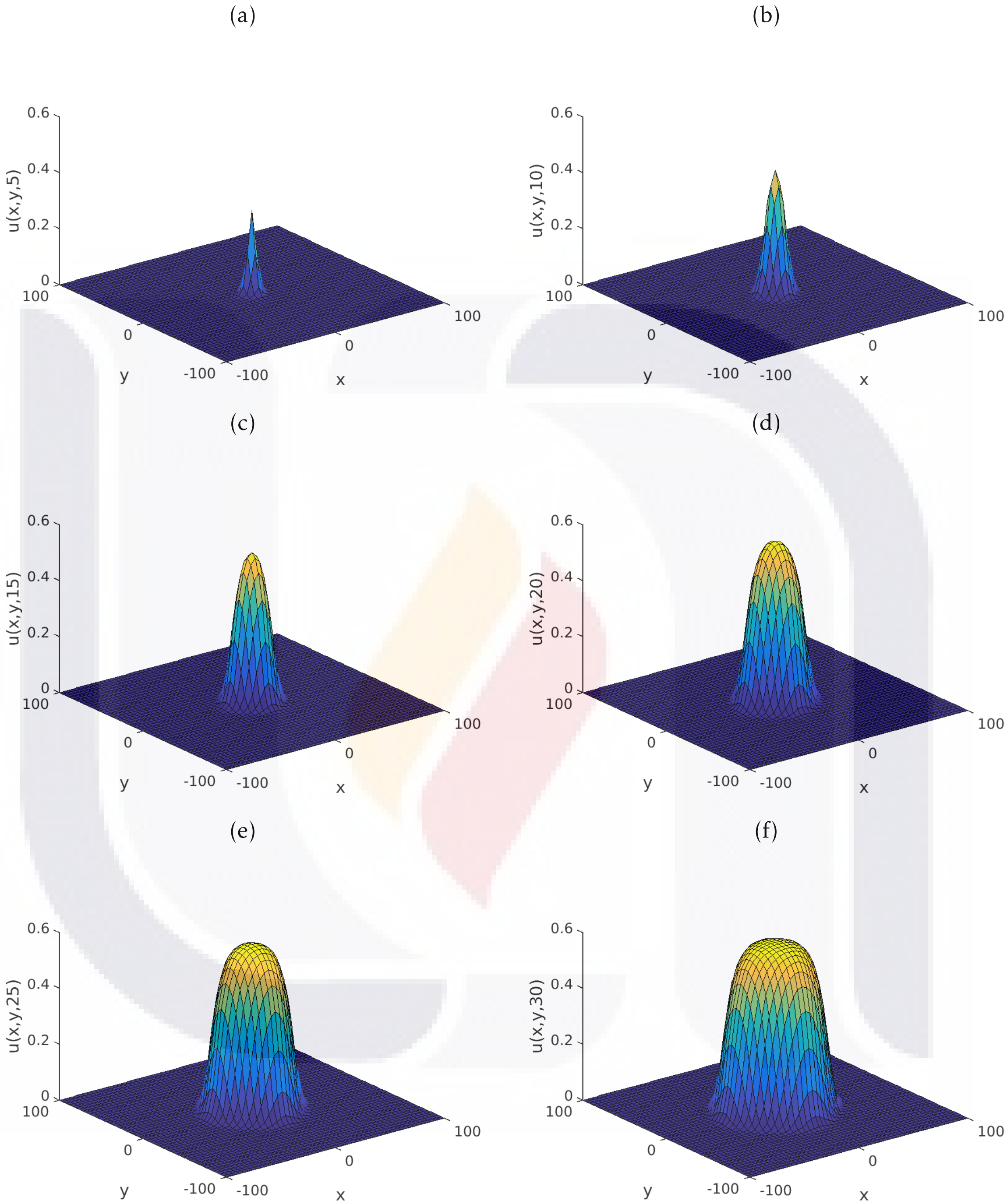

Figure 2.1: Snapshots of the approximate solution u of the model (2.4) as a function of $(x,y) \in \overline{B} = [-200,200] \times [-200,200]$, for the times (a) $t = 5$, (b) $t = 10$, (c) $t = 15$, (d) $t = 20$, (e) $t = 25$ and (f) $t = 30$. The model uses the parameters $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = \lambda_2 = 0$, $p = 1$, $\gamma = 0.6$, and f is given by (2.6). We employed homogeneous Dirichlet conditions on the boundary of B, along with the initial profiles (2.99). Computationally, we let $h_{x_1} = h_{x_2} = 4$ and $\tau = 0.05$.

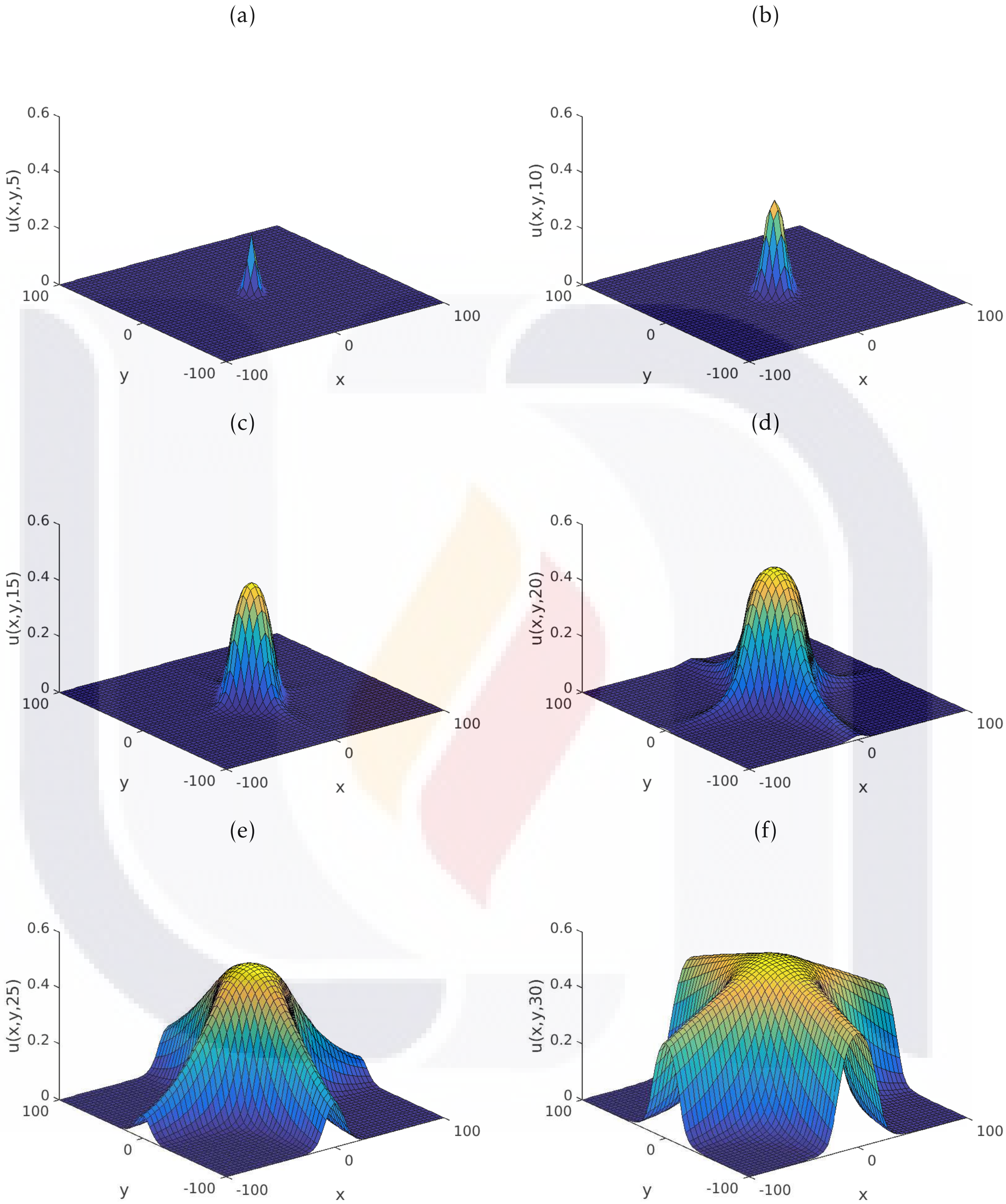

Figure 2.2: Snapshots of the approximate solution u of the model (2.4) as a function of $(x,y) \in \overline{B} = [-200,200] \times [-200,200]$, for the times (a) $t = 5$, (b) $t = 10$, (c) $t = 15$, (d) $t = 20$, (e) $t = 25$ and (f) $t = 30$. The model uses the parameters $\alpha_1 = 1.9$, $\alpha_2 = 1.95$, $\beta_1 = 0.8$, $\beta_2 = 0.9$, $\lambda_1 = \lambda_2 = 1$, $p = 1$, $\gamma = 0.6$, and f is given by (2.6). We employed homogeneous Dirichlet conditions on the boundary of B, along with the initial profiles (2.99). Computationally, we let $h_{x_1} = h_{x_2} = 4$ and $\tau = 0.05$.

in the x and y directions. It is worth pointing out that the properties of positivity and boundedness of the approximate solutions are preserved, in agreement with Theorem 2.2. □

It is important to mention that we have conducted more simulations with different model parameters and different initial-boundary conditions. The results are not presented in this work in view of their repetitiveness: they also confirm the capability of the finite-difference method (2.43) to preserve the analytical features of the solutions of interest of (2.4), namely, the positivity and the boundedness of the solutions.

3. A structure-preserving Bhattacharya method

In this chapter, we investigate a parabolic equation with nonlinear reaction, and fractional diffusion and advection terms of the Riesz type. The model under investigation is a fractional generalization of the well-known Burgers–Fisher and Burgers–Huxley models from population and fluid dynamics, which are equations that admit positive, bounded and monotone solutions, some of them being traveling waves. A variable-step Bhattacharya-type finite-difference scheme based on fractional centered differences is proposed to approximate the solutions of the parabolic partial differential equation. The method is an explicit technique which, under suitable parameter conditions, is capable of preserving the positivity, the boundedness and the monotonicity of the approximations. Moreover, the method preserves the constant solutions of the fractional partial differential equation under investigation. The properties of consistency, stability and convergence of the technique are established thoroughly in this manuscript along with some *a priori* bounds for the numerical solutions. Some illustrative simulations are carried out in order to show that the method preserves these features of the approximations.

3.1 Introduction

Nonlinear advection-diffusion-reaction equations have been extensively investigated (both analytically and numerically) in the specialized literature during the past decades. Computationally, many different numerical approaches have been followed to approximate the solutions of those systems [49]. As a result, many finite-difference methodologies have been reported in the literature. In fact, various different approaches have been followed to develop schemes for advection-diffusion-reaction equations, one of them being the exponential method proposed by M. C. Bhattacharya around 1985 to solve some simple diffusion models [50]. This exponential approach was later used in some applications [51] and extended to the investigation of more complicated partial differential equations [52, 53]. Various hybrid methods that employ the Laplace transform were designed at that time with the help of the exponential method [54, 55]. Nowadays this exponential approach has been extended to more complicated nonlinear systems. For instance, the Korteweg–de Vries equation has been investigated under this methodology for small times [56]. It is worth recalling that implicit [57] and Crank–Nicolson [58] forms of the exponential method have been proposed for the solutions of the one-dimensional Burgers' equation. More complicated models were investigated later on, like the Burgers–Huxley equation [59] and the generalized Huxley and Burgers–Huxley equations [60]. It is important to mention that this method has been used mainly due to the simplicity of its implementation, especially the explicit forms of this technique. This is probably why some further variations of this technique have been developed, like some logarithmic versions of the original method proposed by Bhattacharya [61].

In spite of these facts, it is well known that this method presents various important shortcomings. Firstly, this technique is highly sensitive with respect to approximations which are close to zero. Good approximations are obtained in the case of positive solutions, and early studies suggested that the method could be indeed a convergent technique in those cases [51]. However, this singular character in Bhattacharya's approach has been perhaps one of the most important limitations in the applicability of the technique. Nevertheless, it is worth noting that some corrections have been proposed in order to avoid the high sensitivity with respect to zero approximations. In light of this correction, explicit methods have been designed to solve the Burgers–Fisher equation [62], some complex thin-film models [63] and advection-diffusion equations governing the distribution of probability of some random variables [38] among other physical systems. Other important limitation of the exponential approach has been the lack of studies that guarantee the preservation of important features of the solutions. More precisely, there have been very few studies in which the structure-preserving capabilities of Bhattacharya-type methods have been analyzed. It is important to remember here that the design of structure-preserving techniques (also called dynamically consistent techniques if we follow the nomenclature proposed by R. E. Mickens [64]) has been an important avenue of investigation in numerical analysis. Structure-preserving techniques have been proposed as a need to guarantee that the numerical methods reflect the physical context of the problems. Using this perspective, methods that preserve the energy of Hamiltonian systems have been proposed [65]. Other works report on discretizations that preserve the positivity of nonlinear systems describing the dynamics of chemical processes [66], the positivity of Poisson integrators for the Lotka–Volterra equations [67], or the boundedness and monotonicity of solitary-wave solutions of the Burgers–Huxley equation [68].

Indeed, some reports by the author of the present manuscript have focused on the capability of the Bhattacharya approach to preserve the structure of the solutions of some advection-diffusion-reaction systems [62, 63, 38]. However, the investigation of the numerical efficiency of this methodology has been left without investigation. Unfortunately, there are very few reports that focus on the study of the properties of stability, consistency and convergence of the Bhattacharya methodology. Moreover, in light of the interest that fractional models have gathered in recent years, it is worth noting that this exponential approach has not been extended yet to the fractional scenario. In view of these facts, the goal of the present manuscript is to investigate numerically a general parabolic partial differential equation with nonlinear reaction term which considers the presence of fractional diffusion and advection. We will use Riesz space-fractional derivatives in our model, and the Bhattacharya perspective will be employed to provide a finite-difference discretization based on fractional centered differences [45]. The method derived in this manuscript will have the following characteristics, which will depend on some analytical features of the reaction term:

- the method preserves the positivity,
- it also preserves the boundedness of the approximations,

- it is an explicit finite-difference technique,
- it is a non-singular extension of the Bhattacharya approach,
- the existence and uniqueness of approximations is guaranteed,
- it preserves the constant solutions of the continuous model,
- it is a stable technique,
- the consistency of the method is guaranteed under suitable analytical conditions and
- *a priori* bounds are provided for the method.

3.2 Preliminaries

Throughout this work, we will suppose that a, b, c and d are real numbers satisfying $a < b$ and $c < d$, also we will assume that $T > 0$. We will let $\Omega = B \times (0,T)$ with $B = (a,b) \times (c,d)$, and use $\overline{\Omega}$ and $\overline{B}$ to represent respectively the closures of Ω and B under the standard topology of $\mathbb{R}^3$, and we will use ∂B to denote the boundary of B. In this manuscript $u : \overline{\Omega} \to \mathbb{R}$ will represent a function. Moreover, for the sake of simplicity, let $x = (x_1, x_2)$ and we define $u(x,t) = 0$ for each $x \in B^c$ and each $t \in [0,T]$.

Definition 3.1. Let $\alpha > -1$ and suppose that n is a nonnegative integer such that $n - 1 < \alpha \leq n$. The Riesz space-fractional derivative of u of order α at the point $(x,t)$is defined by

$$\frac{\partial^\alpha u}{\partial |x|^\alpha}(x,t) = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi,t)d\xi}{|x-\xi|^{\alpha+1-n}}, \quad \forall (x,t) \in \Omega. \tag{3.1}$$

Here Γ is the gamma function defined in (2.3).

Definition 3.2. Suppose that $\alpha > -1$ and that $n \geq 0$ is an integer such that $n - 1 < \alpha \leq n$. The left and the right Riemann–Liouville fractional derivatives in space of order α of u at (x,t) are given by

$$_{-\infty}D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} \frac{u(\xi,t)d\xi}{(x-\xi)^{\alpha+1-n}}, \tag{3.2}$$

$$_xD_{+\infty}^\alpha u(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{x}^{\infty} \frac{u(\xi,t)d\xi}{(\xi-x)^{\alpha+1-n}}, \tag{3.3}$$

respectively.

It is important to point out that the Riesz space-fractional derivative becomes the one-dimensional spatial Laplacian operator in the case when $\alpha = 2$. Note also that the differential operator (3.1) can be expressed in terms of the Riemann–Liouville fractional derivatives as

$$\frac{\partial^\alpha u}{\partial |x|^\alpha}(x,t) = -\frac{1}{2\cos(\frac{\pi\alpha}{2})} ({}_{-\infty}D_x^\alpha + {}_x D_{+\infty}^\alpha) u(x,t), \quad \forall (x,t) \in \Omega. \tag{3.4}$$

For the remainder of this work and unless we say otherwise, we will use α_i, β_i, γ and λ_i to represent real numbers such that $1 < \alpha_i \leq 2$, $0 < \beta_i < 1$ and $0 < \gamma < 1$, let $p \in \mathbb{N}$, and $i = 1,2$. Let $\phi : \overline{B} \to \mathbb{R}$

and $\psi : \partial B \times (0,T) \to \mathbb{R}$ be functions whose ranges are subsets of some closed and bounded interval $I \subseteq \mathbb{R}$, and assume additionally that the compatibility conditions $\phi(x) = \psi(x,0)$ are satisfied for each $x \in \partial B$. With these conventions, the problem under consideration in this work is the nonlinear initial-boundary-value problem

$$\begin{gathered}\frac{\partial u}{\partial t}(x,t) = \sum_{i=1}^{2} \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) - u^p(x,t) \sum_{i=1}^{2} \lambda_i \frac{\partial^{\beta_i} u}{\partial |x_i|^{\beta_i}}(x,t) + u(x,t) f(u(x,t)), \quad \forall (x,t) \in \Omega, \\ \text{such that } \begin{cases} u(x,0) = \phi(x), & \forall x \in \overline{B}, \\ u(x,t) = \psi(t), & \forall (x,t) \in \partial B \times (0,T). \end{cases}\end{gathered} \tag{3.5}$$

Here, f is in general a real-valued function defined as (2.5) or (2.6) in chapter 2.

An alternative form of the partial differential equation in (3.5) is readily at hand. Indeed, suppose that u is a positive solution of (3.5), and let $\kappa \in \mathbb{R}^+$. Divide both sides of the Burgers–Fisher equation by $u(x,t) + \kappa$ and use the chain rule on the left-hand side to obtain the following equivalent initial-boundary-value problem:

$$\begin{gathered}\frac{\partial}{\partial t} \ln(u(x,t) + \kappa) = \frac{1}{u(x,t) + \kappa} \left[\sum_{i=1}^{2} \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) - u^p(x,t) \sum_{i=1}^{2} \lambda_i \frac{\partial^{\beta_i} u}{\partial |x_i|^{\beta_i}}(x,t) + u(x,t) f(u(x,t)) \right], \\ \text{such that } \begin{cases} u(x,0) = \phi(x), & \forall x \in \overline{B}, \\ u(x,t) = \psi(t), & \forall (x,t) \in \partial B \times (0,T). \end{cases}\end{gathered} \tag{3.6}$$

for each $(x,t) \in \Omega$. This equivalent form will be useful to provide an exponential discretization of our initial-boundary-value problem.

3.3 Numerical model

In this work we follow a finite difference approach to approximate the solutions of (3.5), and use fractional centered differences and their properties defined in chapter 2 with the objective to approximate Riesz space-fractional derivatives.

Let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$, for each $q \in \mathbb{N}$. Assume that $M, N, K \in \mathbb{N}$ and use a space partition norm $h_{x_1} = (b-a)/M$ of the interval $[a,b]$ and $h_{x_2} = (d-c)/N$ of the interval $[c,d]$. More precisely, for each $m \in \{0,1,\ldots,M\}$ and for each $n \in \{0,1,\ldots,N\}$ we define $x_{1,m} = a + mh_{x_1}$ and $x_{2,n} = a + nh_{x_2}$. Let $J = I_{M-1} \times I_{N-1}$, $\overline{J} = \overline{I}_M \times \overline{I}_N$ and $\partial J = \overline{J} \cap \partial B$. Also, we will use a (not necessarily uniform) partition $0 = t_0 < t_1 < \ldots < t_K = T$ of the interval $[0,T]$, and define the constants $\tau_k = t_{k+1} - t_k$, for each $k \in \{0,1,\ldots,K-1\}$. Moreover, for each $m \in \{0,1,\ldots,M\}$, each $n \in \{0,1,\ldots,N\}$ and each $k \in \{0,1,\ldots,K\}$ define $u_{m,n}^k = u(x_{m,n}, t_k)$ with $x_{m,n} = (x_{1,m}, x_{2,n})$. Under these circumstances, note that

$$\frac{\partial^\alpha u}{\partial |x_1|^\alpha}(x_{m,n}, t_k) = -\frac{1}{h_{x_1}^\alpha} \sum_{i=m-M}^{m} g_i^\alpha u_{m-i,n}^k + \mathcal{O}(h^2) = -\frac{1}{h_{x_1}^\alpha} \sum_{i=0}^{M} g_{m-i}^\alpha u_{m,i}^k + \mathcal{O}(h^2), \tag{3.7}$$

$$\frac{\partial^\alpha u}{\partial |x_2|^\alpha}(x_{m,n},t_k) = -\frac{1}{h_{x_2}^\alpha}\sum_{i=n-N}^{n} g_i^\alpha u_{m,n-i}^k + \mathcal{O}(h^2) = -\frac{1}{h_{x_2}^\alpha}\sum_{i=0}^{N} g_{n-i}^\alpha u_{i,n}^k + \mathcal{O}(h^2), \tag{3.8}$$

for each $m \in I_{M-1}$, each $n \in I_{N-1}$, each $k \in I_{K-1}$ and each $0 < \alpha \leq 2$ with $\alpha \neq 1$. In the following, we will consider the discrete operators

$$\delta_t u_{m,n}^k = \frac{u_{m,n}^{k+1} - u_{m,n}^k}{\tau_k}, \tag{3.9}$$

$$\delta_{x_1}^\alpha u_{m,n}^k = -\frac{1}{h_{x_1}^\alpha}\sum_{i=0}^{M} g_{m-i}^\alpha u_{i,n}^k, \tag{3.10}$$

$$\delta_{x_2}^\alpha u_{m,n}^k = -\frac{1}{h_{x_2}^\alpha}\sum_{i=0}^{N} g_{n-i}^\alpha u_{m,i}^k. \tag{3.11}$$

Using this nomenclature, the fully discrete model used in this work to approximate the solutions of (3.6) is given by the discrete system of equations

$$\begin{aligned}\delta_t \ln(u_{m,n}^k + \kappa) = (u_{m,n}^k + \kappa)^{-1}\left[\sum_{i=1}^{2}\delta_{x_i}^{\alpha_i} u_{m,n}^k - (u_{m,n}^k)^p \sum_{i=1}^{2}\lambda_i \delta_{x_i}^{\beta_i} u_{m,n}^k + u_{m,n}^k f(u_{m,n}^k)\right],\\ \text{such that } \begin{cases} u_{m,n}^0 = \phi(x_{m,n}), & \forall (m,n) \in \overline{J}, \\ u_{m,n}^k = \psi(x_{m,n}^k), & \forall (m,n,k) \in \partial J \times \overline{I}_K. \end{cases}\end{aligned} \tag{3.12}$$

for each $m \in I_{M-1}$, each $n \in I_{N-1}$ and each $k \in I_{K-1}$.

Clearly, the discrete fractional model (3.12) is computationally a two-step exponential discretization of the continuous system (3.6) that considers a variable step-size in time. Moreover, it can be explicitly and equivalently rewritten as

$$\begin{aligned}u_{m,n}^{k+1} = (u_{m,n}^k + \kappa)\exp\left[\frac{\tau_k\left(\sum_{i=1}^{2}\delta_{x_i}^{\alpha_i} u_{m,n}^k - (u_{m,n}^k)^p \sum_{i=1}^{2}\lambda_i \delta_{x_i}^{\beta_i} u_{m,n}^k + u_{m,n}^k f(u_{m,n}^k)\right)}{u_{m,n}^k + \kappa}\right] - \kappa,\\ \text{such that } \begin{cases} u_{m,n}^0 = \phi(x_{m,n}), & \forall (m,n) \in \overline{J}, \\ u_{m,n}^k = \psi(x_{m,n}^k), & \forall (m,n,k) \in \partial J \times \overline{I}_K. \end{cases}\end{aligned} \tag{3.13}$$

for each $m \in I_{M-1}$, each $n \in I_{N-1}$ and each $k \in I_{K-1}$.

Here, it is important to observe that $u_{m,n}^{k+1} = F_{m,n,k}^u(u_{m,n}^k)$, where

$$F_{m,n,k}^u(w) = g(w)\exp(\varphi(w)) - \kappa, \tag{3.14}$$

and the functions g and φ are given by

$$g(w) = w + \kappa, \quad \forall w \in \mathbb{R}, \tag{3.15}$$

$$\varphi(w) = \tau_k\left(\frac{wf(w) + cw^{p+1} + \left[d_{m,n,x_1}^k + d_{m,n,x_2}^k\right]w^p - ew - f_{m,n,x_1}^k - f_{m,n,x_2}^k}{w + \kappa}\right), \tag{3.16}$$

$\forall w \in \mathbb{R} \setminus \{-\kappa\}$ and for the constants $c = \lambda_1 g_0^{\beta_1} h_{x_1}^{-\beta_1} + \lambda_2 g_0^{\beta_2} h_{x_2}^{-\beta_2}$, $e = g_0^{\alpha_1} h_{x_1}^{-\alpha_1} + g_0^{\alpha_2} h_{x_2}^{-\alpha_2}$ and

$$d_{m,n,x_1}^k = \frac{\lambda_1}{h_{x_1}^{\beta_1}} \sum_{\substack{i=0 \\ i\neq m}}^{M} g_{m-i}^{\beta_1} u_{i,n}^k, \qquad f_{m,n,x_1}^k = \frac{1}{h_{x_1}^{\alpha_1}} \sum_{\substack{i=0 \\ i\neq m}}^{M} g_{m-i}^{\alpha_1} u_{i,n}^k. \tag{3.17}$$

$$d_{m,n,x_2}^k = \frac{\lambda_2}{h_{x_2}^{\beta_2}} \sum_{\substack{i=0 \\ i\neq n}}^{N} g_{n-i}^{\beta_2} u_{m,i}^k, \qquad f_{m,n,x_2}^k = \frac{1}{h_{x_2}^{\alpha_2}} \sum_{\substack{i=0 \\ i\neq n}}^{N} g_{n-i}^{\alpha_2} u_{m,i}^k. \tag{3.18}$$

Remark 3.1. Note that we use the letter 'f' for two different mathematical objects. Throughout this work f will represent the factor function of (3.5), while f_{m,n,x_j}^k will represent the constant defined above and which depends on m, n, k and $j = 1,2$. The difference between both notations should be clear from the context. □

Finally, for the sake of convenience we will use

$$u^k = (u_{0,0}^k, u_{0,1}^k, \ldots, u_{0,N}^k,\ u_{1,0}^k, u_{1,1}^k, \ldots, u_{1,N}^k, \ldots,\ u_{M,0}^k, u_{M,1}^k, \ldots, u_{M,N}^k), \tag{3.19}$$

for each $k \in \overline{I}_K$. Moreover, for any real $(M+1)(N+1)$-dimensional vector u we use $u > 0$ to represent the fact that each component of u is positive. If s is any real number we use $u < s$ to denote that each component of u is less than s. Also $0 < u < s$ will represent that $u > 0$ and $u < s$. If u and v are real vectors of the same dimension we will use $u < v$ to mean that $v - u > 0$. As expected, $0 < u < v < s$ means that $u > 0$, $u < v$ and $v < s$ are all satisfied.

3.4 Structural properties

In this section, we prove the most important properties of (3.12). More precisely, we wish to show that the method yields a unique solution under suitable conditions, that it preserves the positivity, the boundedness and the monotonicity of the approximations, and that the constant solutions of (3.5) are also solutions of (3.12). Throughout we will suppose that κ is a positive constant unless we say otherwise.

In a first stage, we establish here a theorem on the existence an uniqueness of the solutions of (3.12) as well as a result on the constant solutions of our numerical method.

Theorem 3.1 (Existence and uniqueness). *Let $k \in \{0,1,\ldots,K-1\}$. If $u^k > 0$ and $\kappa > 0$ then* (3.12) *yields a unique solution u^{k+1}.*

Proof. Note that $u_{m,n}^k + \kappa > 0$ for each $m \in \{1,\ldots,M-1\}$ and $n \in \{1,\ldots,N-1\}$. As a consequence, the real number $u_{m,n}^{k+1} = F_{m,n,k}^u(u_{m,n}^k)$ is defined uniquely, whence the conclusion of the theorem follows. □

Theorem 3.2 (Constant solutions). *The constant sequence $(0)_{k=0}^K$ consisting of zero vectors of dimension $(M+1)\times(N+1)$ is a solution of* (3.13) *if $\phi, \psi \equiv 0$.*

Proof. By assumption $u^0 = 0$, so suppose that $u^k = 0$ for some $k \in \{0,1,\ldots,K-1\}$. Note then that $u_{m,n}^{k+1} = F_{m,n,k}^u(0) = 0$ readily holds for each $m \in \{1,\ldots,M-1\}$ and $n \in \{1,\ldots,N-1\}$. The result readily follows by induction. □

In the following theorem, we will consider the infinite system described by

$$\begin{aligned} u_{m,n}^{k+1} = F_{m,n,k}^{u}(u_{m,n}^{k}), \quad \forall (m,n) \in \mathbb{Z}\times\mathbb{Z}, \forall k \in \{0,1,\ldots,K-1\}, \\ \text{such that } u_{m,n}^{0} = \phi(x_{m,n}), \quad \forall (m,n) \in \mathbb{Z}\times\mathbb{Z}, \end{aligned} \tag{3.20}$$

where $F_{m,n,k}^{u}$ is defined as in (3.14), and the parameters c and e are as before. For each $m \in \mathbb{Z}$, each $n \in \mathbb{Z}$ and each $k \in \{0,1,\ldots,K-1\}$ the constants d_{m,n,x_1}^{k}, d_{m,n,x_2}^{k}, f_{m,n,x_1}^{k} and f_{m,n,x_2}^{k} are respectively given in this case by

$$d_{m,n,x_1}^{k} = \frac{\lambda_1}{h_{x_1}^{\beta_1}} \sum_{\substack{i=-\infty \\ i\neq m}}^{\infty} g_{m-i}^{\beta_1} u_{i,n}^{k}, \qquad f_{m,n,x_1}^{k} = \frac{1}{h_{x_1}^{\alpha_1}} \sum_{\substack{i=-\infty \\ i\neq m}}^{\infty} g_{m-i}^{\alpha_1} u_{i,n}^{k}. \tag{3.21}$$

$$d_{m,n,x_2}^{k} = \frac{\lambda_2}{h_{x_2}^{\beta_2}} \sum_{\substack{i=-\infty \\ i\neq n}}^{\infty} g_{n-i}^{\beta_2} u_{m,i}^{k}, \qquad f_{m,n,x_2}^{k} = \frac{1}{h_{x_2}^{\alpha_2}} \sum_{\substack{i=-\infty \\ i\neq n}}^{\infty} g_{n-i}^{\alpha_2} u_{m,i}^{k}. \tag{3.22}$$

Theorem 3.3 (Constant solutions)**.** *Let $s \neq -\kappa$ be root of f, and let ϵ be the two-sided infinite sequence all of whose terms are equal to s. Then $(\epsilon)_{k=0}^{K}$ is a solution of* (3.20) *with $\phi \equiv s$.*

Proof. The proof is done by induction again. By hypothesis $u^k = \epsilon$ when $k = 0$, so let us suppose that it is also true for some $k \in \{0,1,\ldots,K-1\}$. Note then that for each $(m,n) \in \mathbb{Z}\times\mathbb{Z}$, Lemma **??** guarantees that

$$\begin{aligned} \varphi(s) &= \frac{\tau_n}{s+\kappa}\left(s f(s) + \left[\frac{\lambda_1}{h_{x_1}^{\beta_1}} g_0^{\beta_1} + \frac{\lambda_2}{h_{x_2}^{\beta_2}} g_0^{\beta_2} \right] s^{p+1} + \left[\frac{\lambda_1}{h_{x_1}^{\beta_1}} \sum_{\substack{i=-\infty \\ i\neq m}}^{\infty} g_{m-i}^{\beta_1} + \frac{\lambda_2}{h_{x_2}^{\beta_2}} \sum_{\substack{i=-\infty \\ i\neq n}}^{\infty} g_{n-i}^{\beta_2} \right] s^{p+1} \right) \\ &\quad - \frac{\tau_n}{s+\kappa}\left(\left[\frac{g_0^{\alpha_1}}{h_{x_1}^{\alpha_1}} + \frac{g_0^{\alpha_2}}{h_{x_2}^{\alpha_2}} \right] s + \left[\frac{1}{h_{x_1}^{\alpha_1}} \sum_{\substack{i=-\infty \\ i\neq m}}^{\infty} g_{m-i}^{\alpha_1} + \frac{1}{h_{x_2}^{\alpha_2}} \sum_{\substack{i=-\infty \\ i\neq n}}^{\infty} g_{n-i}^{\alpha_2} \right] s \right) \\ &= \frac{\tau_n}{s+\kappa}\left(\frac{\lambda_1 s^{p+1}}{h_{x_1}^{\beta_1}} \sum_{i=-\infty}^{\infty} g_{m-i}^{\beta_1} + \frac{\lambda_2 s^{p+1}}{h_{x_2}^{\beta_2}} \sum_{i=-\infty}^{\infty} g_{n-i}^{\beta_2} - \frac{s}{h_{x_1}^{\alpha_1}} \sum_{i=-\infty}^{\infty} g_{m-i}^{\alpha_1} - \frac{s}{h_{x_2}^{\alpha_2}} \sum_{i=-\infty}^{\infty} g_{n-i}^{\alpha_2} \right) = 0 \end{aligned} \tag{3.23}$$

As a consequence,

$$u_{m,n}^{k+1} = F_{m,n,k}^{u}(s) = g(s)\exp(\varphi(s)) - \kappa = s, \quad \forall (m,n) \in \mathbb{Z}\times\mathbb{Z}. \tag{3.24}$$

It follows that $u^{k+1} = \epsilon$, and the conclusion of the theorem is reached by induction. □

We will need the following lemma to establish the capability of the method (3.12) to preserve the positivity and the boundedness. The proof is a straightforward application of the mean value theorem of one-variable real analysis.

Lemma 3.1. *Let κ be a real number and suppose that $F, g, \varphi : [0,1] \to \mathbb{R}$ are functions such that $F(w) = g(w)\exp(\varphi(w)) - \kappa$ for each $w \in [0,1]$. Then F is increasing in $[0,1]$ if g and φ are differentiable functions satisfying*

$$g'(w) + g(w)\varphi'(w) > 0, \tag{3.25}$$

for each $0 < w < 1$. □

The following result is the cornerstone of this work.

Lemma 3.2. *Let* $\kappa > 0$ *and* $s \in (0,1]$, *and let* $k \in \{0,1,\ldots,K-1\}$. *Suppose that* f *and* f' *are bounded on* $[0,s]$, *and let* K_0 *be a bound common to both functions. Define*

$$B_0 = (1+2\kappa)K_0 + (\kappa+1)\left(\frac{g_0^{\alpha_1}}{h_{x_1}^{\alpha_1}} + \frac{g_0^{\alpha_2}}{h_{x_2}^{\alpha_2}}\right) + [(2p-1)+(2p+1)\kappa]\left(\frac{|\lambda_1|g_0^{\beta_1}}{h_{x_1}^{\beta_1}} + \frac{|\lambda_2|g_0^{\beta_2}}{h_{x_2}^{\beta_2}}\right), \tag{3.26}$$

and assume that $0 < u^k < s$. *If* $\tau_k B_0 < \kappa$ *holds then* $F^u_{m,n,k} : [0,s] \to \mathbb{R}$ *is an increasing function for each* $m \in \{1,\ldots,M-1\}$ *and each* $n \in \{1,\ldots,N-1\}$.

Proof. Let K_0 be a common bound for f and f' on $[0,s]$, and let $H(w) = g'(w) + g(w)\varphi'(w)$ for each $w \in [0,s]$. After differentiating and simplifying algebraically, we readily check that

$$H(w) = \frac{G(w)}{w+\kappa}, \tag{3.27}$$

where

$$\begin{aligned} G(w) \quad = \quad & w + \kappa + \tau_k\Big\{w(w+\kappa)f'(w) + \kappa f(w) + cpw^{p+1} \\ & + \Big[\Big(d^k_{m,n,x_1} + d^k_{m,n,x_2}\Big)(p-1) + \kappa(p+1)c\Big]w^p \\ & + \kappa\Big(d^k_{m,n,x_1} + d^k_{m,n,x_2}\Big)pw^{p-1} - \kappa e + f^k_{m,n,x_1} + f^k_{m,n,x_2}\Big\}. \end{aligned} \tag{3.28}$$

Here we are omitting again the dependence of H and G on m, n and k. In light of Lemma 3.1, the function $F^u_{m,n,k}$ is increasing in $[0,s]$ if H is positive on $[0,s]$ or, equivalently, if G is positive. Note that the following inequalities are satisfied:

(a) $|w(w+\kappa)f'(w)| \leq (1+\kappa)K_0$.

(b) $|\kappa f(w)| \leq \kappa K_0$.

(c) $|pcw^{p+1}| \leq p\left(|\lambda_1|g_0^{\beta_1}h_{x_1}^{-\beta_1} + |\lambda_2|g_0^{\beta_2}h_{x_2}^{-\beta_2}\right)$.

(d) $|\left(d^k_{m,n,x_1} + d^k_{m,n,x_2}\right)(p-1)w^p| \leq -(p-1)\left(|\lambda_1|h_{x_1}^{-\beta_1}\sum_{\substack{i=0\\ i\neq m}}^{M} g^{\beta_1}_{m-i} + |\lambda_2|h_{x_2}^{-\beta_2}\sum_{\substack{i=0\\ i\neq n}}^{N} g^{\beta_2}_{n-i}\right)$

$\leq (p-1)\left(|\lambda_1|h_{x_1}^{-\beta_1}g_0^{\beta_1} + |\lambda_2|h_{x_2}^{-\beta_2}g_0^{\beta_2}\right)$.

(e) $|\kappa(p+1)cw^p| \leq \kappa(p+1)\left(|\lambda_1|h_{x_1}^{-\beta_1}g_0^{\beta_1} + |\lambda_2|h_{x_2}^{-\beta_2}g_0^{\beta_2}\right)$.

(f) Similarly to (d), $|\kappa d^n_j pw^{p-1}| \leq p\kappa\left(|\lambda_1|h_{x_1}^{-\beta_1}g_0^{\beta_1} + |\lambda_2|h_{x_2}^{-\beta_2}g_0^{\beta_2}\right)$.

(g) $|\kappa e| = \kappa\left(g_0^{\alpha_1}h_{x_1}^{-\alpha_1} + g_0^{\alpha_2}h_{x_2}^{-\alpha_2}\right)$.

(h) Similarly to (d), $|f^k_{m,n,x_1}| \leq g_0^{\alpha_1}h_{x_1}^{-\alpha_1}$.

(i) Similarly to (d), $|f^k_{m,n,x_2}| \leq g_0^{\alpha_2}h_{x_2}^{-\alpha_2}$.

As a consequence of these bounds and the hypotheses, note that $G(w) \geq \kappa - \tau_k B_0 > 0$ for each $w \in [0,s]$. So the function G is positive on $[0,s]$ and Lemma 3.1 guarantees now that $F^u_{m,n,k}$ is increasing on $[0,s]$, as desired. □

In the following, we will use the notation $\mathcal{R}_h$ to represent the range of any real function h. The next result summarizes the capability of the finite-difference method (3.12) to preserve the positivity and the boundedness of the approximations. To that end, it is important to note beforehand that the constants f^k_{m,n,x_1} and f^k_{m,n,x_2} are negative when $u^k > 0$, for each $k \in \{0,1,\ldots,K\}$, each $m \in \{1,\ldots,M-1\}$ and each $n \in \{1,\ldots,N-1\}$.

Theorem 3.4 (Positivity and boundedness). *Let $\kappa > 0$, $\lambda_1 < 0$, $\lambda_2 < 0$ and $s \in (0,1]$, and assume that f and f' are bounded on $[0,s]$. Let B_0 be as in Lemma 3.2, and $\mathcal{R}_\phi, \mathcal{R}_\psi \subseteq (0,s)$. If $f(s) = 0$ and $\tau_n B_0 < \kappa$ for each $k \in \{0,1,\ldots,K-1\}$ then there is a unique solution $(u^k)_{k=0}^K$ of (3.12) that satisfies*

$$0 < u^k < s, \quad \forall k \in \{0,1,\ldots,K\}. \tag{3.29}$$

Proof. We proceed inductively. Note that the conclusion is true when $n = 0$ by hypothesis, so assume that it is valid for some $k \in \{0,1,\ldots,K-1\}$ and let $m \in \{1,\ldots,M-1\}$ and $n \in \{1,\ldots,N-1\}$. Obviously Lemma 3.2 guarantees that the function $F^u_{m,n,k}$ is increasing on $[0,s]$. Moreover, note that

$$F^u_{m,n,k}(0) = \kappa \exp\left(-\frac{\tau_k\left(f^k_{m,n,x_1} + f^k_{m,n,x_2}\right)}{\kappa}\right) - \kappa > \kappa e^0 - \kappa = 0. \tag{3.30}$$

On the other hand, define $g^k_{m,n} = cs^{p+1} + \left(d^k_{m,n,x_1} + d^k_{m,n,x_2}\right)s^p - es - f^k_{m,n,x_1} - f^k_{m,n,x_2}$. Using the properties of Lemma **??** we obtain that

$$\begin{aligned} g^k_{m,n} &= \frac{\lambda_1 s^p}{h^{\beta_1}_{x_1}}\left[g^{\beta_1}_0 s + \sum_{\substack{i=0\\ i\neq m}}^{M} g^{\beta_1}_{m-i} u^k_{i,n}\right] + \frac{\lambda_2 s^p}{h^{\beta_2}_{x_2}}\left[g^{\beta_2}_0 s + \sum_{\substack{i=0\\ i\neq n}}^{N} g^{\beta_2}_{n-i} u^k_{m,i}\right] \\ &\quad -\frac{1}{h^{\alpha_1}_{x_1}}\left[g^{\alpha_1}_0 s + \sum_{\substack{i=0\\ i\neq m}}^{M} g^{\alpha_1}_{m-i} u^k_{i,n}\right] - \frac{1}{h^{\alpha_2}_{x_2}}\left[g^{\alpha_2}_0 s + \sum_{\substack{i=0\\ i\neq n}}^{N} g^{\alpha_2}_{n-i} u^k_{m,i}\right] < 0. \end{aligned} \tag{3.31}$$

As a consequence,

$$F^u_{m,n,k}(s) = (s+\kappa)\exp\left(\frac{\tau_k g^k_{m,n}}{s+\kappa}\right) - \kappa < (s+\kappa)e^0 - \kappa = s. \tag{3.32}$$

Summarizing, we have established that the function $F^u_{m,n,k} : [0,s] \to \mathbb{R}$ is increasing, and that $0 < F^u_{m,n,k}(0) < F^u_{m,n,k}(s) < s$. The fact that $0 < u^k_{m,n} < s$ implies that $u^{k+1}_{m,n} = F^u_{m,n,k}(u^k_{m,n})$ belongs to $(0,s)$ for each $m \in \{1,\ldots,M-1\}$ and each $n \in \{1,\ldots,N-1\}$. Using the boundary data leads to obtain that $0 < u^{k+1} < s$, and the result follows now from induction. □

We would like to establish not that the finite-difference method (3.12) is capable to preserve the monotonicity of the approximations, but the method (3.12) is two–dimensional, in this case, we will consider (3.12) one–dimensional to monotonicity makes sense. In the statement of our next result, we will consider two sets of initial-boundary conditions which will be denoted by (ϕ^u, ψ^u) and (ϕ^v, ψ^v), respectively. The corresponding numerical solutions obtained through (3.12) will be denoted by $(u^k)_{k=0}^K$ and $(v^k)_{k=0}^K$, respectively.

Theorem 3.5 (Monotonicity)**.** *Let $\kappa > 0$, $\lambda < 0$ and $s \in (0,1]$, and assume that f and f' are bounded on $[0,s]$. Let B_0 be as in Lemma 3.2, and let $\mathcal{R}_{\phi^z}, \mathcal{R}_{\psi_1^z}, \mathcal{R}_{\psi_2^z} \subseteq (0,s)$ for $z = u, v$. Suppose that $f(s) = 0$ and $\tau_n B_0 < \kappa$, and that the following are satisfied:*

(a) *$\phi^u(w) < \phi^v(w)$ for each $w \in [0,s]$, and*

(b) *$\psi_i^u(t) < \psi_i^v(t)$ for each $t \in [0,T]$ and $i = 1,2$.*

Then there exist unique solutions $(u^n)_{n=0}^N$ and $(v^n)_{n=0}^N$ of (3.12) corresponding to $(\phi^u, \psi_1^u, \psi_2^u)$ and $(\phi^v, \psi_1^v, \psi_2^v)$, respectively, and they satisfy

$$0 < u^n < v^n < s, \quad \forall n \in \{0,1,\ldots,N\}. \tag{3.33}$$

Proof. Beforehand note that Theorem 2.2 guarantees the existence and the uniqueness of the solutions $(u^n)_{n=0}^N$ and $(v^n)_{n=0}^N$, and that they satisfy the inequalities $0 < u^n < s$ and $0 < v^n < s$ for each $n \in \{0,1,\ldots,N\}$. It remains to prove that $u^n < v^n$ is also satisfied (which is true for $n = 0$ by hypothesis), so suppose that this inequality holds for some $n \in \{0,1,\ldots,N-1\}$ and fix $j \in \{1,\ldots,M-1\}$. Let $\Psi_j : \mathbb{R}^{M+1} \to \mathbb{R}$ be given by

$$\Psi_j(w) = w_j f(w_j) + c(w_j)^{p+1} + d_j(w_j)^p - e w_j - f_j, \tag{3.34}$$

for each $w = (w_0, w_1, \ldots, w_M) \in [0,s]^{M+1}$. Here the parameters c and e are as before, and we use the constants

$$d_j = \frac{\lambda}{h^\beta} \sum_{\substack{k=0 \\ k \neq j}}^{M} g_{j-k}^{\beta} w_k, \qquad f_j = \frac{1}{h^\alpha} \sum_{\substack{k=0 \\ k \neq j}}^{M} g_{j-k}^{\alpha} w_k. \tag{3.35}$$

A simple calculation shows that if $k \in \{0,1,\ldots,M\}$ and $w \in (0,s)^{M+1}$ then

$$\frac{\partial \Psi_j}{\partial w_k}(w) = \left[\frac{\lambda g_{j-k}^{\beta}}{h^\beta}(w_j)^p - \frac{g_{j-k}^{\alpha}}{h^\alpha} \right] > 0, \quad \text{if } k \neq j. \tag{3.36}$$

This implies in particular that for each $v, w \in (0,s)^{M+1}$ satisfying $v_k < w_k$ for each $k \neq j$, and $v_j = w_j$ then $\Psi_j(v) < \Psi_j(w)$. Let $v_u^n = (v_0^n, v_1^n, \ldots, v_{j-1}^n, u_j^n, v_{j+1}^n, \ldots, v_M^n)$ now. The following identities and inequalities are now trivial in light of these facts and Lemma 3.2, and they are satisfied for each $j \in \{1,\ldots,M-1\}$:

$$\begin{aligned} u_j^{n+1} &= F_{j,n}^u(u_j^n) = (u_j^n + \kappa)\exp\left(\frac{\tau_n \Psi_j(u^n)}{u_j^n + \kappa} \right) - \kappa \\ &< (u_j^n + \kappa)\exp\left(\frac{\tau_n \Psi_j(v_u^n)}{u_j^n + \kappa} \right) - \kappa = F_{j,n}^v(u_j^n) < F_{j,n}^v(v_j^n) = v_j^{n+1}. \end{aligned} \tag{3.37}$$

This and the boundary conditions imply that $u^{n+1} < v^{n+1}$. The conclusion of the theorem is reached now using induction. □

The following corollaries are easy consequences of Theorem 3.5.

Corollary 3.1 (Temporal monotonicity)**.** *Let $\kappa > 0$, $\lambda < 0$ and $s \in (0,1]$, and assume that f and f' are bounded on $[0,s]$. Let B_0 be as in Lemma 3.2, and let $\mathcal{R}_\phi, \mathcal{R}_{\psi_1}, \mathcal{R}_{\psi_2} \subseteq (0,s)$. Suppose that $f(s) = 0$ and $\tau_n B_0 < \kappa$. Then*

$$0 < u^n < u^{n+1} < 1, \quad \forall n \in \{0,1,\ldots,N-1\} \tag{3.38}$$

whenever $0 < u^0 < u^1 < 1$ and the functions ψ_1 and ψ_2 are increasing. □

Definition 3.3. A real vector $u = (u_0, u_1, \ldots, u_M)$ is *increasing* if $u_j < u_{j+1}$ for each $j \in \{0,1,\ldots,M-1\}$. If $-u$ is increasing then we say that u is *decreasing*.

Corollary 3.2 (Spatial monotonicity)**.** *Let $\kappa > 0$, $\lambda < 0$ and $s \in (0,1]$, and assume that f and f' are bounded on $[0,s]$. Let B_0 be as in Lemma 3.2, and let $\mathcal{R}_\phi, \mathcal{R}_{\psi_1}, \mathcal{R}_{\psi_2} \subseteq (0,s)$. Suppose that $f(s) = 0$ and $\tau_n B_0 < \kappa$.*

(a) *If ϕ, ψ_1 and ψ_2 are increasing, and if $u_0^n < u_1^n$ and $u_{M-1}^n < u_M^n$ for each $n \in \{0,1\ldots,N\}$ then u^n is increasing for all $n \in \{0,1,\ldots,N\}$.*

(b) *If ϕ, ψ_1 and ψ_2 are decreasing, and if $u_1^n < u_0^n$ and $u_M^n < u_{M-1}^n$ for each $n \in \{0,1\ldots,N\}$ then u^n is decreasing for all $n \in \{0,1,\ldots,N\}$.* □

3.5 Numerical results

The present section is devoted to establish some numerical properties of (3.12) and to provide simulations that illustrate the capability of the method to preserve the positivity, the boundedness and the monotonicity of the approximations. In a first stage, we show that our method is consistent with the partial differential equation of (3.5). To that end, we introduce the continuous and discrete operators

$$\begin{aligned} \mathcal{L}u(x,t) &= (u(x,t)+\kappa)\frac{\partial}{\partial t}\ln(u(x,t)+\kappa) - \sum_{i=1}^{2}\frac{\partial^{\alpha_i} u}{\partial|x_i|^{\alpha_i}}(x,t) \\ &\quad + u^p(x,t)\sum_{i=1}^{2}\lambda_i\frac{\partial^{\beta_i} u}{\partial|x_i|^{\beta_i}}(x,t) - u(x,t)f(u(x,t)), \end{aligned} \tag{3.39}$$

$$\begin{aligned} Lu_{m,n}^k &= (u_{m,n}^k+\kappa)\delta_t \ln(u_{m,n}^k+\kappa) - \sum_{i=1}^{2}\delta_{x_i}^{\alpha_i} u_{m,n}^k + (u_{m,n}^k)^p \sum_{i=1}^{2}\lambda_i \delta_{x_i}^{\beta_i} u_{m,n}^k, \\ &\quad - u_{m,n}^k f(u_{m,n}^k), \end{aligned} \tag{3.40}$$

defined for each $(x,t) \in \Omega$, each $m \in \{1,\ldots,M-1\}$, each $n \in \{1,\ldots,N-1\}$ and each $k \in \{0,1,\ldots,K-1\}$. Also, we will employ the symbols $\|\cdot\|_2$ and $\|\cdot\|_\infty$ to represent the Euclidean norm and the maximum norm in $\mathbb{R}^{(M+1)(N+1)}$, respectively.

Theorem 3.6 (Consistency)**.** *If $u \in \mathcal{C}_{x_1,x_2,t}^{5,5,2}(\overline{\Omega})$ is a positive function and $\kappa > 0$ then there exists a constant $C_0 > 0$ which is independent of h_{x_1}, h_{x_2} and τ_k such that for each $m \in \{1,\ldots,M-1\}$, each $n \in \{1,\ldots,N-1\}$ and each $k \in \{1,\ldots,K-1\}$,*

$$\left|\mathcal{L}u(x_{m,n},t_k) - Lu_{m,n}^k\right| \leq C_0(\tau_k + h_{x_1}^2 + h_{x_2}^2). \tag{3.41}$$

Proof. We employ here Lemma **??** and the usual argument with Taylor polynomials. Using the hypotheses on the continuous differentiability of u there exist positive constants C_1, C_2, C_3, C_4 and C_5

such that

$$\left|(u(x_{m,n},t_k)+\kappa)\frac{\partial}{\partial t}\ln(u(x_{m,n},t_k)+\kappa)-(u_{m,n}^k+\kappa)\delta_t\ln(u_{m,n}^k+\kappa)\right| \leq C_1\tau_k, \tag{3.42}$$

$$\left|\frac{\partial^{\alpha_1}u}{\partial|x_1|^{\alpha_1}}(x_{m,n},t_k)-\delta_{x_1}^{\alpha_1}u_{m,n}^k\right| \leq C_2h_{x_1}^2, \tag{3.43}$$

$$\left|\frac{\partial^{\alpha_2}u}{\partial|x_2|^{\alpha_2}}(x_{m,n},t_k)-\delta_{x_2}^{\alpha_2}u_{m,n}^k\right| \leq C_3h_{x_2}^2, \tag{3.44}$$

$$\left|u^p(x_{m,n},t_k)\frac{\partial^{\beta_1}u}{\partial|x_1|^{\beta_1}}(x_{m,n},t_k)-(u_{m,n}^k)^p\delta_{x_1}^{\beta_1}u_{m,n}^k\right| \leq C_4h_{x_1}^2, \tag{3.45}$$

$$\left|u^p(x_{m,n},t_k)\frac{\partial^{\beta_2}u}{\partial|x_2|^{\beta_2}}(x_{m,n},t_k)-(u_{m,n}^k)^p\delta_{x_2}^{\beta_2}u_{m,n}^k\right| \leq C_5h_{x_2}^2, \tag{3.46}$$

for each $m\in\{1,\dots,M-1\}$, each $n\in\{1,\dots,N-1\}$ and each $k\in\{1,\dots,K-1\}$. Note also that

$$|u(x_{m,n},t_k)f(u(x_{m,n},t_k))-u_{m,n}^kf(u_{m,n}^k)|=0. \tag{3.47}$$

The conclusion of this theorem is readily reached using the triangle inequality and defining the constant $C_0=\max\{C_1,C_2,C_3|\lambda_1|C_4,|\lambda_2|C_5\}$. □

Next, we tackle the problem of stability of the method (3.12). In the following, for any real numbers x and y we use $x\vee y$ to represent the maximum of x and y.

Theorem 3.7 (Stability)**.** *Let $\kappa>0$, $\lambda_1<0$, $\lambda_2<0$ and $s\in(0,1]$, and let f and f' be bounded on $[0,s]$ with $f(s)=0$. Let B_0 be as in Lemma 3.2 and $\tau_kB_0<\kappa$ for each $k\in\{0,1,\dots,K-1\}$. There exists a constant C such that for any sets of initial-boundary conditions (ϕ^u,ψ^u) and (ϕ^v,ψ^v) with ranges bounded in $(0,s)$, the corresponding solutions satisfy*

$$\|u^k-v^k\|_\infty\leq C\|u^0-v^0\|_\infty,\quad\forall k\in\{0,1,\dots,K\}. \tag{3.48}$$

Proof. The hypotheses guarantee the existence and the uniqueness of the solutions $(u^k)_{k=0}^K$ and $(v^k)_{k=0}^K$, satisfying $0<u^k<s$ and $0<v^k<s$ for each $k\in\{0,1,\dots,K\}$. To derive the constant C, let us define the function $\Phi_{m,n}^k:[0,s]^{(M+1)(N+1)}\to\mathbb{R}$ for each $m\in\{1,\dots,M-1\}$, each $n\in\{1,\dots,N-1\}$ and each $k\in\{0,1,\dots,K-1\}$ by

$$\Phi_{m,n}^k(u)=(u_{m,n}+\kappa)\exp\left(\frac{\tau_k\Psi(u)}{u_{m,n}+\kappa}\right)-\kappa,\quad\forall u\in[0,s]^{(M+1)(N+1)}. \tag{3.49}$$

Here we are using the nomenclature employed in Theorem 3.5. It is obvious that each of the functions $\Phi_{m,n}^k$ is of class $\mathcal{C}^1([0,s]^{(M+1)(N+1)})$, so the numbers $C_{m,n,k}=\max_{[0,s]^{(M+1)(N+1)}}\|\nabla\Phi_{m,n}^k\|_2$ exist in $\mathbb{R}$. Moreover, the mean value theorem guarantees that for each $u,v\in[0,s]^{(M+1)(N+1)}$ there exists $\xi\in[0,s]^{(M+1)(N+1)}$ such that

$$|\Phi_{m,n}^k(u)-\Phi_{m,n}^k(v)|\leq\|\nabla\Phi_{m,n}^k(\xi)\|_2\|u-v\|_2\leq C_{m,n,k}\sqrt{(M+1)(N+1)}\|u-v\|_\infty. \tag{3.50}$$

As a consequence, note that for each $m \in \{1,\ldots,M-1\}$ and each $n \in \{1,\ldots,N-1\}$,

$$|u_{m,n}^{k+1} - v_{m,n}^{k+1}| = |\Phi_{m,n}^k(u^k) - \Phi_{m,n}^k(v^k)| \leq C_k \|u^k - v^k\|_\infty, \tag{3.51}$$

where

$$C_k = 1 \vee \max\{C_{m,n,k}\sqrt{(M+1)(N+1)} : 1 \leq m \leq M-1, \quad 1 \leq n \leq N-1\}. \tag{3.52}$$

It is clear then that $\|u^{k+1} - v^{k+1}\|_\infty \leq C_k\|u^k - v^k\|_\infty$ for each $k \in \{0,1,\ldots,K-1\}$. Using recursion we obtain that the inequality $\|u^k - v^k\|_\infty \leq C\|u^0 - v^0\|_\infty$ is satisfied for each $k \in \{0,1,\ldots,K\}$, where

$$C = (M+1)^{K/2}(N+1)^{K/2}\prod_{k=0}^{K-1} C_k, \tag{3.53}$$

as desired. □

Next, we tackle the property of convergence of the numerical model (3.12).

Theorem 3.8 (Convergence)**.** *Let $u \in \mathcal{C}^{5,5,2}_{x_1,x_2,t}(\overline{\Omega})$ be a solution of* (3.5) *satisfying* $0 \leq u(x,t) \leq 1$ *for each* $\overline{\Omega}$*. Let* $(v^k)_{k=0}^K$ *be a solution of* (3.13) *satisfying* $0 \leq v^k \leq 1$, *for each* $k \in \{0,1,\ldots,K\}$*. If*

$$\exp(\tau_k/\lambda) - 1 \leq 2\tau_k/\lambda, \tag{3.54}$$

then there exists a constant C' independent of τ_k, h_{x_1} and h_{x_2}, such that the following is satisfied for each $k \in \{0,1,\ldots,K\}$:

$$\|u^k - v^k\|_\infty \leq C'(\tau_k + h_{x_1}^2 + h_{x_2}^2) \tag{3.55}$$

Proof. Throughout, we will use the notation used in Theorem 3.7. Let $e_{m,n}^k = u_{m,n}^k - v_{m,n}^k$ for each $m \in \{0,1,\ldots,M\}$, $n \in \{0,1,\ldots,N\}$ and $k \in \{0,1,\ldots,K\}$. The exact solution $u(x,t)$ of (3.5) satisfies the finite–difference method (3.13) in the point $(x_{m,n}, t_k)$ with truncation error $R_{n,m}^k$, for each $m \in \{0,1,\ldots,M-1\}$, $n \in \{0,1,\ldots,N-1\}$ and $k \in \{0,1,\ldots,K-1\}$. We have that the exact and the numerical solutions satisfy the equations

$$(u_{m,n}^k + \lambda)Lu_{m,n}^k = R_{m,n}^k, \tag{3.56}$$

$$(v_{m,n}^k + \lambda)Lv_{m,n}^k = 0, \tag{3.57}$$

respectively, for each $m \in \{0,1,\ldots,M-1\}$, $n \in \{0,1,\ldots,N-1\}$ and $k \in \{0,1,\ldots,K-1\}$. By Theorem 3.6, there exists $C_0 > 0$ such that $\left|R_{m,n}^k\right| \leq C_0(\tau_k + h_{x_1}^2 + h_{x_2}^2)$ for each m,n and k. Equations (3.56) and (3.57) could be seen as

$$u_{m,n}^{k+1} = (u_{m,n}^k + \lambda)\exp\left(\frac{\tau_k R_{m,n}^k}{u_{m,n}^k + \lambda}\right)\exp\left(\varphi(u_{m,n}^k)\right) - \lambda, \tag{3.58}$$

$$v_{m,n}^{k+1} = (v_{m,n}^k + \lambda)\exp\left(\varphi(v_{m,n}^k)\right) - \lambda, \tag{3.59}$$

for each $m \in \{0,1,\ldots,M-1\}$, $n \in \{0,1,\ldots,N-1\}$ and $k \in \{0,1,\ldots,K-1\}$. Subtracting these identities, we obtain

$$\begin{aligned}\left|e_{m,n}^{k+1}\right| &\leq (u_{m,n}^k+\lambda)\left[\exp\left(\frac{\tau_k R_{m,n}^k}{u_{m,n}^k+\lambda}\right)-1\right]\exp\left(\varphi(u_{m,n}^k)\right)\\ &\quad+\left|\Phi_{m,n}^k(u)-\Phi_{m,n}^k(v)\right|\\ &\leq (\lambda+1)D_{m,n}^k\left[\exp(\tau_k R_{m,n}^k/\lambda)-1\right]+C_{m,n,k}\|u^k-v^k\|_2\\ &\leq D\tau_k R_{m,n}^k+C\|e^k\|_\infty, \end{aligned} \tag{3.60}$$

where

$$e^k=(e_{0,0}^k,e_{0,1}^k,\ldots,e_{0,N}^k,\ e_{1,0}^k,e_{1,1}^k,\ldots,e_{1,N}^k,\ldots,\ e_{M,0}^k,e_{M,1}^k,\ldots,e_{M,N}^k), \tag{3.61}$$

$$D_{m,n}^k=\max\left\{\exp\left(\varphi(u_{m,n}^k)\right):u\in[0,1]^{(M+1)(N+1)}\right\}, \tag{3.62}$$

for each $m\in\{0,1,\ldots,M-1\}$, $n\in\{0,1,\ldots,N-1\}$ and $k\in\{0,1,\ldots,K-1\}$, and

$$C=\max\left\{C_{m,n,k}\sqrt{(M+1)(N+1)}:m=1,\ldots,M-1;n=1,\ldots,N-1;k=1,\ldots,K-1\right\}, \tag{3.63}$$

$$D=\max\left\{\frac{2(\lambda+1)D_{m,n}^k}{\lambda}:m=1,\ldots,M-1;n=1,\ldots,N-1;k=1,\ldots,K-1\right\}. \tag{3.64}$$

Here, $\Phi_{m,n}^k$ and $C_{m,n,k}$ are as in the proof of Theorem 3.7. Moreover, all the constants $C_{m,n,k}$ are elements of $[0,1]$, whence is follows that $0\leq C\leq 1$. Therefore, by Theorem 3.6 and by the inequality (3.60), we have that

$$\begin{aligned}\|e^{k+1}\|_\infty-\|e^k\|_\infty &\leq \|e^{k+1}\|_\infty-C\|e^k\|_\infty && (3.65)\\ &\leq C_0D\tau_k(\tau_k+h_{x_1}^2+h_{x_2}^2) && (3.66)\\ &\leq C_0D\tau_k(\tau+h_{x_1}^2+h_{x_2}^2) && (3.67)\\ & && (3.68)\end{aligned}$$

for each $k\in\{0,1,\ldots,K-1\}$ and

$$\tau=\max\{\tau_k:k=0,1,\ldots,K-1\}. \tag{3.69}$$

Taking summation on both sides of this inequality and using the exactness of the numerical initial conditions, we have

$$\|e^{l+1}\|_\infty=\|e^{l+1}\|_\infty-\|e^0\|_\infty\leq C_0DT(\tau+h_{x_1}^2+h_{x_2}^2)\leq C'(\tau+h_{x_1}^2+h_{x_2}^2), \tag{3.70}$$

for each $l\in 0,1,\ldots,K-1$, where we have used $C'=C_0DT$. □

Recall that K_0 represented a common bound for f and f' in the context of Lemma 3.2. This notation will be observed in the next result, which provides some *a priori* bound for the numerical solutions of (3.12).

Theorem 3.9 (*A priori* bounds)**.** *Let $\kappa > 0$, $\lambda < 0$ and $s \in (0,1]$, and let f and f' be bounded on $[0,s]$ with $f(s) = 0$. Let B_0 be as in Lemma 3.2 and $\tau_k B_0 < \kappa$ for each $k \in \{0,1,\ldots,K-1\}$. If $\mathcal{R}_\phi, \mathcal{R}_{\psi_1}, \mathcal{R}_{\psi_2} \subseteq (0,s)$ then there exists a constant $C > 1$ such that*

$$\|u^k\|_\infty \leq C\|u^0\|_\infty + \kappa(C-1), \quad \forall k \in \{0,1,\ldots,K\}. \tag{3.71}$$

Proof. Beforehand note that the hypotheses and Theorem 3.1 guarantee the existence and the uniqueness of positive and bounded solutions for (3.12). On the other hand, observe that $\delta_{x_i}^{\alpha_i} u_{m,n}^k \leq g_0^{\alpha_i}/h_{x_i}^{\alpha_i}$ and $\delta_{x_i}^{\beta_i} u_{m,n}^k \leq g_0^{\beta_i}/h_{x_i}^{\beta_i}$ hold for each $m \in \{0,1,\ldots,M-1\}$, $n \in \{0,1,\ldots,N-1\}$, $k \in \{1,\ldots,K-1\}$ and $i = 1,2$. Note then that

$$\begin{aligned}
\exp(\phi(u_{m,n}^k)) &= \exp\left[\frac{\tau_k\left(\sum_{i=1}^2 \delta_{x_i}^{\alpha_i} u_{m,n}^k - (u_{m,n}^k)^p \sum_{i=1}^2 \lambda_i \delta_{x_i}^{\beta_i} u_{m,n}^k + u_{m,n}^k f(u_{m,n}^k)\right)}{u_{m,n}^k + \kappa}\right] \\
&= \exp\left(\frac{\tau_k \sum_{i=1}^2 \delta_{x_i}^{\alpha_i} u_{m,n}^k}{u_{m,n}^k + \kappa}\right) \exp\left(-\frac{\tau_k (u_{m,n}^k)^p \sum_{i=1}^2 \lambda_i \delta_{x_i}^{\beta_i} u_{m,n}^k}{u_{m,n}^k + \kappa}\right) \\
&\quad \exp\left(\frac{\tau_k u_{m,n}^k f(u_{m,n}^k)}{u_{m,n}^k + \kappa}\right) \\
&\leq \exp\left(\tau_k \sum_{i=1}^2 \frac{g_0^{\alpha_i}}{h_{x_i}^{\alpha_i}}\right) \exp\left(\tau_k \sum_{i=1}^2 \frac{|\lambda_i| g_0^{\beta_i}}{h_{x_i}^{\beta_i}}\right) \exp\left(\tau_k f(u_{m,n}^k)\right) \leq C_k,
\end{aligned} \tag{3.72}$$

where

$$C_n = \exp\left(\tau_k\left(\sum_{i=1}^2 \frac{g_0^{\alpha_i}}{h_{x_i}^{\alpha_i}} + \sum_{i=1}^2 \frac{|\lambda_i| g_0^{\beta_i}}{h_{x_i}^{\beta_i}} + K_0\right)\right). \tag{3.73}$$

Let $C = \prod_{k=0}^{K-1} C_k$, and observe that the following is satisfied for each $m \in \{0,1,\ldots,M-1\}$, $n \in \{0,1,\ldots,N-1\}$ and each $k \in \{1,\ldots,K-1\}$:

$$u_{m,n}^{k+1} = (u_{m,n}^k + \kappa)\exp\left(\phi(u_{m,n}^k)\right) - \kappa \leq C_k u_{m,n}^k + \kappa(C_k - 1) \leq C_k\|u^k\|_\infty + \kappa(C_k - 1). \tag{3.74}$$

Once that this inequality is established, it is easy to prove now the validity of the conclusion of this theorem using induction. □

In this part, we do some computer simulations to show that the finite-difference method (3.12) is capable of preserving the main analytical features of the solutions of interest of (3.5). Concretely, we illustrate the capability of the method to preserve the positivity and the boundedness. The simulations were obtained using our own implementation of the method in ©Matlab 8.1.0.604 (R2013a), on a ©ASUS Tp501ua laptop computer with Windows 10 as operating system.

Example 3.1. Let us consider the continuous model (3.5) with parameters $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = \lambda_2 = 0$, $p = 1$, $\kappa = 1$, and f is given by (**??**). We will consider the spatial domain $B = (-200,200) \times (-200,200) \subseteq \mathbb{R}^2$, and the computational constants $h_{x_1} = h_{x_2} = 1$ and $\tau = 0.05$. Let us fix homogeneous Dirichlet conditions on the boundary of B, and consider the initial profiles

$$\phi^1(x,y) = \phi^2(x,y) = \begin{cases} 0.2, & \text{if } (x,y) = (0,0), \\ 0, & \text{otherwise.} \end{cases} \tag{3.75}$$

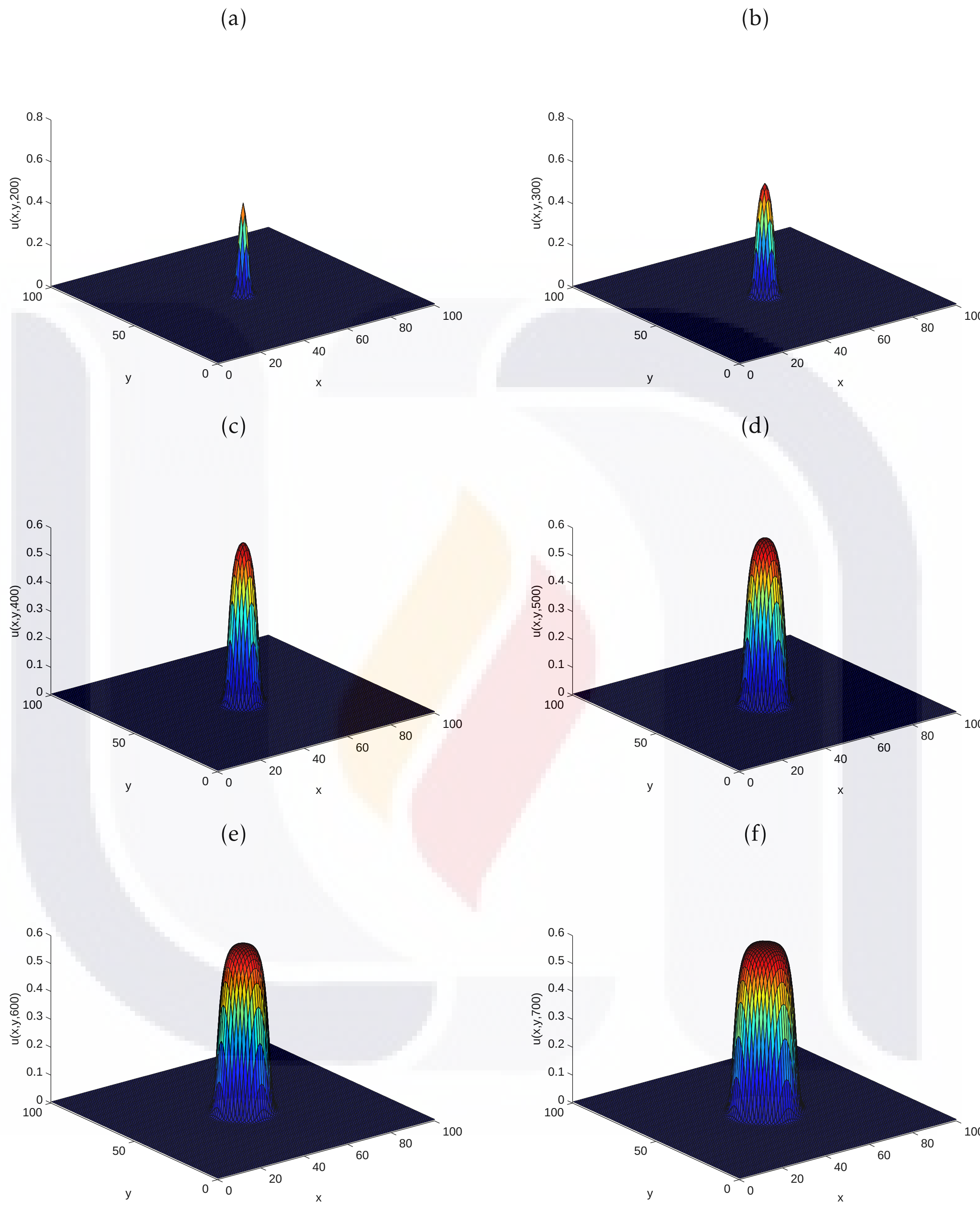

Figure 3.1: Snapshots of the approximate solution u of the model (3.5) as a function of $(x, y) \in \overline{B} = [-200, 200] \times [-200, 200]$, for the times (a) $t = 200$, (b) $t = 300$, (c) $t = 400$, (d) $t = 500$, (e) $t = 600$ and (f) $t = 700$. The model uses the parameters $\alpha_1 = \alpha_2 = 2$, $\lambda_1 = \lambda_2 = 0$, $p = 1$, $\kappa = 1$, and f is given by (**??**). We employed homogeneous Dirichlet conditions on the boundary of B, along with the initial profiles (3.75). Computationally, we let $h_{x_1} = h_{x_2} = 4$ and $\tau = 0.05$.

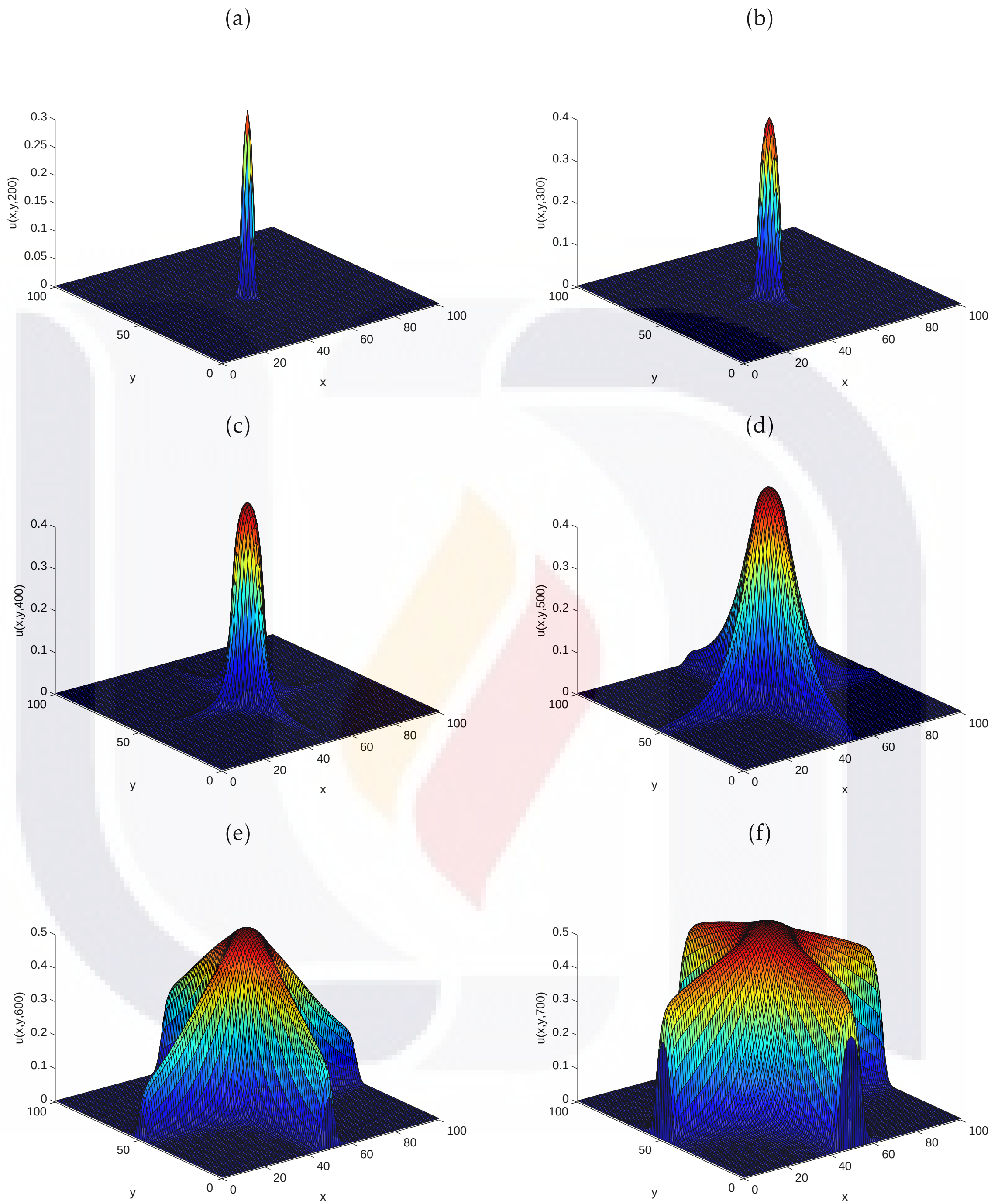

Figure 3.2: Snapshots of the approximate solution u of the model (3.5) as a function of $(x, y) \in \overline{B} = [-200, 200] \times [-200, 200]$, for the times (a) $t = 200$, (b) $t = 300$, (c) $t = 400$, (d) $t = 500$, (e) $t = 600$ and (f) $t = 700$. The model uses the parameters $\alpha_1 = 1.9$, $\alpha_2 = 1.95$, $\beta_1 = 0.8$, $\beta_2 = 0.9$, $\lambda_1 = \lambda_2 = 1$, $p = 1$, $\kappa = 1$, and f is given by (**??**). We employed homogeneous Dirichlet conditions on the boundary of B, along with the initial profiles (3.75). Computationally, we let $h_{x_1} = h_{x_2} = 4$ and $\tau = 0.05$.

We can note that we consider a problem without convective effects and with partial derivatives of integer order. In such situation, the classical solution of the initial-boundary-value problem (3.5) is nonnegative and bounded. Figure 3.1 shows snapshots of the approximate solution u as a function of x and y, for the times (a) $t = 200$, (b) $t = 300$, (c) $t = 400$, (d) $t = 500$, (e) $t = 600$ and (f) $t = 700$. The solutions suggest that the method is capable of preserving the positivity and the boundedness of the approximations, in agreement with Theorem 3.4. □

Example 3.2. Let us consider the same problem as in Example 3.1, using the constants $\alpha_1 = 1.9$, $\alpha_2 = 1.95$, $\beta_1 = 0.8$, $\beta_2 = 0.9$, $\lambda_1 = \lambda_2 = 1$, $p = 1$ and $\kappa = 1$, together with the same computational parameters and the same initial-boundary conditions as in the previous example. The results of our simulations are shown in Figure 2.2. In this case, anomalous diffusion and convection are considered in the x and y directions. It is worth pointing out that the properties of positivity and boundedness of the approximate solutions are preserved, in agreement with Theorem 3.4. □

It is important to mention that we have conducted more simulations with different model parameters and different initial-boundary conditions. The results are not presented in this work in view of their repetitiveness: they also confirm the capability of the finite-difference method (3.12) to preserve the analytical features of the solutions of interest of (3.5), namely, the positivity and the boundedness of the solutions.

4. Conclusions and discussions

In chapter 1, we worked with a discrete system formed by interacting particles. We considered the particles are arranged in a linear pattern and the distance between particle is the same. This system was modeled by a discrete motion equation. We defined the concept of α-interaction. We defined a transform operation to obtain the continuous motion equation of the discrete system. The transform operation is the continuous limit process that involves the Fourier series transform, the limit when the distance between particles tends to zero and the inverse Fourier transform. In the continuous motion equation, we found the Riesz fractional derivative in space. We considered the generalization of the discrete system in the three-dimensional case.

In chapter 2, we investigated numerically a multidimensional generalization of the well–known Burgers–Fisher and Burgers–Huxley equations with anomalous diffusion and convection terms. The fractional derivatives considered are of the Riesz type, and the model was discretized using fractional centered differences in a linear approach. The method proposed in this work is a Crank–Nicolson–type implicit finite–difference scheme which is capable of preserving many features of some solutions of the Burgers–Fisher and the Burgers–Huxley models with derivatives of integer order. For instance, the methodology proposed in this work is capable of preserving the positivity and the boundedness of the approximations. A theorem on the existence and the uniqueness of approximations was also established in this work using the theory of Minkowski matrices. Moreover, the technique has the same constant solutions as its continuous counterpart. The consistency, the stability and the convergence of the technique were proved thoroughly in this work. As two of the most important results, we showed that the method is quadratically convergent and unconditionally stable under suitable scenarios. Moreover, we derived suitable *a priori* bounds for the numerical solutions of our finite–difference–scheme. Simulations carried out with a computational implementation of our method show that the technique is capable of preserving the analytical features of the solutions.

In chapter 3, we investigated numerically a generalization of the well-known the Burgers–Fisher and Burgers–Huxley equations with fractional diffusion and advection terms. The fractional derivatives considered are of the Riesz type, and the model was discretized using fractional centered differences and an exponential approach. The method proposed in this work is an explicit finite-difference scheme which is capable of preserving many features of some solutions of the Burgers–Fisher and

Burgers–Huxley models with derivatives of integer order. For instance, the methodology proposed in this work is capable of preserving the positivity, the boundedness and the monotonicity of the approximations. Moreover, we observed that our the technique and its continuous counterpart both have the same constant solutions. We showed in this work the numerical properties of our technique which are consistency, stability and convergence, along with some *a priori* bounds for the numerical solutions. We did some simulations of our technique using a computational implementation to show that our technique is capable of preserving the structural properties of the solutions.

Finally, we would like to add that the initial objective of this work was to apply fractional methods to the processing of images. We applied the methods reported in this thesis, but the results that we obtained did not yield satisfactory outcomes. More precisely, we expected to obtain better results than those reported on the literature, however, that was not the case. We are convinced that the problem lies in our use of the diffusion factors. We would expect to have better results using different diffusion coefficients. Such task could be the topic of research in a future doctoral dissertation.

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