CENTRO DE CIENCIAS BÁSICAS

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA

TESIS

DISSIPATION-PRESERVING METHODS FOR MULTIDIMENSIONAL NONLINEAR DAMPED WAVE EQUATIONS OF FRACTIONAL ORDER

PRESENTA

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PARA OPTAR POR EL GRADO DE MAESTRO EN CIENCIAS EN MATEMÁTICAS APLICADAS

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Aguascalientes, Ags., 8 de Abril de 2019

UNIVERSIDAD AUTONOMA
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FORMATO DE CARTA DE VOTO APROBATORIO

M. en C. José de Jesús Ruiz Gallegos
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Por medio de la presente, en mi calidad de sinodal designado del estudiante **ADÁN JAIR SERNA REYES** con ID 105842 quien realizó la tesis titulada: **DISSIPATION-PRESERVING METHODS FOR MULTIDIMENSIONAL NONLINEAR DAMPED WAVE EQUATIONS OF FRACTIONAL ORDER**, y con fundamento en el Artículo 175, Apartado II del Reglamento General de Docencia, me permito emitir el **VOTO APROBATORIO**, para que él pueda proceder a imprimirla, y así continuar con el procedimiento administrativo para la obtención del grado.

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Estimado alumno:

Por medio de este conducto me permito comunicar a Usted que habiendo recibido los votos aprobatorios de los revisores de su trabajo de tesis y/o caso práctico titulado: ***"DISSIPATION-PRESERVING METHODS FOR MULTIDIMENSIONAL NONLINEAR DAMPED WAVE EQUATIONS OF FRACTIONAL ORDER"*** hago de su conocimiento que puede imprimir dicho documento y continuar con los trámites para la presentación de su examen de grado.

Sin otro particular me permito saludarle muy afectuosamente.

ATENTAMENTE
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EL DECANO

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Adán Jair Serna Reyes

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Resumen

En este documento, trabajamos con varios modelos: la ecuación sine-Gordon, además de una ecuación de onda no lineal con p dimensiones espaciales, amortiguamiento y derivadas espaciales fraccionarias y finalmente el caso $p = 2$ para un método particular. El primer modelo tuvo fines prácticos pues sirvió de base para el desarrollo numérico de los siguientes modelos, sine-Gordon se caracteriza por tener una función densidad de energía constante, usamos un método de discretización explícito que nos permite conservar las cantidades antes mencionadas con un buen orden de consitencia, que además es estable y convergente. Usando estas ideas, usamos la generalización de la ecuación de onda con p dimensiones espaciales y derivadas fraccionarias al estilo de Riesz de ordenes en $(1, 2]$, una función de densidad de energía propuesta en la literatura, además del método de diferencias centradas fraccionarias para aproximar las derivadas de Riesz y notamos que las propiedades descritas en sine-Gordon siguen presentes, aumentando la disipación de la energía si consideramos un término de amortiguamiento, un orden cuadrático de consitencia, estabilidad y convergencia además de existencia de una solución al ser un método explícito, para este método se desarrolló un código de Matlab en el caso unidimensional, presentamos además algunas simulaciones. En el último modelo consideramos la misma ecuación con $p = 2$ dimensiones espaciales, aplicamos un operador compacto en el sentido de análisis funcional, el cual acelera el método hacia la solución, como el método es implícito demostramos existencia de una solución bajo ciertas condiciones, mostrando estabilidad, consistencia y convergencia.

Abstract

In this document, we work with several models: The sine-Gordon equation, a nonlinear wave equation with p spatial dimensions and fractional derivatives and finally the case $p = 2$ for a particular method. The first model has practical meaning, because it worked as inspiration for the following methods, the sine-Gordon equation is characterized for having an energy density function which conserves the energy through time, we use an explicit discretization method which allows us to keep such quantity with a good order of consistency, besides it is stable and convergent. We use the ideas from sine-Gordon to work with a generalization of the wave equation, considering p spatial dimensions and Riesz fractional derivatives of orders in $(1, 2]$, an energy density function proposed in the literature, and an explicit method with fractional centered differences for the Riesz derivatives, we notice that the properties present on sine-Gordon still remain, besides when we add a damping constant, the energy dissipates. We obtained a method with quadratic order of consistency, proved stability and convergence, and a solution for the method always exist since it is explicit. For this method, we developed a Matlab code for the unidimensional case and shown some simulations. In the last model, we consider the same equation but with $p = 2$, we apply a compact operator, in the functional analysis sense, which accelerates the method towards the solution, since the method is implicit we proved the existence of a solution under some conditions, stability, consistency, and convergence.

Introduction

Humankind has the desire to predict, predict movement, reactions, events, and even life fit into the description, mathematics help to fulfill that desire, with mathematics we can create a model, and a model predicts. Differential equations have helped when algebraic and transcendental equations could not keep going, an example of a model with differential equations is the wave equation, which is represented mathematically with a partial differential equation described first by Jean Le Rond d'Alembert in 1747 [1] and also developed by Euler, Bernoulli, and Lagrange. Ever since, waves have been found in all kinds of physical phenomena.

In some cases it is complicated to find the analytical solution of a differential equation, which is why there are some special techniques developed to approximate the solution, such techniques are called numerical methods. Numerical methods have been used previous its formal definition by investigators like Newton, Jacobi, Runge, and Kutta but some of the firsts researchers to talk about rounding errors are John Von Neumann and Herman Goldstine in 1947 [2] they worked with the inverse of matrices of high order. To find numerical methods that preserve some quantities is not an easy task, but definitely worth, since some physical phenomena are easier to study by a special quantity that depends on the system instead of the solution of the actual system. A good example of this is the phenomenon called Supratransmission.

Nonlinear supratransmission is a physical phenomenon that has been investigated in various nonlinear regimes. This process consists in the sudden increase in the amplitude of wave signals that propagate in a semi-infinite discrete nonlinear chain when one of the ends is perturbed by a harmonic disturbance irradiating at a frequency in the forbidden band gap. The existence of a nonlinear supratransmission threshold for the energy administered into finite chains has been established in continuous-limit cases, though numerical predictions are also available for both continuous and discrete systems. In fact, media governed by spatially discrete sine-Gordon and Klein-Gordon chains [3], double sine-Gordon systems [4], Fermi–Pasta–Ulam discrete chains [5], Bragg media in the nonlinear Kerr regime [6] and continuous media described by undamped sine-Gordon equations [7] have been identified as systems that exhibit the presence of nonlinear supratransmission. In most of these cases, the occurrence of this phenomenon has been predicted with a good degree of approximation.

Many of the partial differential equations that appear in the investigation of wave phenomena have been extended to account for nonlocal effects. Problems that only considered local contributions to the dynamics of discrete or continuous systems have been extended to account for global influences.

In such way, various classical models from mathematical physics that were traditionally described by partial differential equations have been formulated using derivatives of fractional order under different approaches [8]. As expected, among those models that have been extended to the fractional scenario are the classical sine-Gordon and nonlinear Klein–Gordon equations [9].

Here, it is important to point out that the Riesz definition of spatial derivatives of fractional order has been extensively employed in order to account physically for anomalous diffusion [10] and to provide pertinent conservation laws and Hamiltonian equations [11]. In view of these remarks, a natural question that arises is whether the process of nonlinear supratransmission is present in Riesz space-fractional sine-Gordon equations.

The design of techniques to approximate the solution of physical systems has been largely enriched with the study of partial differential equations of fractional orders. Indeed, fractional calculus has found interesting applications in many fields of the natural sciences and engineering, including the theory of viscoelasticity [12], the theory of thermo-elasticity [13], financial problems under a continuous time frame [14], self-similar protein dynamics [15] and quantum mechanics [16]. Distributed-order fractional diffusion-wave equations are used in groundwater flow modeling to and from wells [17]. A vast amount of nonequivalent approaches have been followed, and new criteria of fractional differentiation have been proposed constantly in the last decades. However, the problem in those cases is the common lack of a physically meaningful formulation of the Euler–Lagrange formality for fractional variational systems [18]. As expected, this has been a major problem in the design of energy-preserving method for fractional partial differential equations.

This thesis starts with the non-fractional wave equation, studying a method first developed by Vázquez, principal source of inspiration for the fractional wave equation method, this work had a warm up purpose. Then we aimed to study the p spatial dimension wave equation with Riesz derivatives in space, this method conserves the energy of the system which allows us to study even further the properties of this particular equation. Last but not least we used a compact operator in the sense of functional analysis to reach for the solution of the equation with much better accuracy. It is worth mention that for practical reasons, some of the definitions in Chapter 1 and Chapter 2 will be written again in Chapter 2 and Chapter 3, as well as some theorems.

Summary

This thesis is sectioned as follows.

- Chapter 1 provides a full development study of the well known Fei and Vázquez article [19] using the sine-Gordon wave equation and an explicit method to solve it, such method preserves the quantity of energy through time and also has a quadratic order of consistency, allowing us to take this idea into the fractional system.

- In Chapter 2 we generalize the idea from Chapter 1, instead of using the sine-Gordon wave equation, we use a generalized equation, with p spatial dimensions, we also changed the differential operators in time to Riesz operators of order $\alpha \in (1, 2]$ this equation comes with an extension of the energy functional proposed in the literature [9]. The Riesz fractional differential operator has a square root, this is a cornerstone in our investigation, we study some theorems which allow us to make sure that our discrete model has important properties like conservation of energy in the damped scenario, square root of the discrete Riesz operator and a good enough order of consistency, this model is based in fractional centered differences. Then we proved some important numerical properties of our model such as the existence of a solution, (which was an implicit property since the system is explicit) stability, and convergence. Finally, we provided a MatLab code used for simulation.

- In Chapter 3, we investigate numerically a model governed by a two dimensional wave equation, but in this case we used a compact operator to accelerate the method towards the solution, this is a substantial improvement over the scheme proposed in Chapter 2, we did notice that the use of this operator has an important impact on the properties of the solution to exist, also the reason for two instead of p spatial dimensions is the complexity of the method every time we added a dimension, we proved that this scheme preserves variational properties as in the previous chapter and also the scheme is stable an convergent. Then we do simulations to illustrate the preservation of the Hamiltonian in the equation, this last chapter is a continuation of Chapter 2.
- This thesis closes with a section of conclusions for each chapter and a list of relevant references.

1. Energy Conserving Scheme for the sine-Gordon Equation

The sine-Gordon equation is a second-order partial differential equation which possesses an important property: the energy of the system is constant through time, with that special interest, in this chapter we developed a numerical method that preserves the energy, also we provide proof that the method is second order consistent, stable and converges under suitable conditions, this chapter is the base of our investigation, since further investigations and analysis are first used in this part of the thesis.

1.1 Introduction

This chapter is motivated by the nonlinear sine-Gordon equation from quantum mechanics, which is one of the basic equations of modern nonlinear wave theory and it arises in many different areas of physics, such as Josephson junction theory, field theory, and lattices theory. This equation was introduced and studied by Edmond Bour in 1862 [20], as the result of the study of deformation of surfaces. The equation has constant energy through time under suitable conditions, one of our goals is to find a finite difference scheme that preserves the energy, a pioneer in this topic is L. Vazquez and coworkers, they proposed some schemes in 1991 [19], we used one of those schemes because it conserves the energy and is also completely explicit. The reason for the development of the numerical analysis of the scheme is as practice for the fractional scheme.

1.2 Preliminaries

In this section we will consider $T \in \mathbb{R}^+$, let us define the next system

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + \sin(u(x,t)) = 0, \quad \forall x \in \mathbb{R}, \\ \text{such that } \begin{cases} u(x,0) = \phi(x), & \forall x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x), & \forall x \in \mathbb{R}. \end{cases}\end{gathered} \tag{1.1}$$

For each pair $f, g \in L_2(\mathbb{R})$, the inner product of f and g is the function of t defined by

$$\langle f, g \rangle_x = \int_{-\infty}^{\infty} f(x,t) g(x,t) dx, \quad \forall t \in [0,T]. \tag{1.2}$$

In turn, the Euclidean norm of $f \in L_2(\mathbb{R})$ is the function of t defined by $\|f\|_{x,2} = \sqrt{\langle f, f \rangle}$. The set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, t) \in L_1(\mathbb{R})$ for each $t \in [0, T]$ will be denoted by $L_{x,1}(\mathbb{R})$, and for each such f we define its norm as the function of t given by

$$\|f\|_{x,1} = \int_{-\infty}^{\infty} |f(x,t)| dx, \quad \forall t \in [0, T]. \tag{1.3}$$

The literature on mathematical physics proved that the model (1.1) has an infinite number of conserved quantities, among the conserved quantities is the Hamiltonian

$$\mathcal{H}(x,t) = \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 + G(u(x,t)), \tag{1.4}$$

which integrating in $\mathbb{R}$ gives us the energy of the system

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 + G(u(x,t))\right] dx, \tag{1.5}$$

with $G(u(x,t))$ defined as

$$G(u(x,t)) = 1 - \cos(u(x,t)). \tag{1.6}$$

Here we provide a small proof for the conservation of the energy.

Theorem 1.1. *The just defined functional of energy is constant through time.*

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 + G(u(x,t))\right] dx = C. \tag{1.7}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 + G(u(x,t))\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}(x,t)\right)^2 + \frac{\partial}{\partial t}\left(1 - cos(u(x,t))\right)\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}2\frac{\partial u}{\partial t}(x,t)\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{1}{2}2\frac{\partial u}{\partial x}(x,t)\frac{\partial^2 u}{\partial x \partial t}(x,t) + sin(u(x,t))\frac{\partial u}{\partial t}(x,t)\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{\partial u}{\partial t}(x,t)\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial u}{\partial t}(x,t) + \frac{\partial u}{\partial t}(x,t) sin(u(x,t))\right] dx \\
&= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\left[\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + sin(u(x,t))\right] dx \\
&= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\,[0]\, dx = 0.
\end{aligned}$$

□

In the following sections, we will develop a numerical method to approximate both the solutions of (1.1) and the energy function (1.5) in such way that the discrete version of Theorem 1.1 is still satisfied. Various additional numerical properties of our methodology will be derived in the way, including the consistency, the stability and the convergence of the method.

1.3 Numerical method

For the remainder of this chapter we let $h > 0$ and $\tau > 0$ be fixed step-sizes in space and time, respectively, and assume that $N = T/\tau$ and $M = (b-a)/h$ are positive integers. Consider uniform partitions of $[a, b]$ and $[0, T]$, respectively, given by $x_j = a + jh$ and $t_n = n\tau$ for each $j \in J_M = \{0, 1, \ldots, M\}$ and each $n \in I_N = \{0, 1, \ldots, N\}$. In this chapter, the symbol v_j^n will represent a numerical approximation to the exact value of $u_j^n = u(x_j, t_n)$, that is, the solution of the initial-boundary-value problem (1.1) at the point (x_j, t_n), for each $j \in J_M$ and $n \in I_N$. Moreover, we will use the discrete linear operators

$$\mu_t u_j^n = \frac{u_j^{n+1} + u_j^n}{2}, \tag{1.8}$$

$$\delta_t^{(1)} u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \tag{1.9}$$

$$\delta_t^{(2)} u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2}, \tag{1.10}$$

$$\delta_x^{(1)} u_j^n = \frac{u_{j+1}^n - u_j^n}{h}, \tag{1.11}$$

$$\delta_x^{(2)} u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \tag{1.12}$$

$$\delta_{u,t}^{(1)} G(u_j^n) = \begin{cases} \dfrac{G(u_j^{n+1}) - G(u_j^n)}{u_j^{n+1} - u_j^n}, & \text{if } u_j^{n+1} \neq u_j^n, \\ G'(u_j^n), & \text{if } u_j^{n+1} = u_j^n, \end{cases} \tag{1.13}$$

for each $j \in J_M$ and $n \in I_{N-1}$.

After this, using the defined nomenclature, the finite-difference method to approximate the solution of (1.1) is given by

$$\mu_t \delta_t^{(2)} u_j^{n+1} - \mu_t \delta_x^{(2)} u_j^{n+1} + \delta_{u,t}^{(1)} G(u_j^n) = 0. \tag{1.14}$$

Note that the scheme is completely explicit, u_j^{n+3} can be directly calculated from the difference equation (1.14), and the starting data u_j^1, u_j^2, and u_j^3. The scheme is also spatially and temporally symmetric, and the consistency of the operators can be proved by Taylor expansion, which will be the next thing we show.

Lemma 1.2. *Let $u \in \mathcal{C}_t^4(\mathbb{R})$ then*

$$\mu_t \delta_t^{(2)} u_j^{n+1} = \frac{u_j^{n+3} - (u_j^{n+2} + u_j^{n+1}) + u_j^n}{2\tau^2} = \frac{\partial^2 u}{\partial t^2}\left(jh, \left(n + \frac{3}{2}\tau\right)\right) + \mathcal{O}\left(\tau^2\right). \tag{1.15}$$

Proof. We use Taylor's expansion

$$
\begin{aligned}
u_j^{n+3} \quad = \quad & u(x_j, t_{n+\frac{3}{2}}) + \frac{\partial}{\partial t}u(x_j, t_{n+\frac{3}{2}})(t_{n+3} - t_{n+\frac{3}{2}}) + \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+3} - t_{n+\frac{3}{2}})^2}{2} \\
+ \quad & \frac{\partial^3}{\partial t^3}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+3} - t_{n+\frac{3}{2}})^3}{6} + \frac{\partial^4}{\partial t^4}u(x_j, t^{1*})\frac{(t_{n+3} - t^{1*})^4}{24},
\end{aligned} \tag{1.16}
$$

$$
\begin{aligned}
u_j^{n+2} \quad = \quad & u(x_j, t_{n+\frac{3}{2}}) + \frac{\partial}{\partial t}u(x_j, t_{n+\frac{3}{2}})(t_{n+2} - t_{n+\frac{3}{2}}) + \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+2} - t_{n+\frac{3}{2}})^2}{2} \\
+ \quad & \frac{\partial^3}{\partial t^3}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+2} - t_{n+\frac{3}{2}})^3}{6} + \frac{\partial^4}{\partial t^4}u(x_j, t^{2*})\frac{(t_{n+2} - t^{2*})^4}{24},
\end{aligned} \tag{1.17}
$$

$$
\begin{aligned}
u_j^{n+1} \quad = \quad & u(x_j, t_{n+\frac{3}{2}}) + \frac{\partial}{\partial t}u(x_j, t_{n+\frac{3}{2}})(t_{n+1} - t_{n+\frac{3}{2}}) + \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+1} - t_{n+\frac{3}{2}})^2}{2} \\
+ \quad & \frac{\partial^3}{\partial t^3}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n+1} - t_{n+\frac{3}{2}})^3}{6} + \frac{\partial^4}{\partial t^4}u(x_j, t^{3*})\frac{(t_{n+1} - t^{3*})^4}{24},
\end{aligned} \tag{1.18}
$$

$$
\begin{aligned}
u_j^{n} \quad = \quad & u(x_j, t_{n+\frac{3}{2}}) + \frac{\partial}{\partial t}u(x_j, t_{n+\frac{3}{2}})(t_{n} - t_{n+\frac{3}{2}}) + \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n} - t_{n+\frac{3}{2}})^2}{2} \\
+ \quad & \frac{\partial^3}{\partial t^3}u(x_j, t_{n+\frac{3}{2}})\frac{(t_{n} - t_{n+\frac{3}{2}})^3}{6} + \frac{\partial^4}{\partial t^4}u(x_j, t^{4*})\frac{(t_{n} - t^{4*})^4}{24},
\end{aligned} \tag{1.19}
$$

then substituting we get

$$
\begin{aligned}
\frac{u_j^{n+3} - (u_j^{n+2} + u_j^{n+1}) + u_j^n}{2\tau^2} \quad = \quad & \frac{1}{2}\frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\frac{\left(\frac{3}{2}^2 - \left(\frac{1}{2}^2 + \frac{1}{2}^2\right) + \frac{3}{2}^2\right)\tau^2}{2\tau^2} \\
& + \sum_{i=1}^{4}\frac{\partial^4}{\partial t^4}u(x_j, t^{i*})\frac{(t_n - t^{i*})^4}{24 * 2\tau^2} \\
= \quad & \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}}) + \sum_{i=1}^{4}\frac{\partial^4}{\partial t^4}u(x_j, t^{i*})\frac{(t_n - t^{i*})^4}{48\tau^2},
\end{aligned} \tag{1.20}
$$

finally using $u \in \mathcal{C}_t^4(\mathbb{R})$ we obtain

$$
\begin{aligned}
\left|\mu_t\delta_t^{(2)}u_j^{n+1} - \frac{\partial^2}{\partial t^2}u(x_j, t_{n+\frac{3}{2}})\right| \quad = \quad & \left|\sum_{i=1}^{4}\frac{\partial^4}{\partial t^4}u(x_j, t^{i*})\frac{(t_n - t^{i*})^4}{48\tau^2}\right| \\
\leq \quad & \frac{1}{48\tau^2}\left(\left(\frac{3\tau}{2}\right)^4\left|\frac{\partial^4}{\partial t^4}u(x_j, t^{1*})\right| + \left(\frac{\tau}{2}\right)^4\left|\frac{\partial^4}{\partial t^4}u(x_j, t^{2*})\right|\right) \\
+ \quad & \frac{1}{48\tau^2}\left(\left(\frac{-\tau}{2}\right)^4\left|\frac{\partial^4}{\partial t^4}u(x_j, t^{3*})\right| + \left(\frac{-3\tau}{2}\right)^4\left|\frac{\partial^4}{\partial t^4}u(x_j, t^{4*})\right|\right) \\
\leq \quad & C\tau^2.
\end{aligned} \tag{1.21}
$$

$\square$

Lemma 1.3. *Let $u \in \mathcal{C}^{4,4}_{x,t}(\mathbb{R})$ then*

$$\begin{aligned}
\mu_t \delta_x^{(2)} u_j^{n+1} &= \frac{u_{j+1}^{n+2} - 2u_j^{n+2} + u_{j-1}^{n+2}}{2\tau^2} - \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2\tau^2} \\
&= \frac{\partial^2 u}{\partial x^2}\left(jh, \left(n + \frac{3}{2}\tau\right)\right) + \mathcal{O}\left(\tau^\alpha h^{\beta-2}\right). \qquad (1.22)
\end{aligned}$$

Proof. We omitted the Taylor extension due the long terms in it, but after the eliminations and since $u \in \mathcal{C}^{4,4}_{x,t}(\mathbb{R})$ we get

$$\begin{aligned}
\left|\mu_t \delta_x^{(2)} u_j^{n+1} - \frac{\partial^2 u}{\partial x^2}\left(jh, \left(n + \frac{3}{2}\tau\right)\right)\right| &\leq \frac{1}{48h^2}\left[4C_1 h^4 + 4C_2\left(\frac{\tau}{2}\right)h^3 + 4C_3\left(\frac{\tau}{2}\right)^2 h^2\right. \\
&+ \left. 4C_2\left(\frac{\tau}{2}\right)^3 h + 4C_2\left(\frac{\tau}{2}\right)^4\right], \qquad (1.23)
\end{aligned}$$

where we notice that the sum of powers in each term inside the brackets is equal to 4, and h^2 in the denominator, using this

$$\left|\mu_t \delta_x^{(2)} u_j^{n+1} - \frac{\partial^2 u}{\partial x^2}\left(jh, \left(n + \frac{3}{2}\tau\right)\right)\right| \leq C\tau^\alpha h^{\beta-2} \qquad (1.24)$$

with $\alpha \geq 0$, $\beta \geq 0$ integers, $\alpha + \beta = 4$.

$\square$

Lemma 1.4. *Let $G \in \mathcal{C}^3(\mathbb{R})$, $u \in \mathcal{C}^3_t(\mathbb{R})$ and $u_j^{n+2} \neq u_j^{n+1}$, then*

$$\delta_{u,t}^{(1)} G(u_j^{n+1}) = \frac{G(u_j^{n+1}) - G(u_j^n)}{u_j^{n+1} - u_j^n} = \sin\left(u\left(jh, \left(n + \frac{3}{2}\tau\right)\right)\right) + \mathcal{O}\left(\tau^2\right). \qquad (1.25)$$

Proof. We use Taylor's expansions:

$$\begin{aligned}
G(u_j^{n+2}) &= G(u_j^{n+\frac{3}{2}}) + G'(u_j^{n+\frac{3}{2}})\frac{\partial}{\partial t}u_j^{n+\frac{3}{2}}(t_{n+2} - t_{n+\frac{3}{2}}) \\
&+ \frac{1}{2}\left[G''(u_j^{n+\frac{3}{2}})\left(\frac{\partial}{\partial t}u_j^{n+\frac{3}{2}}\right)^2 + G'(u_j^{n+\frac{3}{2}})\frac{\partial^2}{\partial t^2}u_j^{n+\frac{3}{2}}\right](t_{n+2} - t_{n+\frac{3}{2}})^2 \qquad (1.26) \\
&+ \frac{1}{6}\left[G'''(u_j^{1*})\left(\frac{\partial}{\partial t}u_j^{1*}\right)^3 + 3G''(u_j^{1*})\frac{\partial}{\partial t}u_j^{1*}\frac{\partial^2}{\partial t^2}u_j^{1*} + G'(u_j^{1*})\frac{\partial^3}{\partial t^3}u_j^{1*}\right](t^{1*} - t_{n+\frac{3}{2}})^3,
\end{aligned}$$

$$\begin{aligned}
G(u_j^{n+1}) &= G(u_j^{n+\frac{3}{2}}) + G'(u_j^{n+\frac{3}{2}})\frac{\partial}{\partial t}u_j^{n+\frac{3}{2}}(t_{n+1} - t_{n+\frac{3}{2}}) \\
&+ \frac{1}{2}\left[G''(u_j^{n+\frac{3}{2}})\left(\frac{\partial}{\partial t}u_j^{n+\frac{3}{2}}\right)^2 + G'(u_j^{n+\frac{3}{2}})\frac{\partial^2}{\partial t^2}u_j^{n+\frac{3}{2}}\right](t_{n+1} - t_{n+\frac{3}{2}})^2 \qquad (1.27) \\
&+ \frac{1}{6}\left[G'''(u_j^{2*})\left(\frac{\partial}{\partial t}u_j^{2*}\right)^3 + 3G''(u_j^{2*})\frac{\partial}{\partial t}u_j^{2*}\frac{\partial^2}{\partial t^2}u_j^{2*} + G'(u_j^{2*})\frac{\partial^3}{\partial t^3}u_j^{2*}\right](t^{2*} - t_{n+\frac{3}{2}})^3,
\end{aligned}$$

then after some calculations:

$$\begin{aligned}\left|\delta_{u,t}^{(1)}G(u_j^{n+1})-G'(u_j^{n+\frac{3}{2}})\right| &\leq \frac{\tau^3}{6}\frac{\left|G'''(u_j^{1*})\left(\frac{\partial}{\partial t}u_j^{1*}\right)^3+3G''(u_j^{1*})\frac{\partial}{\partial t}u_j^{1*}\frac{\partial^2}{\partial t^2}u_j^{1*}+G'(u_j^{1*})\frac{\partial^3}{\partial t^3}u_j^{1*}\right|}{\left|u_j^{n+2}-u_j^{n+1}\right|}\\ &+ \frac{\tau^3}{6}\frac{\left|G'''(u_j^{2*})\left(\frac{\partial}{\partial t}u_j^{2*}\right)^3+3G''(u_j^{2*})\frac{\partial}{\partial t}u_j^{2*}\frac{\partial^2}{\partial t^2}u_j^{2*}+G'(u_j^{2*})\frac{\partial^3}{\partial t^3}u_j^{2*}\right|}{\left|u_j^{n+2}-u_j^{n+1}\right|},\end{aligned} \tag{1.28}$$

finally if $G \in \mathcal{C}^3(\Omega)$, $u \in \mathcal{C}_t^3(\Omega)$ and $u_j^{n+2} \neq u_j^{n+1}$

$$\begin{aligned}\left|\delta_{u,t}^{(1)}G(u_j^{n+1})-G'(u_j^{n+\frac{3}{2}})\right| &\leq \frac{\tau^2}{6}\frac{\left|G'''(u_j^{1*})\left(\frac{\partial}{\partial t}u_j^{1*}\right)^3+3G''(u_j^{1*})\frac{\partial}{\partial t}u_j^{1*}\frac{\partial^2}{\partial t^2}u_j^{1*}+G'(u_j^{1*})\frac{\partial^3}{\partial t^3}u_j^{1*}\right|}{\left|\frac{\partial}{\partial t}u_j^{n+1}\right|}\\ &+ \frac{\tau^2}{6}\frac{\left|G'''(u_j^{2*})\left(\frac{\partial}{\partial t}u_j^{2*}\right)^3+3G''(u_j^{2*})\frac{\partial}{\partial t}u_j^{2*}\frac{\partial^2}{\partial t^2}u_j^{2*}+G'(u_j^{2*})\frac{\partial^3}{\partial t^3}u_j^{2*}\right|}{\left|\frac{\partial}{\partial t}u_j^{n+1}\right|}\\ &\leq 3C\tau^2.\end{aligned} \tag{1.29}$$

$\square$

Therefore the truncation error for the system is $\mathcal{O}\left(\tau^2+h^2\right)$ provided $\frac{\tau}{h} \leq C$, C a constant.

1.4 Energy invariants

In this section we show that the finite-difference method (1.14) satisfies physical properties similar to those satisfied by (1.1). More precisely, we will propose a numerical energy functional associated to the scheme (1.14) that is preserved under suitable parameter conditions. For that reason we will suppose that the initial boundary conditions satisfy

$$\begin{cases} u(x,0)=\phi(x)=0, & \forall x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial t}(x,0)=\psi(x)=0, & \forall x \in \mathbb{R}. \end{cases} \tag{1.30}$$

Throughout this section, we will employ the spatial mesh

$$R_h\left\{(x_j)|x_j=kh, \text{ for each } j \in J_{M-1}\right\}, \tag{1.31}$$

Let $\mathcal{V}_h$ be the real vector space of all real grid functions on R_h. For any $u \in \mathcal{V}_h$ and $j \in J_{M-1}$ convey that $u_j = u(x_j)$. Moreover, define respectively the inner product $\langle\cdot,\cdot\rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ and the norm $\|\cdot\|_1 : \mathcal{V}_h \to \mathbb{R}$ by

$$\begin{aligned} \langle u, v\rangle &= h \sum_{j\in J_{M-1}} u_j v_j, &(1.32)\\ \|u\|_1 &= h \sum_{j\in J_{M-1}} |u_j|, &(1.33)\end{aligned}$$

for any $u, v \in \mathcal{V}_h$. The Euclidean norm induced by $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|_2$. In the following, we will represent the solutions of the finite-difference method (1.14) by $(v^n)_{n=0}^N$, where we convey that $v^n = (v_j^n)_{j\in J_M}$ for each $n \in I_N$.

It is important to note here that the operator $\delta_x^{(2)} u_j^n$ has the next property:

Lemma 1.5. *The operator $\delta_x^{(2)} u_j^n$ satisfies*

$$\left\langle \delta_x^{(1)} u_j^n, \delta_x^{(1)} v_j^n \right\rangle = \left\langle -\delta_x^{(2)} u_j^n, v_j^n \right\rangle, \tag{1.34}$$

when $u_M^n = u_0^n = u_{-1}^n = 0$.

Proof.

$$\begin{aligned}
\left\langle \delta_x^{(1)} u_j^n, \delta_x^{(1)} u_j^n \right\rangle &= h \sum_{j=0}^{M-1} \left(\frac{u_{j+1}^n - u_j^n}{h} \right)^2 \\
&= \frac{1}{h}\left[\sum_{j=0}^{M-1} \left(u_{j+1}^n - u_j^n\right)\left(u_{j+1}^n\right) - \left(u_{j+1}^n - u_j^n\right)\left(u_j^n\right) \right] \\
&= \frac{1}{h}\left[\sum_{j=0}^{M-1} \left(u_{j+1}^n - u_j^n\right)\left(u_{j+1}^n\right) - \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\left(u_j^n\right) + \left(-u_j^n + u_{j-1}^n\right)\left(u_j^n\right) \right] \\
&= -h \sum_{j=0}^{M-1} \frac{\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}{h^2}\left(u_j^n\right) \\
&\quad + \frac{1}{h}\left[\sum_{j=0}^{M-1} \left(u_{j+1}^n - u_j^n\right)\left(u_{j+1}^n\right) + \left(-u_j^n + u_{j-1}^n\right)\left(u_j^n\right) \right] \qquad (1.35)\\
&= \left\langle -\delta_x^{(2)} u_j^n, u_j^n \right\rangle + \frac{1}{h}\left[\sum_{j=1}^{M} \left(u_j^n - u_{j-1}^n\right)\left(u_{j+1}^n\right) + \sum_{j=0}^{M-1} \left(-u_j^n + u_{j-1}^n\right)\left(u_j^n\right) \right] \\
&= \left\langle -\delta_x^{(2)} u_j^n, u_j^n \right\rangle + (u_M^n)^2 - (u_0^n)^2 - u_M^n u_{M-1}^n + u_0^n u_1^n,
\end{aligned}$$

by making $u_M^n = u_0^n = u_{-1}^n = 0$ then we get:

$$\left\langle \delta_x^{(1)} u_j^n, \delta_x^{(1)} u_j^n \right\rangle = \left\langle -\delta_x^{(2)} u_j^n, u_j^n \right\rangle. \tag{1.36}$$

□

Following the same idea, then $\left\langle \delta_x^{(1)} u_j^n, \delta_x^{(1)} v_j^n \right\rangle = \left\langle -\delta_x^{(2)} u_j^n, v_j^n \right\rangle$, finally we can assure $\delta_x^{(1)} u_j^n$ is the square root operator of $-\delta_x^{(2)} u_j^n$.

For the scheme there is a discrete Hamiltonian operator for each $1 \leq n \leq N-1$ and $1 \leq j \leq M-1$:

$$\begin{aligned} H_j^n &= \frac{1}{2}\frac{\left(u_j^{n+1}-u_j^n\right)\left(u_j^n-u_j^{n-1}\right)}{\tau^2}+\frac{1}{2}\left(\frac{\left(u_{j+1}^n-u_j^n\right)}{h^2}\right)^2+G(u_j^n) \\ &= \frac{1}{2}\left(\delta_t u^n * \delta_t u^{n-1}\right)+\frac{1}{2}(\delta_x u^n)^2+G(u_j^n), \end{aligned} \tag{1.37}$$

subsequently summing over j and multiplying by h, we get the also constant discrete energy operator for each $1 \leq n \leq N-1$:

$$\begin{aligned} E^n &= h\sum_{j=0}^{M-1}\left[\frac{1}{2}\frac{\left(u_j^{n+1}-u_j^n\right)\left(u_j^n-u_j^{n-1}\right)}{\tau^2}+\frac{1}{2}\left(\frac{\left(u_{j+1}^n-u_j^n\right)}{h^2}\right)^2+G(u_j^n)\right] \\ &= \frac{1}{2}\langle\delta_t u^n,\delta_t u^{n-1}\rangle+\frac{1}{2}\|\delta_x u^n\|_2^2+\|G(u^n)\|_1. \end{aligned} \tag{1.38}$$

Now we calculate the next inner products:

$$\begin{aligned} \left\langle\delta_t^{(1)}u_j^{n+1},\mu_t\delta_t^{(2)}u_j^{n+1}\right\rangle &= \frac{1}{2}\left\langle\delta_t^{(1)}u_j^{n+1},\delta_t^{(2)}\left(u_j^{n+2}+u_j^{n+1}\right)\right\rangle \\ &= \frac{1}{2}\left\langle\delta_t^{(1)}u_j^{n+1},\frac{u_j^{n+3}-\left(u_j^{n+2}+u_j^{n+1}\right)+u_j^n}{\tau^2}\right\rangle \\ &= \frac{1}{2}\left\langle\delta_t^{(1)}u_j^{n+1},\frac{\left(u_j^{n+3}-u_j^{n+2}\right)}{\tau^2}-\frac{\left(u_j^{n+1}-u_j^n\right)}{\tau^2}\right\rangle \\ &= \frac{1}{2\tau}\left\langle\delta_t^{(1)}u_j^{n+1},\delta_t^{(1)}u_j^{n+2}-\delta_t^{(1)}u_j^n\right\rangle \\ &= \frac{1}{2\tau}\left[\left\langle\delta_t^{(1)}u_j^{n+2},\delta_t^{(1)}u_j^{n+1}\right\rangle-\left\langle\delta_t^{(1)}u_j^{n+1},\delta_t^{(1)}u_j^n\right\rangle\right], \end{aligned} \tag{1.39}$$

we ought to remember that $\left\langle-\delta_x^{(2)}u_j^n,v_j^n\right\rangle=\left\langle\delta_x^{(1)}u_j^n,\delta_x^{(1)}v_j^n\right\rangle$.

$$\begin{aligned} \left\langle\delta_t^{(1)}u_j^{n+1},-\mu_t\delta_x^{(2)}u_j^{n+1}\right\rangle &= \frac{1}{2\tau}\left\langle-\delta_t^{(2)}\left(u_j^{n+2}+u_j^{n+1}\right),\left(u_j^{n+2}-u_j^{n+1}\right)\right\rangle \\ &= \frac{1}{2\tau}\left\langle\delta_t^{(1)}\left(u_j^{n+2}+u_j^{n+1}\right),\delta_t^{(1)}\left(u_j^{n+2}-u_j^{n+1}\right)\right\rangle \\ &= \frac{1}{2\tau}[\left\langle\delta_t^{(1)}u_j^{n+2},\delta_t^{(1)}u_j^{n+2}\right\rangle-\left\langle\delta_t^{(1)}u_j^{n+2},\delta_t^{(1)}u_j^{n+1}\right\rangle \\ &+ \left\langle\delta_t^{(1)}u_j^{n+1},\delta_t^{(1)}u_j^{n+2}\right\rangle-\left\langle\delta_t^{(1)}u_j^{n+1},\delta_t^{(1)}u_j^{n+1}\right\rangle] \\ &= \frac{1}{2\tau}\left[\left\|\delta_t^{(1)}u_j^{n+2}\right\|_2^2-\left\|\delta_t^{(1)}u_j^{n+1}\right\|_2^2\right], \end{aligned} \tag{1.40}$$

$$\begin{aligned}
\left\langle \delta_t^{(1)} u_j^{n+1}, \mu_t \delta_{u,t}^{(1)} u_j^{n+1} \right\rangle &= h \sum_{j=2}^{M-1} \left(\frac{u_j^{n+2} - u_j^{n+1}}{\tau} \right) \left(\frac{G(u_j^{n+1}) - G(u_j^n)}{u_j^{n+2} - u_j^{n+1}} \right) \\
&= h \sum_{j=2}^{M-1} \left(\frac{G(u_j^{n+1}) - G(u_j^n)}{\tau} \right) = \frac{h}{\tau} \sum_{j=2}^{M-1} \left(G(u_j^{n+1}) - G(u_j^n) \right) \quad (1.41) \\
&= \frac{1}{\tau} \left[\left\| G(u_j^{n+2}) \right\|_1 - \left\| G(u_j^{n+1}) \right\|_1 \right].
\end{aligned}$$

The next theorem establishes the existence of invariants for the discrete system.

Theorem 1.6 (Dissipation of energy)**.** *Let* $(v^n)_{n=0}^N$ *be a solution of* (1.14)*, the energy of the system is:*

$$E^n = h \sum_{j=0}^{M-1} \left[\frac{1}{2} \frac{\left(v_j^{n+1} - v_j^n\right)\left(v_j^n - v_j^{n-1}\right)}{\tau^2} + \frac{1}{2} \left(\frac{\left(v_{j+1}^n - v_j^n\right)}{h^2} \right)^2 + G(v_j^n) \right]. \quad (1.42)$$

Then $\delta_t E^{n+1} = 0$ *for* $n \in I_{N-1}$.

Proof. Let Θ_j^{n+1} represent the left-hand side of the difference equations in (1.14) for each $n \in I_{N-1}$, and let $\Theta^{n+1} = (\Theta_j^{n+1})_{j \in J_M}$. Suppose that $(v^n)_{n=0}^N$ is a solution of (1.14). Calculating the inner product of Θ^{n+1} with $\delta_t^{(1)} v^{n+1}$, using the identities above and collecting terms, we note that

$$\begin{aligned}
0 = \langle \Theta^{n+1}, \delta_t^{(1)} v^{n+1} \rangle &= \langle \mu_t \delta_t^{(2)} v^{n+1} - \mu_t \delta_x^{(2)} v^{n+1} + \delta_{v,t}^{(1)} G(v^{n+1}), \delta_t^{(1)} v^{n+1} \rangle \\
&= \langle \mu_t \delta_t^{(2)} v^{n+1}, \delta_t^{(1)} v^{n+1} \rangle + \langle -\mu_t \delta_x^{(2)} v^{n+1}, \delta_t^{(1)} v^{n+1} \rangle + \langle \delta_{v,t}^{(1)} G(v^{n+1}), \delta_t^{(1)} v^{n+1} \rangle \\
&= \tfrac{1}{2\tau} \left[\left\langle \delta_t^{(1)} v^{n+2}, \delta_t^{(1)} v^{n+1} \right\rangle - \left\langle \delta_t^{(1)} v^{n+1}, \delta_t^{(1)} v^n \right\rangle \right] \\
&\quad + \tfrac{1}{2\tau} \left[\left\| \delta_t^{(1)} v^{n+2} \right\|_2^2 - \left\| \delta_t^{(1)} v^{n+1} \right\|_2^2 \right] \\
&\quad + \tfrac{1}{\tau} \left[\left\| G(v^{n+2}) \right\|_1 - \left\| G(v^{n+1}) \right\|_1 \right] = \delta_t E^{n+1} \quad \forall n \in I_{N-1},
\end{aligned} \quad (1.43)$$

so the conclusion of this result is obtained. □

Theorem 1.7. *The discrete quantities* (1.38) *may be rewritten alternatively as*

$$E^n = \frac{1}{2} \mu_t \| \delta_t^{(1)} v^{n-1} \|_2^2 - \frac{\tau^2}{4} \| \delta_t^{(2)} v^n \|_2^2 + \frac{1}{2} \| \delta_x v^n \|_2^2 + \| G(v^n) \|_1, \quad \forall n \in I_{N-1}. \quad (1.44)$$

Proof. Note that

$$\begin{aligned}
\langle \delta_t v^n, \delta_t v^{n-1} \rangle &= \| \delta_t^{(1)} v^n \|_2^2 - \frac{1}{\tau^2} \langle v^{n+1} - v^n, v^{n+1} - 2v^n + v^{n-1} \rangle \\
&= \| \delta_t^{(1)} v^n \|_2^2 - \tau^2 \| \delta_t^{(2)} v^n \|_2^2 - \frac{1}{\tau^2} \langle v^n - v^{n-1}, v^{n+1} - 2v^n + v^{n-1} \rangle \quad (1.45) \\
&= \| \delta_t^{(1)} v^n \|_2^2 + \| \delta_t^{(1)} v^{n-1} \|_2^2 - \tau^2 \| \delta_t^{(2)} v^n \|_2^2 - \langle \delta_t v^n, \delta_t v^{n-1} \rangle,
\end{aligned}$$

holds for each $n \in I_{N-1}$. It follows that

$$\langle \delta_t v^n, \delta_t v^{n-1} \rangle = \mu_t \| \delta_t^{(1)} v^{n-1} \|_2^2 - \frac{\tau^2}{2} \| \delta_t^{(2)} v^n \|_2^2, \quad \forall n \in I_{N-1}, \quad (1.46)$$

whence the conclusion of the theorem is reached. □

1.5 Auxiliary lemmas

In this section, we prove some propositions needed to establish the properties of numerical efficiency of the finite-difference method (1.14), we will use the fact that $n\tau < T$. To start with, we will require the following elementary facts which will be employed in the sequel without an explicit reference:

(A) If v and w are real vectors of the same dimension then $|2\langle v, w\rangle| \leq \|v\|_2^2 + \|w\|_2^2$.

(B) As a consequence, $\|v+w\|_2^2 \leq 2\|v\|_2^2 + 2\|w\|_2^2$ for any two real vectors v and w of the same dimension.

(C) More generally, if $k \in \mathbb{N}$ and $v_1, \ldots, v_k$ are real vectors of the same dimension then

$$\left\| \sum_{n=1}^{k} v_n \right\|_2^2 \leq k \sum_{n=1}^{k} \|v_n\|_2^2. \tag{1.47}$$

(D) If $(v^n)_{n=0}^N$ is a finite sequence in $\mathcal{V}_h$ and $n \in I_N$ then $v^n = v^0 + \tau \sum_{k=0}^{n-1} \delta_t^{(1)} v^k$. It follows that

$$\|v^n\|_2^2 \leq 2\|v^0\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)} v^k\|_2^2, \quad \forall n \in I_N. \tag{1.48}$$

Proof. (A) We notice that:

$$\begin{aligned} \|v+w\|^2 &= \langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \geq 0, \\ \|v-w\|^2 &= \langle v-w, v-w\rangle = \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle \geq 0. \end{aligned}$$

Then

$$\begin{aligned} \|v\|^2 + \|w\|^2 &\geq -2\langle v, w\rangle, \\ \|v\|^2 + \|w\|^2 &\geq 2\langle v, w\rangle. \end{aligned}$$

By the definition of absolute value we get the result

$$|2\langle v, w\rangle| \leq \|v\|_2^2 + \|w\|_2^2 \tag{1.49}$$

(B) Now by using (A), we obtain

$$\|v+w\|^2 = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \leq 2\|v\|^2 + 2\|w\|^2. \tag{1.50}$$

(C) By generalizing like in (A), we obtain

$$\begin{aligned} \left\| \sum_{i=1}^{k} v_i \right\|^2 &= \sum_{i=1}^{k} \|v_i\|^2 + 2\left[\langle v_1, v_2\rangle + \ldots + \langle v_1, v_k\rangle + \langle v_2, v_3\rangle \right. \\ &\quad \left. + \ldots + \langle v_2, v_k\rangle + \ldots + \langle v_k, v_k\rangle\right], \end{aligned} \tag{1.51}$$

we notice that each v_j appears exactly $k-1$ times inside the brackets, by applying (A) in every one of them we get

$$\left\|\sum_{i=1}^{k} v_i\right\|^2 = \sum_{i=1}^{k} \|v_i\|^2 + (k-1)\sum_{i=1}^{k} \|v_i\|^2 = k\sum_{i=1}^{k} \|v_i\|^2 . \tag{1.52}$$

(D) Finally we just make some calculations:

$$\begin{aligned} v^n &= \tau\left[\frac{v^0}{\tau} + \frac{v^1 - v^0}{\tau} + \frac{v^2 - v^1}{\tau} + \dots + \frac{v^{n-1} - v^{n-2}}{\tau} + \frac{v^n - v^{n-1}}{\tau}\right] \\ &= v^0 + \tau \sum_{k=0}^{n-1} \delta_t^{(1)} v_k, \end{aligned} \tag{1.53}$$

using (C)

$$\begin{aligned} \|v^n\|_2^2 &= \left\|v^0 + \tau\sum_{k=0}^{n-1} \delta_t^{(1)} v_k\right\|_2^2 \leq 2\left\|v^0\right\|_2^2 + 2\left\|\tau\sum_{k=0}^{n-1} \delta_t^{(1)} v_k\right\|_2^2 \\ &\leq 2\left\|v^0\right\|_2^2 + 2n\tau^2\sum_{k=0}^{n-1} \left\|\delta_t^{(1)} v_k\right\|_2^2, \end{aligned} \tag{1.54}$$

but in our hypothesis we have $n\tau \leq T$ so

$$\|v^n\|_2^2 \leq 2\left\|v^0\right\|_2^2 + 2T\tau\sum_{k=0}^{n-1} \left\|\delta_t^{(1)} v_k\right\|_2^2 . \tag{1.55}$$

□

Lemma 1.8. *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that $(u^n)_{n=0}^N$, $(v^n)_{n=0}^N$ and $(R^n)_{n=0}^N$ are sequences in $\mathcal{V}_h$. Let $\varepsilon^n = v^n - u^n$ and $\widetilde{G}^n = \delta_{v,t}G(v^n) - \delta_{w,t}G(w^n)$ for each $n \in I_{N-1}$ and let $k \in I_{N-1}$. Then the following is satisfied.*

(a)

$$\|\widetilde{G}^n\|_2^2 \leq 2(\|\varepsilon^n\|_2^2 + \|\varepsilon^{n-1}\|_2^2), \quad \forall n \in I_{N-1}. \tag{1.56}$$

(b)

$$2\sum_{n=1}^{k} |\langle R^n, \delta_t^{(1)}\varepsilon^{n-1}\rangle| \leq \sum_{n=1}^{k}\left(\|R^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^{n-1}\|_2^2\right), \quad \forall k \in I_{N-1}. \tag{1.57}$$

(c)

$$2\sum_{n=1}^{k} \left|\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^{n-1}\rangle\right| \leq 8k\|\varepsilon^0\|_2^2 + \left(8T^2 + 1\right)\sum_{n=0}^{k-1} \|\delta_t^{(1)}\varepsilon^n\|_2^2, \quad \forall k \in I_{N-1}. \tag{1.58}$$

Proof.

(a) As a consequence of the Mean Value Theorem and a direct integration we obtain that $|\widetilde{G}_j^n| \leq$

$(|\varepsilon_j^{n+1}| + |\varepsilon_j^n|)$ for each $j \in J_M$ and each $n \in I_{N-1}$. Raising both sides of this inequality to the second power and using the inequalities at the beginning of this section we readily reach.

(b) Note that for each $n \in I_{N-1}$,

$$2|\langle R^n, \delta_t^{(1)} \varepsilon^{n-1} \rangle| \leq \|R^n\|_2^2 + \|\delta_t^{(1)} \varepsilon^{n-1}\|_2^2, \tag{1.59}$$

then summing from 1 to k we get:

$$2 \sum_{n=1}^{k} |\langle R^n, \delta_t^{(1)} \varepsilon^{n-1} \rangle| \leq \sum_{n=1}^{k} \left(\|R^n\|_2^2 + \|\delta_t^{(1)} \varepsilon^{n-1}\|_2^2 \right). \tag{1.60}$$

(c) Using the inequality (2.2) and the remarks at the beginning of the present section we obtain that

$$\begin{aligned}
\|\widetilde{G}^n\|_2^2 \leq 2\left(\|\varepsilon^n\|_2^2 + \|\varepsilon^{n-1}\|_2^2\right) &\leq 2\left(2\|\varepsilon^0\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)}\varepsilon^k\|_2^2 + 2\|\varepsilon^0\|_2^2 + 2T\tau \sum_{k=0}^{n-2} \|\delta_t^{(1)}\varepsilon^k\|_2^2\right) \\
&\leq 2\left(4\|\varepsilon^0\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)}\varepsilon^k\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)}\varepsilon^k\|_2^2\right) \\
&= 8\|\varepsilon^0\|_2^2 + 8T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)}\varepsilon^k\|_2^2,
\end{aligned} \tag{1.61}$$

for each $k \in I_{N-1}$. Now we evaluate the sum:

$$\begin{aligned}
\sum_{n=1}^{k} \left|\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^{n-1}\rangle\right| &\leq \sum_{n=0}^{k} \left(\|\widetilde{G}^n\|_2 + \|\delta_t\varepsilon^{n-1}\|_2^2\right) \\
&\leq \sum_{n=1}^{k} \left(8\|\varepsilon^0\|_2^2 + 8T\tau \sum_{l=0}^{n-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2\right) + \sum_{n=1}^{k} \|\delta_t\varepsilon^{n-1}\|_2^2 \\
&= 8k\|\varepsilon^0\|_2^2 + 8T\tau \sum_{n=0}^{k-1}\sum_{l=0}^{n-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2 + \sum_{n=0}^{k-1} \|\delta_t\varepsilon^n\|_2^2 \\
&\leq 8k\|\varepsilon^0\|_2^2 + 8T\tau \sum_{n=0}^{k-1}\sum_{l=0}^{k-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2 + \sum_{n=0}^{k-1} \|\delta_t\varepsilon^n\|_2^2 \\
&= 8k\|\varepsilon^0\|_2^2 + 8T\tau k \sum_{n=0}^{k-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2 + \sum_{n=0}^{k-1} \|\delta_t\varepsilon^n\|_2^2 \\
&= 8k\|\varepsilon^0\|_2^2 + 8T^2 \sum_{n=0}^{k-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2 + \sum_{n=0}^{k-1} \|\delta_t\varepsilon^n\|_2^2 \\
&= 8k\|\varepsilon^0\|_2^2 + \left(8T^2 + 1\right) \sum_{n=0}^{k-1} \|\delta_t^{(1)}\varepsilon^l\|_2^2,
\end{aligned} \tag{1.62}$$

which is what we wanted to prove. □

Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that $(u^n)_{n=0}^N$, $(v^n)_{n=0}^N$ and $(R^n)_{n=0}^N$ are sequences in $\mathcal{V}_h$. As in our last result, let $\varepsilon^n = v^n - u^n$ and $\widetilde{G}^n = \delta_{v,t}G(v^n) - \delta_{w,t}G(w^n)$ for each $n \in I_{N-1}$. Suppose also that

$$\mu_t\delta_t^2\varepsilon^{n-1} - \mu_t\delta_x^2\varepsilon^{n-1} + \widetilde{G}^n = R^n, \quad \forall n \in I_{N-1}. \tag{1.63}$$

Using this identity and mathematical induction it follows that

$$\delta_t^{(2)}\varepsilon^k = (-1)^{k-1}\delta_t^{(2)}\varepsilon^1 + \delta_x^2\varepsilon^k + (-1)^{k-1}\delta_x^2\varepsilon^1 + 2\sum_{n=2}^{k}(-1)^{n-1}\left[\gamma\delta_t^{(1)}\varepsilon^n + \widetilde{G}^n - R^n\right], \quad \forall k \in I_{N-2}. \tag{1.64}$$

Moreover, calculating the square of the Euclidean norm of $\delta_t^{(2)}\varepsilon^k$, using the inequalities at the beginning of this section

$$\begin{aligned}
\|\delta_t^{(2)}\varepsilon^k\|_2^2 &\leq 4\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^k\|_2^2 + 4(4)\left\|\sum_{n=1}^{k}\widetilde{G}^n - R^n\right\|_2^2 \\
&\leq 4\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^k\|_2^2 + 16\left(2\left\|\sum_{n=1}^{k}\widetilde{G}^n\right\|_2^2 + 2\left\|\sum_{n=1}^{k}R^n\right\|_2^2\right) \\
&\leq 4\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^k\|_2^2 + 32\left(k\sum_{n=1}^{k}\left\|\widetilde{G}^n\right\|_2^2 + k\sum_{n=1}^{k}\|R^n\|_2^2\right) \\
&= 4\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^1\|_2^2 + 4\|\delta_x^2\varepsilon^k\|_2^2 + 32k\sum_{n=1}^{k}\left(\left\|\widetilde{G}^n\right\|_2^2 + \|R^n\|_2^2\right).
\end{aligned} \tag{1.65}$$

Multiplying by $\frac{\tau^2}{2}$, applying some definitions and simplifying

$$\begin{aligned}
\frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^k\|_2^2 &\leq 2\tau^2\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 2\tau^2\|\delta_x^2\varepsilon^1\|_2^2 + 2\tau^2\|\delta_x^2\varepsilon^k\|_2^2 + 16k\tau^2\sum_{n=0}^{k}\left(\|\widetilde{G}^n\|_2^2 + \|R^n\|_2^2\right) \\
&\leq 2\tau^2\left\|\delta_t^1\left(\tfrac{\varepsilon^1-\varepsilon^0}{\tau}\right)\right\|_2^2 + 2\tau^2\left\|\delta_x\left(\tfrac{\varepsilon_j^k-\varepsilon_{j-1}^k}{h}\right)\right\|_2^2 + 2\tau^2\left\|\delta_x\left(\tfrac{\varepsilon_j^1-\varepsilon_{j-1}^1}{h}\right)\right\|_2^2 \\
&\quad +16k\tau^2\sum_{n=0}^{k}\left(\|\widetilde{G}^n\|_2^2 + \|R^n\|_2^2\right) \\
&\leq 2\frac{\tau^2}{\tau^2}\left(2\|\delta_t^1\varepsilon^1\|_2^2 + 2\|\delta_t^1\varepsilon^0\|_2^2\right) + 2\frac{\tau^2}{h^2}\left(2\|\delta_x\varepsilon^k\|_2^2 + 2\|\delta_x\varepsilon^k\|_2^2\right) \\
&\quad +2\frac{\tau^2}{h^2}\left(2\|\delta_x\varepsilon^1\|_2^2 + 2\|\delta_x\varepsilon^1\|_2^2\right) + 16k\tau^2\sum_{n=0}^{k}\left(\|\widetilde{G}^n\|_2^2 + \|R^n\|_2^2\right) \\
&\leq 4\|\delta_t^1\varepsilon^1\|_2^2 + 4\|\delta_t^1\varepsilon^0\|_2^2 + 2a\|\delta_x\varepsilon^k\|_2^2 + 2a\|\delta_x\varepsilon^1\|_2^2 \\
&\quad +16T\tau\sum_{n=0}^{k}\left(\|\widetilde{G}^n\|_2^2 + \|R^n\|_2^2\right).
\end{aligned} \tag{1.66}$$

Applying Lemma 1.8 (a) and simplifying we readily obtain that

$$\begin{aligned}
&\leq 4\|\delta_t^1\varepsilon^1\|_2^2 + 4\|\delta_t^1\varepsilon^0\|_2^2 + 2a\|\delta_x\varepsilon^k\|_2^2 + 2a\|\delta_x\varepsilon^1\|_2^2 \\
&\quad +16T\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 16T\tau\sum_{n=0}^{k}\left(8\|\varepsilon^0\|_2^2 + 8T\tau\sum_{i=0}^{n-1}\|\delta_t\varepsilon^i\|_2^2\right) \\
&\leq 4\|\delta_t^1\varepsilon^1\|_2^2 + 4\|\delta_t^1\varepsilon^0\|_2^2 + 2a\|\delta_x\varepsilon^k\|_2^2 + 2a\|\delta_x\varepsilon^1\|_2^2 \\
&\quad +16T\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 16T\tau\left(8k\|\varepsilon^0\|_2^2 + 8T\tau k\sum_{n=0}^{k}\|\delta_t\varepsilon^i\|_2^2\right) \\
&\leq 4\|\delta_t^1\varepsilon^1\|_2^2 + 4\|\delta_t^1\varepsilon^0\|_2^2 + 2a\|\delta_x\varepsilon^k\|_2^2 + 2a\|\delta_x\varepsilon^1\|_2^2 \\
&\quad +16T\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 128T^2\|\varepsilon^0\|_2^2 + 128T^3\tau\sum_{n=0}^{k}\|\delta_t\varepsilon^n\|_2^2.
\end{aligned} \tag{1.67}$$

This inequality will be used in the following section to establish the stability and the convergence of the finite-difference method (1.14).

The following result will be useful to prove the stability and convergence properties of (1.14). It is obviously a discrete version of the well-known Gronwall inequality.

Lemma 1.9 (Pen-Yu [21]). *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative mesh functions, and suppose that there exists $C \geq 0$ such that*

$$\omega^k \leq \rho^k + C\tau \sum_{n=0}^{k-1} \omega^k, \quad \forall k \in I_{N-1}. \tag{1.68}$$

Then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in I_N$. □

1.6 Numerical results

The main numerical properties of the finite-difference method (1.14) as well as some illustrative computational simulations are presented in this stage. Here we show that our scheme is a consistent, stable and convergent technique under suitable conditions on the parameters of the model. In a first stage, we show that (1.14) is a second-order consistent technique, and that the discrete energy density (1.37) also provides a consistent approximation to the continuous Hamiltonian (1.4). For practical purposes we define the following continuous and discrete functionals:

$$\mathcal{L}u(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + \sin(u(x,t)) \in \Omega, \tag{1.69}$$

$$Lu_j^n = \mu_t \delta_t^{(2)} u_j^{n+1} - \mu_t \delta_x^{(2)} u_j^{n+1} + \delta_{u,t}^{(1)} G(u_j^n), \quad \forall (j,n) \in J \times I_{N-2}. \tag{1.70}$$

Theorem 1.10 (Consistency). *If $u \in \mathcal{C}^4(\mathbb{R})$ then there exist constants $C, C^{'} > 0$ which are independent of h and τ such that for each $j \in J_M$ and each $n \in I_{N-2}$,*

$$\left|Lu_j^n - \mathcal{L}u(x_j,t_n)\right| \leq C(\tau^2 + h^2), \tag{1.71}$$

$$\left|Hu_j^n - \mathcal{H}u(x_j,t_n)\right| \leq C^{'}(\tau + h^2). \tag{1.72}$$

Proof. We employ here the usual arguments with Taylor polynomials shown in Section 1.3. Using the hypotheses of continuous differentiability, there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$ such that

$$\left|\mu_t \delta_t^{(2)} u_j^n - \frac{\partial^2 u}{\partial t^2}(x_j, t_{n+\frac{1}{2}})\right| \leq C_1 \tau^2, \tag{1.73}$$

$$\left|\mu_t \delta_x^2 u_j^n - \frac{\partial^2 u}{\partial x^2}(x_j, t_{n+\frac{1}{2}})\right| \leq C_2(\tau^2 + h^2), \tag{1.74}$$

$$\left|\delta_t^{(1)} u_j^n - \frac{\partial u}{\partial t}(x_j, t_{n+\frac{1}{2}})\right| \leq C_3 \tau^2, \tag{1.75}$$

$$\left|\delta_{u,t}^{(1)} G(u_j^n) - G'(u(x_j, t_{n+\frac{1}{2}}))\right| \leq C_4 \tau^2, \tag{1.76}$$

for each $j \in J_M$ and each $n \in I_{N-2}$. The first inequality in the conclusion of this theorem is readily reached using the triangle inequality and defining $C = \max\{C_1, C_2, \gamma C_3, C_4\}$. To establish the second inequality, note that the consistency of the forward-difference operators, the Mean Value Theorem and the smoothness of the function u guarantee that there exists a constant C_5 independent of τ such that

$$\begin{aligned}\left|\delta_t^{(1)}u_j^n\delta_t^{(1)}u_j^{n-1}-\left(\frac{\partial u}{\partial t}(x_j,t_n)\right)^2\right| &\leq \left|\delta_t^{(1)}u_j^{n-1}\right|\left|\delta_t^{(1)}u_j^n-\frac{\partial u}{\partial t}(x_j,t_n)\right| \\ &+ \left|\frac{\partial u}{\partial t}(x_j,t_n)\right|\left|\delta_t^{(1)}u_j^{n-1}-\frac{\partial u}{\partial t}(x_j,t_n)\right| \leq C_5\tau, \end{aligned} \tag{1.77}$$

for each $j \in J_M$ and each $n \in I_{N-1}$. Likewise, there exist constants C_6 such that

$$\left|u_j^n\delta_x^2u_j^n-u(x_j,t_n)\frac{\partial^2 u}{\partial x^2}(x_j,t_n)\right| \leq C_6h^2, \tag{1.78}$$

for each $j \in J_M$ and each $n \in I_{N-1}$. The second inequality of the conclusion follows again using the triangle inequality and letting $C' = \max\{C_5, C_6\}$. □

We turn our attention to the stability and the convergence properties of (1.14). In the following, the constants C_1, C_2 and C_3 are as in Lemma 1.8, and (ϕ_v,ψ_v) and (ϕ_w,ψ_w) will denote two sets of initial conditions of (1.14).

Theorem 1.11 (Stability)**.** *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that τ and h satisfy*

$$\frac{\tau}{h} = \gamma_0 \leq \frac{\sqrt{a}}{\sqrt{8}}, \quad a < 1. \tag{1.79}$$

Let $\mathbf{v} = (v^n)_{n=0}^N$ and $\mathbf{w} = (w^n)_{n=0}^N$ be solutions of (1.14) for (ϕ_v,ψ_v) and (ϕ_w,ψ_w), respectively, and let $\varepsilon^n = v^n - w^n$ for each $n \in I_N$. Then there exist constants $C_4, C_5 \in \mathbb{R}^+$ independent of $\mathbf{v}$ and $\mathbf{w}$ such that

$$\|\delta_t^{(2)}\varepsilon^n\|_2^2 + 2(1-a)\|\delta_x^1\varepsilon^n\|_2^2 \leq C_4\left(\|\varepsilon^0\|_2^2+\|\delta_t^{(1)}\varepsilon^0\|_2^2+\|\delta_t^{(1)}\varepsilon^1\|_2^2+\|\delta_x^1\varepsilon^1\|_2^2\right)e^{C_5n\tau}, \quad \forall n \in I_{N-1}. \tag{1.80}$$

Proof. Let $\frac{\tau}{h} = \gamma_0 \leq \frac{\sqrt{a}}{2}$ with $a < 1$. Obviously, the sequence $(\varepsilon^n)_{n=0}^N$ satisfies the initial-boundary-value problem

$$\begin{aligned}&\mu_t\delta_t^{(2)}\varepsilon_j^n-\mu_t\delta_x^{(2)}\varepsilon_j^n+\delta_{v,t}^{(1)}G(v_j^n)-\delta_{w,t}G(w_j^n)=0, \quad \forall (j,n) \in J \times I_{N-2}, \\ &\text{such that } \begin{cases}\varepsilon_j^0 = \phi_v(x_j)-\phi_w(x_j), & \forall j \in J, \\ \delta_t\varepsilon_j^0 = \psi_v(x_j)-\psi_w(x_j), & \forall j \in J, \\ \varepsilon_j^n = 0, & \forall (j,n) \in \partial J \times I_N.\end{cases}\end{aligned} \tag{1.81}$$

For the sake of convenience, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)}G(v_j^n) - \delta_{w,t}G(w_j^n)$ for each $j \in J$ and each $n \in I_{N-1}$. From the identities preceding Theorem 1.6 and those after its proof, we readily obtain that

$$\left\langle \mu_t\delta_t^{(2)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\right\rangle = \frac{1}{2}\delta_t^{(1)}\mu_t\|\delta_t^{(1)}\varepsilon^{n-1}\|_2^2 - \frac{\tau^2}{4}\delta_t^{(1)}\|\delta_t^{(2)}\varepsilon^n\|_2^2,$$

$$\langle -\mu_t\delta_x^2\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle = \frac{1}{2}\delta_t^{(1)}\|\delta_x^1\varepsilon^n\|_2^2, \quad \forall i \in I_{N-1}, \tag{1.82}$$

$$|2\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle| \leq 2\left(\|\varepsilon^{n+1}\|_2^2+\|\varepsilon^n\|_2^2+\|\delta_t^{(1)}\varepsilon^n\|_2^2\right), \tag{1.83}$$

for each $n \in I_{N-1}$ and for some $C_1 \in \mathbb{R}^+$. Let $k \in I_{N-1}$. Taking the inner product of $\delta_t^{(1)}\varepsilon^n$ with both sides of the respective difference equation of (1.81), substituting the identities above, calculating then the sum of the resulting identity for all $n \in I_k$, multiplying by 2τ on both sides, applying Lemma 1.8 and simplifying algebraically yields

$$\begin{aligned}
\|\delta_t^{(1)}\varepsilon^k\|_2^2 + 2\|\delta_x^1\varepsilon^k\|_2^2 &= -\|\delta_t\varepsilon^{k-1}\|_2^2 + \tau^2\|\delta_t^2\varepsilon^k\|_2^2 + \|\delta_t\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 \\
&\quad -\tau^2\|\delta_t^2\varepsilon^0\|_2^2 + \|\delta_x\varepsilon^1\|_2^2 + 2\tau\sum_{n=1}^{k}\left|\langle \widetilde{G}^n - R^n, \delta_t^{(1)}\varepsilon^n\rangle\right| \\
&\leq \tau^2\|\delta_t^2\varepsilon^k\|_2^2 + \|\delta_t\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 + \|\delta_x\varepsilon^1\|_2^2 \\
&\quad +2\tau\sum_{n=1}^{k}\left|\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle\right| + 2\tau\sum_{n=1}^{k}\left|\langle R^n, \delta_t^{(1)}\varepsilon^n\rangle\right| \\
&\leq 2a\|\delta_x\varepsilon^k\|_2^2 + 8\|\delta_t\varepsilon^1\|_2^2 + 8\|\delta_t\varepsilon^0\|_2^2 + 2a\|\delta_x\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 \\
&\quad +\|\delta_x\varepsilon^1\|_2^2 + 256T^2\|\varepsilon^0\|_2 + 32T\tau\sum_{n=1}^{k}\|R^n\|_2^2 + 256T^3\tau\sum_{n=0}^{k-1}\|\delta_t\varepsilon^n\|_2^2 \\
&\quad +2\tau\left(8k\|\varepsilon^0\|_2^2 + (8T^2+1)\sum_{n=0}^{k-1}\|\delta_t\varepsilon^n\|_2^2\right) + 2\tau\left(\sum_{n=1}^{k}\|R^n\|_2^2 + \|\delta_t\varepsilon^{n-1}\|_2^2\right).
\end{aligned} \tag{1.84}$$

Now taking the derivate with respect the space to the other side of the inequality, with $R^n = 0$ and simplifying we get:

$$\begin{aligned}
\|\delta_t^{(1)}\varepsilon^k\|_2^2 + 2(1-a)\|\delta_x^1\varepsilon^k\|_2^2 &\leq 9\|\delta_t\varepsilon^1\|_2^2 + 9\|\delta_t\varepsilon^0\|_2^2 + 2(1-a)\|\delta_x\varepsilon^1\|_2^2 + 16T(16T+1)\|\varepsilon^0\|_2 \\
&+ \tau(128T^3 + 2T^2 + 1)\sum_{n=1}^{k}\|\delta_t\varepsilon^n\|_2^2 \\
&\leq C_4\left(\|\delta_t\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 + \|\delta_x\varepsilon^1\|_2^2 + \|\varepsilon^0\|_2\right) \\
&+ C_5\sum_{n=1}^{k}\left(\tau\|\delta_t\varepsilon^n\|_2^2 + 2(1-a)\|\delta_x\varepsilon^k\|_2^2\right),
\end{aligned} \tag{1.85}$$

where

$$C_4 = \max\{9, 2(1-2a), 16T(16T-1)\}, \tag{1.86}$$
$$C_5 = 2(128T^3 + 2T^2 + 1). \tag{1.87}$$
$$\tag{1.88}$$

Then we can rename elements

$$\omega^k \leq \rho + C_5\tau\sum_{n=0}^{k-1}\omega^n, \tag{1.89}$$

where

$$\rho = C_4\left(\|\delta_t\varepsilon^1\|_2^2 + \|\delta_t\varepsilon^0\|_2^2 + \|\delta_x\varepsilon^1\|_2^2 + \|\varepsilon^0\|_2\right), \tag{1.90}$$
$$\omega^n = \|\delta_t^{(1)}\varepsilon^n\|_2^2 + 2(1-a)\|\delta_x\varepsilon^n\|_2^2, \quad \forall n \in I_{N-1}. \tag{1.91}$$

We note that the hypotheses of Lemma 1.9 are readily satisfied with $C = C_5$ and $\rho^k = \rho$ for each $k \in I$, whence the conclusion of Theorem 1.11 follows. □

Note that the inequality (1.79) is satisfied for sufficiently small values of τ and of the components of h. Finally, we tackle the problem of the convergence of the numerical method (1.14). The proof of the following result is similar to that of Theorem 1.11. For that reason we provide only a sketch of the proof.

Theorem 1.12 (Convergence)**.** *Let $u \in \mathcal{C}^5(\mathbb{R})$ be a solution of* (1.1) *with $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and let $(v^n)_{n=0}^N$ be a solution of* (1.14) *for the initial conditions (ϕ, ψ). Assume that $\epsilon^n = v^n - u^n$ for each $n \in I_N$. If* (1.79) *holds then the method* (1.14) *is convergent of order $\mathcal{O}(\tau^2 + h^2)$.*

Proof. Let a be as in the proof of Theorem 1.11, and let R_j^n be the truncation error at the point (x_j, t_n) for each $j \in J$ and each $n \in I_N$. Then $(\epsilon^n)_{n=0}^N$ satisfies

$$\begin{aligned} \mu_t \delta_t^{(2)} \epsilon_j^n - \mu_t \delta_x^2 \epsilon_j^n + \delta_{v,t}^{(1)} G(v_j^n) - \delta_{w,t} G(w_j^n) = R_j^n, \quad \forall (j,n) \in J \times I_{N-2}, \\ \text{such that} \begin{cases} \epsilon_j^0 = \delta_t \epsilon_j^0 = 0, & \forall j \in J, \\ \epsilon_j^n = 0, & \forall (j,n) \in \partial J \times I_N. \end{cases} \end{aligned} \tag{1.92}$$

Following the proof of Theorem 1.11, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)} G(v_j^n) - \delta_{w,t} G(w_j^n)$ for each $j \in J$ and each $n \in I_{N-1}$. Proceeding as in the proof of that theorem, we readily obtain

$$\|\delta_t^{(1)} \varepsilon^k\|_2^2 + 2(1-a)\|\delta_x^1 \varepsilon^k\|_2^2 \leq \rho + C_5 \tau \sum_{n=0}^{k-1} \omega^n, \tag{1.93}$$

where

$$\begin{aligned} \rho &= C_4 \left(\|\delta_t \varepsilon^1\|_2^2 + \|\delta_t \varepsilon^0\|_2^2 + \|\delta_x \varepsilon^1\|_2^2 + \|\varepsilon^0\|_2 + \tau \sum_{n=1}^{k} \|R^n\|_2^2 \right), && (1.94) \\ \omega^n &= \|\delta_t^{(1)} \varepsilon^n\|_2^2 + 2(1-a)\|\delta_x \varepsilon^n\|_2^2, \quad \forall n \in I_{N-1}. && (1.95) \end{aligned}$$

then the hypotheses of Lemma 1.9 are satisfied. Using the conclusion of that result, the consistency property of our method and the homogeneous initial-boundary conditions of (1.92) we obtain that

$$\|\delta_t^{(1)} \varepsilon^k\|_2^2 + 2(1-a)\|\delta_x^1 \varepsilon^k\|_2^2 \leq C_4 e^{C_5 k \tau} \tau \sum_{n=0}^{k-1} \|R^n\|_2^2 \leq C_6 (\tau^2 + h^2)^2, \quad \forall k \in I_{N-1}. \tag{1.96}$$

Here $C_6 = C_4 C^2 e^{C_5 T} T$ and C is the constant of Theorem 1.10. The conclusion of the theorem readily follows from the last inequality. $\square$

2. Energy Conserving Scheme for the Fractional Wave Equation

In this chapter, we investigate numerically a model governed by a multidimensional nonlinear wave equation with damping and fractional diffusion. The governing partial differential equation considers the presence of Riesz space-fractional derivatives of orders in $(1, 2]$, and homogeneous Dirichlet boundary data are imposed on a closed and bounded spatial domain. The model under investigation possesses an energy function which is preserved in the undamped regime. In the damped case, we establish the property of energy dissipation of the model using arguments from functional analysis. Motivated by these results, we propose an explicit finite-difference discretization of our fractional model based on the use of fractional centered differences. Associated to our discrete model, we also propose a discretization of the energy quantity. We establish that the discrete energy is conserved in the undamped regime, and that it dissipates in the damped scenario. Among the most important numerical features of our scheme, we show that the method has a consistency of second order, that it is stable and that it has a quadratic order of convergence. Some one- and two-dimensional simulations are shown in this chapter to illustrate the fact that the technique is capable of preserving the discrete energy in the undamped regime. For the sake of convenience, we provide a Matlab implementation of our method for the one-dimensional scenario.

2.1 Introduction

In the last decades, the investigation of physical media described by systems with long-range interactions has been a fruitful avenue of research. In particular, a wide variety of physical phenomena described by linear systems with long-range interactions have been reported in the literature. One of the typical examples in classical physics is the linear interaction of particles in a three-dimensional gravitational system [22]. Other examples are the interactions of vortices in two dimensions, engineering problems on elasticity arising from the study of planar stress, systems of electric charges and systems that consider dipolar forces [23]. Moreover, there are several well-characterized cases of long-range interactions involved in the activation and the repression of transcription in chromosomal and gene regulation [24], and many other applications have been proposed to polymer science (including some applications to microscopic models of polymer dynamics and rheological constitutive equations), to regular variations in thermodynamics and to Hamiltonian chaotic systems [25]. It is worth pointing out that various works have been devoted to the physical and mathematical investigation of generalized forms of these models [26, 27], including systems which exhibit the presence of the phenomenon of nonlinear supratransmission of energy in fractional sine-Gordon-type equations [28], models of Josephson transmission lines [29] and extensions of the Fermi–Pasta–Ulam chains [30]. Additionally, some models of oscillators with long-range interactions possess conservation laws that resemble those quantities preserved by classical systems [31, 11].

It is important to recall that certain long-range interactions (namely, the so-called α-*interactions*) yield fractional derivatives in some continuous-limit process. This process involves the Fourier series transform, the inverse Fourier transform and the limiting process when the distance between consecutive particles tends to zero [32]. In such way, fractional models in the form of ordinary or partial differential equations are obtained from discrete physical systems. In fact, there are various types of long-range interactions which lead to systems that include fractional derivatives of the Riesz type. From this perspective, the use of the Riesz differential operator is physically justified, at least as the continuous limit of physically meaningful discrete systems appearing in various branches of sciences. Obviously, this fact has encouraged the mathematical modeling using fractional differential equations, as well as the analytical and the physical investigation of these models. Needless to mention that the specialized literature has benefited from the investigation of fractional equations. Indeed various interesting reports have been published on the existence and the uniqueness of solutions of fractional forms of parabolic models, like the porous media equation [33], the nonlinear diffusion equation in multiple dimensions [34] and nonlinear degenerate diffusion equations in bounded domains [35].

On the other hand, the recent advances of fractional calculus have led to the development of numerical techniques to approximate the solution of fractional partial differential equations. As examples, numerical models have been proposed to solve a time-space fractional Fokker–Planck equation with variable force field and diffusion [36] and nonlinear fractional-order Volterra integro-differential equations [37]. Some fractional models that extend well-known equations from mathematical physics have been the motivation to develop suitable numerical schemes. For instance, some highly accurate numerical schemes have been proposed for variable-order fractional Schrödinger equations [38], a new technique based on Legendre polynomials has been reported to solve the fractional two-dimensional heat equation [39], some improvements of the sub-equation method have been designed to solve a $(3+1)$-dimensional generalization of the Korteweg–de Vries–Zakharov–Kuznetsov equation [40] and some novel methods have been constructed to approximate the solutions of a two-dimensional variable-order fractional percolation equation in non-homogeneous porous media [41]. Like these reports, the literature shows that the numerical investigation of fractional partial differential equations has been a fruitful avenue of current research in numerical analysis [42]. However, it is important to point out that the design of structure-preserving methods to solve such models is still a direction of research which has not been sufficiently exploited.

In this context, the notion of structure-preserving method refers to those numerical techniques which are capable of preserving physical features of the solutions of interest. As opposed to numerical efficiency which is typically associated to the computational properties inherent to those techniques (consistency, stability and convergence), the properties of preservation of the structure of solutions depend on each physical problem itself. As examples of those properties, we can quote the conservation of physical quantities like energy, momentum or mass [43]. Mathematical characteristics such as positivity, boundedness, monotonicity and convexity are also considered in this chapter as structural properties [44]. Some structure-preserving methods have been designed for the numerical solution of partial differential equations of fractional order. For instance, some energy-preserving schemes have been proposed for the nonlinear fractional Schrödinger equation [45], and some finite-difference scheme based on fractional centered differences has been used to approximate positive and bounded solutions of a fractional population model [46]. However, it is important to note that there are very few reports in the literature on energy-conserving methods for fractional partial differential equations which are consistent, stable and convergent. In particular, fractional extensions of hyperbolic models like the sine-Gordon and the Klein–Gordon equations have been practically left without investigation. This is a topic that merits deeper investigation in view of all the potential applications of those equations to the continuous mathematical modeling of nonlinear systems with long-range interactions [32, 47, 48].

The purpose of the present chapter is to study numerically a multidimensional Riesz space-fractional generalization of the nonlinear and damped wave equation that extends various models from mathematical physics, including the sine-Gordon and the Klein–Gordon equations. It is well known that these two models possess an energy functional that dissipates or is conserved, depending on suitable analytical and parameter conditions. Thus the design of dissipation and conserving schemes to approximate its solution is pragmatically justified. The method reported in this chapter has some associated energy density functionals along with a function of total energy which is capable of resembling this property of the continuous model. Moreover, we will show that our methodology is an explicit technique which is second-order consistent, stable and quadratically convergent. Some simulations will be provided to illustrate the capability of the scheme to preserve the energy when the damping coefficient is equal to zero. Evidently, the explicit nature of our approach makes the technique an ideal tool in the investigation of multidimensional systems governed by Riesz space-fractional nonlinear wave equations.

2.2 Preliminaries

In this chapter we let $p \in \mathbb{N}$, $T \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+ \cup \{0\}$. Let us define the set $I_n = \{1, \dots, n\}$ for each natural number n, and let $\overline{I}_n = I_n \cup \{0\}$. Suppose that $a_i, b_i \in \mathbb{R}$ satisfy $a_i < b_i$ for each $i \in I_p$. Throughout we will assume that $1 < \alpha_i \leq 2$ for each $i \in I_p$, and we let $B = \prod_{i=1}^{p} (a_i, b_i) \subseteq \mathbb{R}^P$ and $\Omega = B \times (0, T) \subseteq \mathbb{R}^{p+1}$. We introduce the symbols $\overline{B}$ and $\overline{\Omega}$ to denote the closures of B and Ω in $\mathbb{R}^{p+1}$ under the standard topology, respectively, and let ∂B represent the boundary of B. Assume that $G : \mathbb{R} \to \mathbb{R}$ and that $\phi, \psi : B \to \mathbb{R}$ are sufficiently smooth functions that satisfy $\phi(x) = \psi(x) = 0$ for each $x \in \partial B$. Additionally, we will suppose that G is nonnegative and that $u : \overline{\Omega} \to \mathbb{R}$ is a sufficiently smooth function that satisfies the initial-boundary-value problem

$$
\begin{gathered}
\frac{\partial^2 u}{\partial t^2}(x,t) - \sum_{i=1}^{p} \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) + \gamma \frac{\partial u}{\partial t}(x,t) + G'(u(x,t)) = 0, \quad \forall (x,t) \in \Omega, \\
\text{such that } \begin{cases} u(x,0) = \phi(x), & \forall x \in B, \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x), & \forall x \in B, \\ u(x,t) = 0, & \forall (x,t) \in \partial B \times (0,T). \end{cases}
\end{gathered}
\tag{2.1}
$$

Here we assume that $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$, and the Riesz differential operators are defined for each $i \in I_p$ by

$$\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) = \frac{-1}{2\cos(\frac{\pi\alpha_i}{2})\Gamma(2-\alpha_i)} \frac{\partial^2}{\partial x_i^2} \int_{a_i}^{b_i} \frac{u(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_p, t)}{|x_i - \xi|^{\alpha_i - 1}} d\xi, \quad \forall (x,t) \in \Omega. \quad (2.2)$$

Let $L_{x,2}(\overline{\Omega})$ denote the set of all functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f(\cdot, t) \in L_2(\overline{B})$ for each $t \in [0,T]$. For each pair $f, g \in L_{x,2}(\overline{\Omega})$, the inner product of f and g is the function of t defined by

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x,t) g(x,t) dx, \quad \forall t \in [0,T]. \quad (2.3)$$

In turn, the Euclidean norm of $f \in L_{x,2}(\overline{\Omega})$ is the function of t defined by $\|f\|_{x,2} = \sqrt{\langle f, f \rangle}$. The set of all functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f(\cdot, t) \in L_1(\overline{B})$ for each $t \in [0,T]$ will be denoted by $L_{x,1}(\overline{\Omega})$, and for each such f we define its norm as the function of t given by

$$\|f\|_{x,1} = \int_{\overline{B}} |f(x,t)| dx, \quad \forall t \in [0,T]. \quad (2.4)$$

The literature on mathematical physics has proposed various functionals to calculate the energy of one-dimensional systems governed by (2.1) when $\gamma = 0$ (see [11], for instance). For purposes of this chapter, we will use the following dimensional extension of the energy integral employed in [9]:

$$\mathcal{E}(t) = \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \frac{1}{2} \sum_{i=1}^{p} \left\langle u, -\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}} \right\rangle_x + \|G(u)\|_{x,1}, \quad \forall t \in [0,T]. \quad (2.5)$$

It is important to note here that the Riesz fractional derivative of order α_i in the ith component is a self-adjoint and negative operator [49] for each $i \in I_p$. This fact implies that the additive inverse of the Riesz fractional derivative has a unique square root operator [50] which will be denoted by $\Xi_{x_i}^{\alpha_i}$. Moreover, the following holds for any two functions u and v:

$$\left\langle -\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}, v \right\rangle_x = \left\langle \Xi_{x_i}^{\alpha_i} u, \Xi_{x_i}^{\alpha_i} v \right\rangle_x. \quad (2.6)$$

The next result is now easy to verify.

Lemma 2.1. *The energy function* (2.5) *may be rewritten alternatively as*

$$\mathcal{E}(t) = \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \frac{1}{2} \sum_{i=1}^{p} \left\| \Xi_{x_i}^{\alpha_i} u \right\|_{x,2}^2 + \|G(u)\|_{x,1}, \quad (2.7)$$

for each $t \in (0,T)$. □

Obviously, the associated energy density is defined for each $(x,t) \in \Omega$ by

$$\begin{aligned} \mathcal{H}(x,t) &= \frac{1}{2} \left[\frac{\partial u}{\partial t}(x,t) \right]^2 - \frac{1}{2} \sum_{i=1}^{p} u(x,t) \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x,t) + G(u(x,t)) \\ &= \frac{1}{2} \left[\frac{\partial u}{\partial t}(x,t) \right]^2 + \frac{1}{2} \sum_{i=1}^{p} \left[\Xi_{x_i}^{\alpha_i} u(x,t) \right]^2 + G(u(x,t)). \end{aligned} \quad (2.8)$$

The following result is the cornerstone of our investigation. It is a generalization of Theorem 1.1 of [51], which is a result valid for fractional wave equations in one spatial variable. A proof is provided here for completeness.

Theorem 2.2 (Macías-Díaz [51]). *If u is a solution of* (2.1) *then*

$$\mathcal{E}'(t) = -\gamma \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2, \quad \forall t \in (0,T). \tag{2.9}$$

Proof. Note that the following hold:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 &= \frac{1}{2}\frac{d}{dt}\int_B \left(\frac{\partial u}{\partial t}(\xi,t)\right)^2 d\xi = \frac{1}{2}\int_B \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}(\xi,t)\right)^2 d\xi \\
&= \frac{1}{2}\int_B 2\left(\frac{\partial u}{\partial t}(\xi,t)\right)\left(\frac{\partial^2 u}{\partial t^2}(\xi,t)\right) d\xi = \int_B \left(\frac{\partial u}{\partial t}(\xi,t)\right)\left(\frac{\partial^2 u}{\partial t^2}(\xi,t)\right) d\xi \\
&= \left\langle \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right\rangle, \qquad (2.10) \\
\frac{1}{2}\frac{d}{dt}\|\Xi_{x_i}^{\alpha_i} u\|_{x,2}^2 &= \frac{1}{2}\frac{d}{dt}\int_B \left(\Xi_{x_i}^{\alpha_i} u(\xi,t)\right)^2 d\xi = \frac{1}{2}\int_B \frac{\partial}{\partial t}\left(\Xi_{x_i}^{\alpha_i} u(\xi,t)\right)^2 d\xi \\
&= \frac{1}{2}\int_B 2\left(\Xi_{x_i}^{\alpha_i} u(\xi,t)\right)\left(\frac{\partial}{\partial t}\Xi_{x_i}^{\alpha_i} u(\xi,t)\right) d\xi = \left\langle \frac{\partial}{\partial t}\left(\Xi_{x_i}^{\alpha_i} u\right), \Xi_{x_i}^{\alpha_i} u\right\rangle_x \\
&= \left\langle \Xi_{x_i}^{\alpha_i}\left(\frac{\partial u}{\partial t}\right), \Xi_{x_i}^{\alpha_i} u\right\rangle_x = \left\langle \frac{\partial u}{\partial t}, -\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}\right\rangle, \quad \forall i \in I_p, \qquad (2.11) \\
\frac{d}{dt}\|G(u)\|_{x,1} &= \frac{d}{dt}\int_a^b G(u(\xi,t))d\xi = \int_a^b \frac{\partial}{\partial t} G(u(\xi,t))d\xi \\
&= \int_a^b G'(u(\xi,t))\left(\frac{\partial}{\partial t}u(\xi,t)\right) d\xi = \left\langle \frac{\partial u}{\partial t}, G'(u)\right\rangle. \qquad (2.12)
\end{aligned}$$

Taking derivative with respect to t on both sides of (2.7), using the identities above and the partial differential equation of (2.1), and simplifying algebraically we obtain

$$\begin{aligned}
\mathcal{E}'(t) &= \frac{d}{dt}\left(\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \frac{1}{2}\sum_{i=1}^p \left\|\Xi_{x_i}^{\alpha_i} u\right\|_{x,2}^2 + \|G(u)\|_{x,1}\right) \\
&= \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \frac{1}{2}\frac{d}{dt}\sum_{i=1}^p \left\|\Xi_{x_i}^{\alpha_i} u\right\|_{x,2}^2 + \frac{d}{dt}\|G(u)\|_{x,1} \\
&= \int_B \left(\frac{\partial u}{\partial t}(\xi,t)\right)\left(\frac{\partial^2 u}{\partial t^2}(\xi,t)\right) d\xi - \int_B \frac{\partial u}{\partial t}(\xi,t)\sum_{i=1}^p \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(\xi,t)d\xi + \int_B \frac{\partial u}{\partial t}(\xi,t)G'(u(\xi,t))d\xi \\
&= \int_B \frac{\partial u}{\partial t}(\xi,t)\left[\frac{\partial^2 u}{\partial t^2}(\xi,t) - \sum_{i=1}^p \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(\xi,t) + G'(u(\xi,t))\right] d\xi = -\gamma \int_B \left[\frac{\partial u}{\partial t}(\xi,t)\right]^2 d\xi,
\end{aligned} \tag{2.13}$$

whence the result readily follows. □

Corollary 2.3. *If u is a solution of* (2.1) *then*

$$\mathcal{E}(t) = \mathcal{E}(0) - \gamma \int_0^t \left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 dt, \quad \forall t \in [0,T]. \tag{2.14}$$

In particular, if $\gamma = 0$ then the system (2.1) *is conservative.* $\square$

In the following section, we will propose an explicit numerical method to approximate the solutions of (2.1) and the energy function (2.5). Our numerical technique will satisfy discrete versions of Theorem 2.2 and Corollary 2.3, along with the numerical properties of consistency, stability and convergence.

2.3 Numerical method

For the remainder of the chapter we let h_i and τ be positive step-sizes for each $i \in I_p$, and assume that $N = T/\tau$ and $M_i = (b_i - a_i)/h_i$ are positive integers for each $i \in I_p$. Consider uniform partitions of $[a_i, b_i]$ and $[0, T]$ given by

$$x_{i,j_i} = a_i + j_i h_i, \quad \forall i \in I_p, \forall j_i \in \overline{I}_{M_i}, \tag{2.15}$$

$$t_n = n\tau, \quad \forall n \in \overline{I}_N. \tag{2.16}$$

Let $J = \prod_{i=1}^{p} I_{M_i - 1}$ and $\overline{J} = \prod_{i=1}^{p} \overline{I}_{M_i}$, and let ∂J represent the boundary of the mesh $\overline{J}$. Define $x_j = (x_{1,j_1}, \ldots, x_{p,j_p})$ for each multi-index $j = (j_1, \ldots, j_p) \in \overline{J}$. In this manuscript, the symbol v_j^n will represent a numerical approximation to the exact value of $u_j^n = u(x_j, t_n)$ for each $j \in \overline{J}$ and each $n \in \overline{I}_N$. Define the discrete linear operators

$$\mu_t u_j^n = \frac{u_j^{n+1} + u_j^n}{2}, \tag{2.17}$$

$$\delta_t^{(1)} u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \tag{2.18}$$

$$\delta_t^{(2)} u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2}, \tag{2.19}$$

$$\delta_{u,t}^{(1)} G(u_j^n) = \begin{cases} \dfrac{G(u_j^{n+1}) - G(u_j^n)}{u_j^{n+1} - u_j^n}, & \text{if } u_j^{n+1} \neq u_j^n, \\ G'(u_j^n), & \text{if } u_j^{n+1} = u_j^n, \end{cases} \tag{2.20}$$

for each $j \in J$ and $n \in I_{N-1}$.

Definition 2.4. For any function $f : \mathbb{R} \to \mathbb{R}$, any $h > 0$ and any $\alpha > -1$ we define the *fractional centered difference* of order α of f at the point x as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{2.21}$$

where

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{2.22}$$

Lemma 2.5 (Çelik and Duman [52])**.** *If $1 < \alpha \leq 2$ then*

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ *for all $k \geq 1$, and*

(iii) $\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0.$ □

Proof. We first notice that:

$$\begin{aligned}
g_{k+1}^{(\alpha)} &= \frac{(-1)^{k+1}\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-(k+1)+1)\Gamma(\frac{\alpha}{2}+(k+1)+1)}, \quad \forall k \in \mathbb{N}\cup\{0\} \\
&= \frac{(-1)(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k)\Gamma(\frac{\alpha}{2}+k+2)} = \frac{(-1)(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k)(\frac{\alpha}{2}+k+1)\Gamma(\frac{\alpha}{2}+k+1)} \\
&= \frac{(\frac{\alpha}{2}-k)(-1)(-1)^k\Gamma(\alpha+1)}{(\frac{\alpha}{2}-k)\Gamma(\frac{\alpha}{2}-k)(\frac{\alpha}{2}+k+1)\Gamma(\frac{\alpha}{2}+k+1)} = \frac{(k-\frac{\alpha}{2})(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-k+1)(\frac{\alpha}{2}+k+1)\Gamma(\frac{\alpha}{2}+k+1)} \\
&= \frac{(k-\frac{\alpha}{2})}{(\frac{\alpha}{2}+k+1)}g_k^{(\alpha)} = \frac{(\frac{\alpha}{2}+k+1)-(\alpha+1)}{(\frac{\alpha}{2}+k+1)}g_k^{(\alpha)} = \left(1-\frac{\alpha+1}{\frac{\alpha}{2}+k+1}\right)g_k^{(\alpha)}. \qquad (2.23)
\end{aligned}$$

(i) We know $\gamma(z) > 0$ for $z > 0$, and $1 < \alpha < 2$ then $\frac{3}{2} < \frac{\alpha}{2}+1 < \frac{5}{2}$ because of that $\Gamma\left(\frac{\alpha}{2}+1\right) > 0$, with this in mind we calculate g_0^{α}:

$$g_0^{(\alpha)} = \frac{(-1)^0\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-0+1)\Gamma(\frac{\alpha}{2}+0+1)} = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+1)^2} \geq 0. \qquad (2.24)$$

(ii) Now we evaluate $g_{-k}^{(\alpha)}$:

$$\begin{aligned}
g_{-k}^{(\alpha)} &= \frac{(-1)^{-k}\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-(-k)+1)\Gamma(\frac{\alpha}{2}+(-k)+1)} \\
&= \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+k+1)\Gamma(\frac{\alpha}{2}-k+1)} = g_k^{(\alpha)}, \quad \forall k \in \mathbb{N}\cup\{0\}. \qquad (2.25)
\end{aligned}$$

We notice that $1 < \alpha < 2$ then $2 < \alpha+1 < 3$, after some calculations

$$\frac{2k-1}{2k+3} > 1-\frac{\alpha+1}{\frac{\alpha}{2}+k+1} > \frac{k-1}{k+2}. \qquad (2.26)$$

With the last inequality, when $k=0$ then $1-\frac{\alpha+1}{\frac{\alpha}{2}+k+1}$ is bounded between $-\frac{1}{3}$ and $-\frac{1}{2}$, and when $k>0$ it will be bounded by positive numbers, in that way

$$\begin{aligned}
g_{k+1}^{(\alpha)} &= \left(1-\frac{\alpha+1}{\frac{\alpha}{2}+k+1}\right)g_k^{(\alpha)} = \ldots = \\
&= \left(1-\frac{\alpha+1}{\frac{\alpha}{2}+k+1}\right)\ldots\left(1-\frac{\alpha+1}{\frac{\alpha}{2}+1+1}\right)\left(1-\frac{\alpha+1}{\frac{\alpha}{2}+0+1}\right)g_0^{(\alpha)}, \qquad (2.27)
\end{aligned}$$

where only $\left(1-\frac{\alpha+1}{\frac{\alpha}{2}+0+1}\right) < 0$ and all other terms are positive, so $g_{-(k+1)}^{(\alpha)} = g_{k+1}^{(\alpha)} \leq 0$ with $k \geq 0$.

(iii) from Çelik and Duman [52] we get the next equation:

$$\left|2\sin\left(\frac{z}{2}\right)\right|^{\alpha} = \sum_{k=-\infty}^{\infty}\frac{(-1)^k\Gamma(\alpha+1)e^{ikz}}{\Gamma(\frac{\alpha}{2}-k+1)\Gamma(\frac{\alpha}{2}+k+1)}, \qquad (2.28)$$

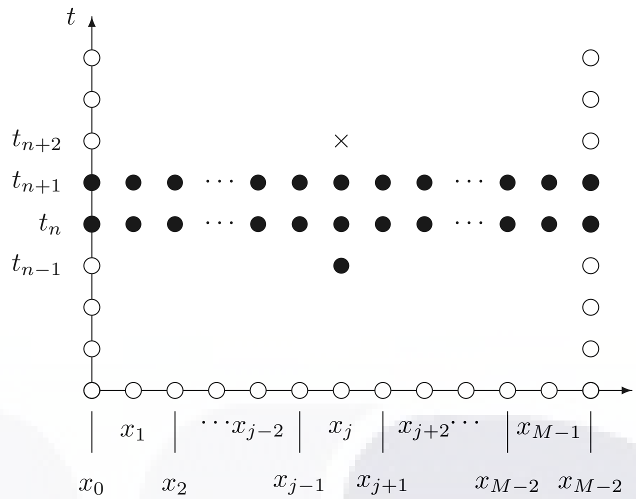

Figure 2.1: Forward-difference stencil for the approximation to the exact solution of the one-dimensional form of (2.1) at the time t_n, using the finite-difference scheme (2.33). The black circles represent the known approximations at the times t_{n-1}, t_n and t_{n+1}, while the cross denotes the unknown approximation at the time t_{n+2}.

substituting $g_k^{(\alpha)}$ and $z = 0$,

$$0 = \left| 2 \sin \left(\frac{0}{2} \right) \right|^\alpha = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} e^{ik*0} = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)}. \tag{2.29}$$

□

As a consequence, the series on the right-hand side of (2.21) converges absolutely for any $f \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. It is easy to see that any $f \in \mathcal{C}^5(\mathbb{R})$ for which all of its derivatives up to order five belong to $L_1(\mathbb{R})$, satisfies

$$-\frac{1}{h^\alpha} \Delta_h^\alpha f(x) = \frac{\partial^\alpha f(x)}{\partial |x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}, \tag{2.30}$$

whenever $1 < \alpha \leq 2$ (see [52]). Moreover, if $u \in \mathcal{C}^5(\overline{B})$ then

$$\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x_j, t_n) = \delta_{x_i}^{(\alpha_i)} u_j^n + \mathcal{O}(h^2), \quad \forall i \in I_p, \forall (j,n) \in J \times \overline{I}_N, \tag{2.31}$$

where

$$\delta_{x_i}^{(\alpha_i)} u_j^n = -\frac{1}{h_i^{\alpha_i}} \sum_{k=0}^{M_i} g_{j_i-k}^{(\alpha_i)} u(x_{1,j_1}, \ldots, x_{i-1,j_{i-1}}, x_{i,k}, x_{i+1,j_{i+1}}, \ldots, x_{p,j_p}, t_n). \tag{2.32}$$

With this nomenclature, the finite-difference method to approximate the solution of (2.1) on Ω is given by

$$\mu_t\delta_t^{(2)}v_j^n - \sum_{i=1}^{p}\mu_t\delta_{x_i}^{(\alpha_i)}v_j^n + \gamma\delta_t^{(1)}v_j^n + \delta_{v,t}^{(1)}G(v_j^n) = 0, \quad \forall(j,n)\in J\times I_{N-2},$$
$$\text{such that }\begin{cases} v_j^0 = \phi(x_j), & \forall j\in\overline{J}, \\ \delta_t v_j^0 = \psi(x_j), & \forall j\in\overline{J}, \\ v_j^n = 0, & \forall(j,n)\in\partial J\times\overline{I}_N. \end{cases} \tag{2.33}$$

For illustration purposes, Figure 2.1 shows the forward-difference stencil in the case that $p = 1$ using the conventions $M = M_1$ and $x_j = x_{1,j}$. Note in general that this scheme is an explicit four-step method, so we can calculate the numerical approximation v_j^{n+2} if we have the values of v_j^{n+1}, v_j^n and v_j^{n-1}, because:

$$\frac{v_j^{n+2} - v_j^{n+1} - v_j^n + v_j^{n-1}}{2\tau} - \sum_{i=1}^{p}\mu_t\delta_{x_i}^{(\alpha_i)}v_j^n + \gamma\delta_t^{(1)}v_j^n + \delta_{v,t}^{(1)}G(v_j^n) = 0, \quad \forall(j,n)\in J\times I_{N-2}. \tag{2.34}$$

so the numerical approximation to the solution at the node x_j and time t_{n+2} is given by the formula

$$v_j^{n+2} = v_j^{n+1} + v_j^n - v_j^{n-1} + 2\tau\left[\sum_{i=1}^{p}\mu_t\delta_{x_i}^{(\alpha_i)}v_j^n - \gamma\delta_t^{(1)}v_j^n - \delta_{v,t}^{(1)}G(v_j^n)\right], \quad \forall(j,n)\in J\times I_{N-2}. \tag{2.35}$$

2.4 Energy invariants

In this section we show that the finite-difference method (2.33) satisfies physical properties similar to those satisfied by (2.1). More precisely, we will propose a numerical energy functional associated to the scheme (2.33) that is preserved under suitable parameter conditions. For the remainder of this chapter we will let $h = (h_1, \dots, h_p)$ and $h_* = \prod_{i=1}^{p} h_i$, and employ the spatial mesh $R_h = \{x_j\}_{j\in J} \subseteq \mathbb{R}^p$. Let $\mathcal{V}_h$ be the real vector space of all real grid functions on R_h. For any $u \in \mathcal{V}_h$ and $j \in I$ convey that $u_j = u(x_j)$. Moreover, define respectively the inner product $\langle\cdot,\cdot\rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ and the norm $\|\cdot\|_1 : \mathcal{V}_h \to \mathbb{R}$ by

$$\langle u, v\rangle = h_*\sum_{j\in I} u_j v_j, \tag{2.36}$$

$$\|u\|_1 = h_*\sum_{j\in I} |u_j|, \tag{2.37}$$

for any $u, v \in \mathcal{V}_h$. The Euclidean norm induced by $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|_2$. In the following, we will represent the solutions of the finite-difference method (2.33) by $(v^n)_{n=0}^N$, where we convey that $v^n = (v_j^n)_{j\in J}$ for each $n \in \overline{I}_N$. The next theorem will be helpful in the demonstration of (2.38).

Theorem 2.6 (Gershgorin's Circle). *If $R_i = \sum_{j\neq i}|a_{ij}|$ and $D(a_{ij}, R_i) = \{x \in \mathbb{C} : |a_{ii} - x| < R_i\}$ then each eigenvalue is inside the set $D(a_{ij}, R_i)$ for $a_{ii}, \in \{1, ..., m\}$.*

Lemma 2.7 (Macías-Díaz [51]). *If $i \in I_p$ then the following are satisfied for the matrix*

$$A_{x_i}^{(\alpha_i)} = \begin{pmatrix} g_0^{(\alpha_i)} & g_{-1}^{(\alpha_i)} & \cdots & g_{2-M_i}^{(\alpha_i)} \\ g_1^{(\alpha_i)} & g_0^{(\alpha_i)} & \cdots & g_{3-M_i}^{(\alpha_i)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{M_i-2}^{(\alpha_i)} & g_{M_i-3}^{(\alpha_i)} & \cdots & g_0^{(\alpha_i)} \end{pmatrix}. \tag{2.38}$$

(a) $A_{x_i}^{(\alpha_i)}$ *is Hermitian.*

(b) $A_{x_i}^{(\alpha_i)}$ *is strictly diagonally dominant.*

(c) *All the eigenvalues of $A_{x_i}^{(\alpha_i)}$ are positive real numbers bounded from above by $2g_0^{(\alpha_i)}$.*

(d) $A_{x_i}^{(\alpha_i)}$ *is positive-definite.* □

Proof.

(a) We know $A_{x_i}^{(\alpha_i)} = [a_{ij}]$ is a real matrix, we then just have to prove that $a_{ij} = a_{ji}, \forall i \neq j$, but $a_{ij} = g_{i-j}^{\alpha}$ if $i > j$ and $a_{ij} = g_{j-i}^{\alpha}$ if $i < j$ we also know by the previous lemma $g_{i-j}^{\alpha} = g_{-(i-j)}^{\alpha} = g_{j-i}^{\alpha}$ then $a_{ij} = a_{ji}, \forall i \neq j$, so $A_{x_i}^{(\alpha_i)}$ is an Hermitian matrix.

(b) The proof continues straight forward from lemma (2.5).

(c) For this part of the proof we need Theorem 2.6, then we define R_i as in the theorem

$$R_i = \sum_{j \neq i} |a_{ij}| = \sum_{j \neq i} \left| g_{i-j}^{\alpha} \right| \leq \sum_{j \neq i} -g_{i-j}^{\alpha} = g_0^{\alpha} = |g_0^{\alpha}|, \tag{2.39}$$

if λ is an eigenvalue of $A_{x_i}^{(\alpha_i)}$ then $|\lambda - g_0^{\alpha}| \leq R_i$ and

$$0 < g_0 - R_i \leq \lambda \leq g_0 + R_i < 2g_0^{\alpha} \tag{2.40}$$

$$\therefore \lambda > 0. \tag{2.41}$$

(d) $A_{x_i}^{(\alpha_i)}$ is a symmetric matrix with positive eigenvalues, then $A_{x_i}^{(\alpha_i)}$ is a positive matrix.

□

Before we state our next result, we require some additional notation. Let v be a grid function, let $j \in J$ and $i \in I_p$. If $k \in I_{M_i-1}$ then we define respectively the real constant and the $(M_i - 1)$-dimensional real vector

$$v_{j|j_i=k} = v(x_{1,j_1}, \ldots, x_{i-1,j_{i-1}}, x_{i,k}, x_{i+1,j_{i+1}}, \ldots, x_{p,j_p}), \tag{2.42}$$

$$v_{j|j_i} = (v_{j|j_i=1}, v_{j|j_i=2}, \ldots, v_{j|j_i=M_i-1})^{\top}. \tag{2.43}$$

These conventions will be required in the next lemma, whose proof will require a well-known result on the existence and the uniqueness of square-root operators from functional analysis [50].

Lemma 2.8. *For each $i \in I_p$ there exists a unique positive linear operator $\Lambda_{x_i}^{(\alpha_i)} : \mathcal{V}_h \to \mathcal{V}_h$ such that*

$$\left\langle -\delta_{x_i}^{(\alpha_i)} u, v \right\rangle = \left\langle \Lambda_{x_i}^{(\alpha_i)} u, \Lambda_{x_i}^{(\alpha_i)} v \right\rangle, \tag{2.44}$$

for each $u, v \in \mathcal{V}_h$.

Proof. Note firstly that if $u, v \in \mathcal{V}_h$ and if $i \in I_p$ then

$$\begin{aligned} \langle u, -\delta_{x_i}^{(\alpha_i)} v \rangle &= -h_* \sum_{j \in J} u_j \delta_{x_i}^{(\alpha_i)} v_j = \frac{h_*}{h^{\alpha_i}} \sum_{j \in J} \sum_{k=1}^{M_i - 1} u_j g_{j_i - k}^{(\alpha_i)} v_{j|j_i = k} \\ &= \frac{h_*}{h^{\alpha_i}} \sum_{j_1=1}^{M_1-1} \cdots \sum_{j_{i-1}=1}^{M_{i-1}-1} \sum_{j_{i+1}=1}^{M_{i+1}-1} \cdots \sum_{j_p=1}^{M_p-1} \sum_{j_i=1}^{M_i-1} \sum_{k=1}^{M_i-1} u_j g_{j_i-k}^{(\alpha_i)} v_{j|j_i=k} \\ &= \frac{h_*}{h^{\alpha_i}} \sum_{j_1=1}^{M_1-1} \cdots \sum_{j_{i-1}=1}^{M_{i-1}-1} \sum_{j_{i+1}=1}^{M_{i+1}-1} \cdots \sum_{j_p=1}^{M_p-1} u_{j|j_i}^\top A_{x_i}^{(\alpha_i)} v_{j|j_i}. \end{aligned} \tag{2.45}$$

Using the symmetry of the matrix $A_{x_i}^{(\alpha_i)}$ we observe that

$$\begin{aligned} \langle u, -\delta_{x_i}^{(\alpha_i)} v \rangle &= \langle u, -\delta_{x_i}^{(\alpha_i)} v \rangle^\top \\ &= \frac{h_*}{h^{\alpha_i}} \sum_{j_1=1}^{M_1-1} \cdots \sum_{j_{i-1}=1}^{M_{i-1}-1} \sum_{j_{i+1}=1}^{M_{i+1}-1} \cdots \sum_{j_p=1}^{M_p-1} v_{j|j_i}^\top A_{x_i}^{(\alpha_i)} u_{j|j_i} = \langle -\delta_{x_i}^{(\alpha_i)} u, v \rangle, \end{aligned} \tag{2.46}$$

holds for each $u, v \in \mathcal{V}_h$, which means that $-\delta_{x_i}^{(\alpha_i)}$ is a self-adjoint operator for each $i \in I_p$. On the other hand, the fact that the matrix $A_{x_i}^{(\alpha_i)}$ is positive definite implies that $u_{j|j_i}^\top A_{x_i}^{(\alpha_i)} u_{j|j_i} \geq 0$ for each $u \in \mathbb{R}^{M_i - 1}$ and $i \in I_p$. As a consequence we note that $\langle u, -\delta_{x_i}^{(\alpha_i)} u \rangle \geq 0$ for each $u \in \mathcal{V}_h$, which means that $-\delta_{x_i}^{(\alpha_i)}$ is positive. We conclude that there exists a unique positive linear square-root operator $\Lambda_{x_i}^{(\alpha_i)}$ for $-\delta_{x_i}^{(\alpha_i)}$ which satisfies the conclusion of the theorem. □

The next theorem establishes the existence of invariants for the discrete system (2.33).

Theorem 2.9 (Dissipation of energy)**.** *Let $(v^n)_{n=0}^N$ be a solution of* (2.33)*, and define*

$$E^n = \frac{1}{2} \langle \delta_t^{(1)} v^n, \delta_t^{(1)} v^{n-1} \rangle + \frac{1}{2} \sum_{i=1}^p \| \Lambda_{x_i}^{(\alpha_i)} v^n \|_2^2 + \| G(v^n) \|_1, \quad \forall n \in I_{N-1}. \tag{2.47}$$

Then $\delta_t E^n = -\gamma \| \delta_t^{(1)} v^n \|_2^2$ for each $n \in I_{N-2}$.

Proof. Recall that G is a nonnegative function and note that the following hold for each $n \in I_{N-2}$:

$$\begin{aligned} \langle \mu_t \delta_t^{(2)} v^n, \delta_t^{(1)} v^n \rangle &= \langle \delta_t^{(2)} \mu_t v^n, \delta_t^{(1)} v^n \rangle = \langle \delta_t^{(2)} \left(\frac{v^{n+1} + v^n}{2} \right), \delta_t^{(1)} v^n \rangle = \frac{1}{2} \langle \delta_t^{(2)} v^{n+1} + \delta_t^{(2)} v^n, \delta_t^{(1)} v^n \rangle \\ &= \frac{1}{2} \left\langle \frac{v^{n+2} - 2v^{n+1} + v^n}{\tau^2} + \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2}, \delta_t^{(1)} v^n \right\rangle \\ &= \frac{1}{2\tau} \langle \frac{v^{n+2} - v^{n+1} - v^n + v^{n-1}}{\tau}, \delta_t^{(1)} v^n \rangle = \frac{1}{2\tau} \langle \frac{v^{n+2} - v^{n+1}}{\tau} - \frac{v^n - v^{n-1}}{\tau}, \delta_t^{(1)} v^n \rangle \\ &= \frac{1}{2\tau} \langle \delta_t^{(1)} v^{n+1} - \delta_t^{(1)} v^{n-1}, \delta_t^{(1)} v^n \rangle = \frac{1}{2\tau} \left[\langle \delta_t^{(1)} v^{n+1}, \delta_t^{(1)} v^n \rangle - \langle \delta_t^{(1)} v^n, \delta_t^{(1)} v^{n-1} \rangle \right], \end{aligned} \tag{2.48}$$

$$
\begin{aligned}
\langle -\mu_t \delta_{x_i}^{(\alpha_i)} v^n, \delta_t^{(1)} v^n \rangle &= \langle -\delta_{x_i}^{(\alpha_i)} \mu_t v^n, \delta_t^{(1)} v^n \rangle = \langle -\delta_{x_i}^{(\alpha_i)} \left(\frac{v^{n+1} + v^n}{2} \right), \left(\frac{v^{n+1} - v^n}{\tau} \right) \rangle \\
&= \frac{1}{2\tau} \langle -\delta_{x_i}^{(\alpha_i)} \left(v^{n+1} + v^n \right), \left(v^{n+1} - v^n \right) \rangle \\
&= \frac{1}{2\tau} \langle \Lambda_{x_i}^{(\alpha_i)} \left(v^{n+1} + v^n \right), \Lambda_{x_i}^{(\alpha_i)} \left(v^{n+1} - v^n \right) \rangle \\
&= \frac{1}{2\tau} \left[\langle \Lambda_{x_i}^{(\alpha_i)} v^{n+1}, \Lambda_{x_i}^{(\alpha_i)} v^{n+1} \rangle - \langle \Lambda_{x_i}^{(\alpha_i)} v^{n+1}, \Lambda_{x_i}^{(\alpha_i)} v^n \rangle \right. \\
&\quad \left. + \langle \Lambda_{x_i}^{(\alpha_i)} v^n, \Lambda_{x_i}^{(\alpha_i)} v^{n+1} \rangle - \langle \Lambda_{x_i}^{(\alpha_i)} v^n, \Lambda_{x_i}^{(\alpha_i)} v^n \rangle \right] \\
&= \frac{1}{2\tau} \left[\langle \Lambda_{x_i}^{(\alpha_i)} v^{n+1}, \Lambda_{x_i}^{(\alpha_i)} v^{n+1} \rangle - \langle \Lambda_{x_i}^{(\alpha_i)} v^n, \Lambda_{x_i}^{(\alpha_i)} v^n \rangle \right] \\
&= \frac{1}{2\tau} \left[\| \Lambda_{x_i}^{(\alpha_i)} v^{n+1} \|_2^2 - \| \Lambda_{x_i}^{(\alpha_i)} v^n \|_2^2 \right], \quad \forall i \in I_p,
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
\langle \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle &= \langle \frac{G(v^{n+1}) - G(v^n)}{v^{n+1} - v^n}, \frac{v^{n+1} - v^n}{\tau} \rangle = h_* \sum_{j \in I} \frac{G(v_j^{n+1}) - G(v_j^n)}{v_j^{n+1} - v_j^n} \frac{v_j^{n+1} - v_j^n}{\tau} \\
&= h_* \sum_{j \in I} \frac{G(v_j^{n+1}) - G(v_j^n)}{\tau} = \frac{1}{\tau} \left[h_* \sum_{j \in I} G(v_j^{n+1}) - h_* \sum_{j \in I} G(v_j^n) \right] \\
&= \frac{1}{\tau} \left[\| G(v^{n+1}) \|_1 - \| G(v^n) \|_1 \right].
\end{aligned} \tag{2.50}
$$

Let Θ_j^n represent the left-hand side of the difference equations in (2.33) for each $j \in J$ and each $n \in I_{N-2}$, and let $\Theta^n = (\Theta_j^n)_{j \in J}$. Suppose that $(v^n)_{n=0}^N$ is a solution of (2.33). Calculating the inner product of Θ^n with $\delta_t^{(1)} v^n$, using the identities above and collecting terms, we note that

$$
\begin{aligned}
0 &= \langle \Theta^n, \delta_t^{(1)} v^n \rangle = \langle \mu_t \delta_t^{(2)} v^n - \mu_t \delta_{x_i}^{(\alpha_i)} v^n + \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle \\
&= \langle \mu_t \delta_t^{(2)} v^n, \delta_t^{(1)} v^n \rangle + \langle -\mu_t \delta_{x_i}^{(\alpha_i)} v^n, \delta_t^{(1)} v^n \rangle + \langle \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle \\
&= \frac{1}{2\tau} \left[\langle \delta_t^{(1)} v^{n+1}, \delta_t^{(1)} v^n \rangle - \langle \delta_t^{(1)} v^n, \delta_t^{(1)} v^{n-1} \rangle \right] + \frac{1}{2\tau} \sum_{i=1}^{p} \left[\| \Lambda_{x_i}^{(\alpha_i)} v^{n+1} \|_2^2 - \| \Lambda_{x_i}^{(\alpha_i)} v^n \|_2^2 \right] \\
&+ \frac{1}{\tau} \left[\| G(v^{n+1}) \|_1 - \| G(v^n) \|_1 \right] + \gamma \left\| \delta_t^{(1)} v^n \right\|_2^2 \\
&= \delta_t E^n + \gamma \| \delta_t^{(1)} v^n \|_2^2, \quad \forall n \in I_{N-2},
\end{aligned} \tag{2.51}
$$

whence the conclusion of this result is obtained. □

Corollary 2.10. *If* $(v^n)_{n=0}^N$ *is a solution of* (2.33) *then*

$$
E^n = E^1 - \gamma \tau \sum_{k=1}^{n-1} \| \delta_t^{(1)} v^k \|_2^2, \quad \forall n \in I_{N-2}. \tag{2.52}
$$

In particular, the quantities E^n *are invariants of* (2.33) *when* $\gamma = 0$.

Proof. It readily follows from Theorem 2.9. □

Theorem 2.9 and Corollary 2.10 are clearly the discrete counterparts of Theorem 2.2 and Corollary 2.3, respectively, and they indicate that our method is a dissipation-preserving technique. Moreover, it is important to point out that the energy quantity E^n defined in Theorem 2.9 has associated the following discrete energy density functions:

$$\begin{aligned} H_j^n &= \frac{1}{2}\left(\delta_t^{(1)} v_j^n\right)\left(\delta_t^{(1)} v_j^{n-1}\right) - \frac{1}{2} v_j^n \sum_{i=1}^{p} \delta_{x_i}^{(\alpha_i)} v_j^n + G(v_j^n) \\ &= \frac{1}{2}\left(\delta_t^{(1)} v_j^n\right)\left(\delta_t^{(1)} v_j^{n-1}\right) + \frac{1}{2} \sum_{i=1}^{p} \left|\Lambda_{x_i}^{(\alpha_i)} v_j^n\right|^2 + G(v_j^n), \quad \forall (j,n) \in J \times I_{N-2}. \end{aligned} \tag{2.53}$$

Theorem 2.11. *The discrete quantities* (2.47) *may be rewritten alternatively as*

$$E^n = \frac{1}{2}\mu_t \|\delta_t^{(1)} v^{n-1}\|_2^2 - \frac{\tau^2}{4}\|\delta_t^{(2)} v^n\|_2^2 + \frac{1}{2}\sum_{i=1}^{p} \|\Lambda_{x_i}^{(\alpha_i)} v^n\|_2^2 + \|G(v^n)\|_1, \quad \forall n \in I_{N-2}. \tag{2.54}$$

Proof. Note that

$$\begin{aligned} \langle \delta_t v^n, \delta_t v^{n-1}\rangle &= \|\delta_t^{(1)} v^n\|_2^2 + \langle \frac{v^{n+1}-v^n}{\tau}, \frac{v^n - v^{n-1}}{\tau}\rangle - \langle \frac{v^{n+1}-v^n}{\tau}, \frac{v^{n+1}-v^n}{\tau}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 - \frac{1}{\tau^2}\left[\langle v^{n+1}-v^n, v^{n+1}-v^n\rangle - \langle v^{n+1}-v^n, v^n - v^{n-1}\rangle\right] \\ &= \|\delta_t^{(1)} v^n\|_2^2 - \frac{1}{\tau^2}\langle v^{n+1}-v^n, v^{n+1} - 2v^n + v^{n-1}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 - \frac{1}{\tau^2}\langle v^{n+1}-v^n, v^{n+1} - 2v^n + v^{n-1}\rangle \\ &\quad + \frac{1}{\tau^2}\langle v^{n+1} - 2v^n + v^{n-1}, v^{n+1} - 2v^n + v^{n-1}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 - \frac{1}{\tau^2}\langle v^n - v^{n-1}, v^{n+1} - 2v^n + v^{n-1}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 + \|\delta_t^{(1)} v^{n-1}\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 \\ &\quad - \frac{1}{\tau^2}\langle v^n - v^{n-1}, v^{n+1} - 2v^n + v^{n-1}\rangle - \langle \frac{v^n - v^{n-1}}{\tau}, \frac{v^n - v^{n-1}}{\tau}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 + \|\delta_t^{(1)} v^{n-1}\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 \\ &\quad - \frac{1}{\tau^2}\left[\langle v^n - v^{n-1}, v^{n+1} - 2v^n + v^{n-1}\rangle + \langle v^n - v^{n-1}, v^n - v^{n-1}\rangle\right] \\ &= \|\delta_t^{(1)} v^n\|_2^2 + \|\delta_t^{(1)} v^{n-1}\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 - \langle \frac{v^n - v^{n-1}}{\tau}, \frac{v^{n+1}-v^n}{\tau}\rangle \\ &= \|\delta_t^{(1)} v^n\|_2^2 + \|\delta_t^{(1)} v^{n-1}\|_2^2 - \tau^2\|\delta_t^{(2)} v^n\|_2^2 - \langle \delta_t v^{n-1}, \delta_t v^n\rangle, \end{aligned} \tag{2.55}$$

holds for each $n \in I_{N-1}$. It follows that

$$\langle \delta_t v^n, \delta_t v^{n-1}\rangle = \mu_t \|\delta_t^{(1)} v^{n-1}\|_2^2 - \frac{\tau^2}{2}\|\delta_t^{(2)} v^n\|_2^2, \quad \forall n \in I_{N-1}, \tag{2.56}$$

whence the conclusion of the theorem is reached. □

2.5 Auxiliary lemmas

In this section, we prove some propositions needed to establish the properties of numerical efficiency of the finite-difference method (2.33). To start with, we will require the following elementary facts which will be employed in the sequel without an explicit reference:

(A) If v and w are real vectors of the same dimension then $|2\langle v, w\rangle| \le \|v\|_2^2 + \|w\|_2^2$.

(B) As a consequence, $\|v+w\|_2^2 \leq 2\|v\|_2^2 + 2\|w\|_2^2$ for any two real vectors v and w of the same dimension.

(C) More generally, if $k \in \mathbb{N}$ and $v_1, \ldots, v_k$ are real vectors of the same dimension then

$$\left\| \sum_{n=1}^{k} v_n \right\|_2^2 \leq k \sum_{n=1}^{k} \|v_n\|_2^2. \tag{2.57}$$

(D) If $(v^n)_{n=0}^N$ is a finite sequence in $\mathcal{V}_h$ and $n \in I_N$ then $v^n = v^0 + \tau \sum_{k=0}^{n-1} \delta_t^{(1)} v^k$. It follows that

$$\|v^n\|_2^2 \leq 2\|v^0\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t^{(1)} v^k\|_2^2, \quad \forall n \in I_N. \tag{2.58}$$

The following lemma summarizes some important properties of the operators $\delta_{x_i}^{(\alpha_i)}$ introduced in Section 2.3 along with their respective square roots.

Lemma 2.12. *Let $v \in \mathcal{V}_h$ and $i \in I_p$.*

(a) $\|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2g_0^{(\alpha_i)} h_* h^{-\alpha_i} \|v\|_2^2$.

(b) $\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 = \|\Lambda_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v\|_2^2$.

(c) $\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2g_0^{(\alpha_i)} h_* h^{-\alpha_i} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 4\left(g_0^{(\alpha_i)} h_* h^{-\alpha_i}\right)^2 \|v\|_2^2$. *It follows then that*

$$\sum_{i=1}^{p} \|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2h_* \sum_{i=1}^{p} g_0^{(\alpha_i)} h^{-\alpha_i} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 4h_*^2 \|v\|_2^2 \sum_{i=1}^{p} (g_0^{(\alpha_i)} h^{-\alpha_i})^2. \tag{2.59}$$

Proof.

(a) The properties of the matrix $A_{x_i}^{(\alpha_i)}$ summarized in Lemma 2.7 guarantee that $v_{j|j_i}^\top A_{x_i}^{(\alpha_i)} v_{j|j_i} \leq 2g_0^{(\alpha_i)} \|v_{j|j_i}\|_2^2$ holds for each $j \in J$. Moreover, Lemma 2.8 yields

$$\begin{aligned}
\|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 &= \langle \Lambda_{x_i}^{(\alpha_i)} v, \Lambda_{x_i}^{(\alpha_i)} v \rangle = \langle v, -\delta_{x_i}^{(\alpha_i)} v \rangle \\
&= \frac{h_*}{h^{\alpha_i}} \sum_{j_1=1}^{M_1-1} \cdots \sum_{j_{i-1}=1}^{M_{i-1}-1} \sum_{j_{i+1}=1}^{M_{i+1}-1} \cdots \sum_{j_p=1}^{M_p-1} v_{j|j_i}^\top A_{x_i}^{(\alpha_i)} v_{j|j_i} \\
&\leq 2g_0^{(\alpha_i)} \frac{h_*}{h^{\alpha_i}} \sum_{j_1=1}^{M_1-1} \cdots \sum_{j_{i-1}=1}^{M_{i-1}-1} \sum_{j_{i+1}=1}^{M_{i+1}-1} \cdots \sum_{j_p=1}^{M_p-1} \|v_{j|j_i}\|_2^2 \\
&= 2g_0^{(\alpha_i)} h_* h^{-\alpha_i} \|v\|_2^2.
\end{aligned} \tag{2.60}$$

(b) Using Lemma 2.8 we readily check that

$$\begin{aligned}
\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 &= \langle -\delta_{x_i}^{(\alpha_i)} v, -\delta_{x_i}^{(\alpha_i)} v \rangle = \langle \Lambda_{x_i}^{(\alpha_i)} v, -\delta_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v \rangle \\
&= \langle \Lambda_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v, \Lambda_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v \rangle = \|\Lambda_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v\|_2^2.
\end{aligned} \tag{2.61}$$

(c) This property is a consequence of (a) and (b), it is easy to verify

$$\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 = \|\Lambda_{x_i}^{(\alpha_i)}\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2g_0^{(\alpha_i)} h_* h^{-\alpha_i} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq \left(2g_0^{(\alpha_i)} h_* h^{-\alpha_i}\right)^2 \|v\|_2^2. \tag{2.62}$$

We get the desired result by taking the sum over i. □

For the remainder of this chapter, we let $\alpha = (\alpha_1, \ldots, \alpha_p)$, and define the constant $g_h^{(\alpha)} = 2h_* \max\{g_0^{(\alpha_i)} h^{-\alpha_i} : i \in I_p\}$. In light of the last lemma, it is clear that $g_h^{(\alpha)}$ is a positive number such that

$$\sum_{i=1}^{p} \|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq g_h^{(\alpha)} \sum_{i=1}^{p} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq \left(g_h^{(\alpha)} \|v\|_2\right)^2. \tag{2.63}$$

Lemma 2.13. *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that $(u^n)_{n=0}^N$, $(v^n)_{n=0}^N$ and $(R^n)_{n=0}^N$ are sequences in $\mathcal{V}_h$. Let $\varepsilon^n = v^n - u^n$ and $\widetilde{G}^n = \delta_{v,t} G(v^n) - \delta_{w,t} G(w^n)$ for each $n \in \overline{I}_{N-1}$. Then the following are satisfied.*

(a) *There exists a constant $C_0 \in \mathbb{R}^+$ that depends only on G such that*

$$\|\widetilde{G}^n\|_2^2 \leq C_0(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2), \quad \forall n \in \overline{I}_{N-1}. \tag{2.64}$$

(b) *There exists $C_1 \in \mathbb{R}^+$ depending only on G such that*

$$2|\langle R^n - \widetilde{G}^n, \delta_t^{(1)} \varepsilon^n\rangle| \leq 2\|R^n\|_2^2 + C_1\left(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2\right), \quad \forall n \in \overline{I}_{N-1}. \tag{2.65}$$

(c) *There exist $C_2, C_3 \in \mathbb{R}^+$ that depend only on G such that for each $k \in I_{N-1}$,*

$$2\tau \sum_{n=1}^{k} \left|\langle R^n - \widetilde{G}^n, \delta_t^{(1)} \varepsilon^n\rangle\right| \leq 2\tau \sum_{n=0}^{k} \|R^n\|_2^2 + C_2 \|\varepsilon^0\|_2^2 + C_3 \tau \sum_{n=0}^{k} \|\delta_t^{(1)} \varepsilon^n\|_2^2. \tag{2.66}$$

(d) *For each $k \in I_{N-1}$,*

$$k\tau^2 \sum_{n=1}^{k} \|\widetilde{G}^n\|_2^2 \leq 4C_0 T^2 \|\varepsilon^0\|_2^2 + 4C_0 T^3 \tau \sum_{n=0}^{k} \|\delta_t^{(1)}\varepsilon^n\|_2^2. \tag{2.67}$$

Proof. Let $C_0' = \sup\{|G''(u)| : u \in \mathbb{R}\}$.

(a) As a consequence of the Mean Value Theorem and a direct integration we obtain that $|\widetilde{G}_j^n| \leq C_0'(|\varepsilon_j^{n+1}| + |\varepsilon_j^n|)$ for each $j \in J$ and each $n \in \overline{I}_{N-1}$. Raising both sides of this inequality to the second power and using the inequalities at the beginning of this section we get

$$\|\widetilde{G}^n\|_2^2 \leq 2C_0'(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2), \tag{2.68}$$

the result is reached with $C_0 = 2C_0'$.

(b) Note that for each $n \in \overline{I}_{N-1}$,

$$\begin{aligned} 2|\langle R^n - \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle| &\leq \|R^n - \widetilde{G}^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2 \\ &\leq 2\|R^n\|_2^2 + 2\|\widetilde{G}^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2 \\ &\leq 2\|R^n\|_2^2 + 2C_0\left(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2\right) + \|\delta_t^{(1)}\varepsilon^n\|_2^2, \end{aligned} \tag{2.69}$$

then we choose $C_1 = \max\{4C_0', 1\}$

$$2|\langle R^n - \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle| \leq 2\|R^n\|_2^2 + C_1\left(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2\right). \tag{2.70}$$

(c) Using the inequality (2.65) and the remarks at the beginning of the present section we obtain that

$$\begin{aligned} &2\tau\sum_{n=1}^{k}\left|\langle R^n - \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle\right| \\ &\leq 2\tau\sum_{n=1}^{k}\|R^n\|_2^2 + C_1\tau\sum_{n=1}^{k+1}\left[\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2\right] \\ &\leq 2\tau\sum_{n=1}^{k}\|R^n\|_2^2 + 2C_1\tau\left[\sum_{n=1}^{k+1}\|\varepsilon^n\|_2^2 + \sum_{n=1}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\right] \\ &\leq 2\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 2C_1\tau\left[\sum_{n=1}^{k+1}\left(2\|\varepsilon^0\|_2^2 + 2T\tau\sum_{l=0}^{n-1}\|\delta_t^{(1)}\varepsilon^l\|_2^2\right) + \sum_{n=1}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\right] \\ &\leq 2\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 2C_1\tau\left[2(k+1)\|\varepsilon^0\|_2^2 + 2T\tau k\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2 + \sum_{n=1}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\right] \\ &\leq 2\tau\sum_{n=0}^{k}\|R^n\|_2^2 + 4C_1T\|\varepsilon^0\|_2^2 + 2C_1\tau(2T^2+1)\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2, \end{aligned} \tag{2.71}$$

for each $k \in I_{N-1}$. The conclusion of this result follows for $C_2 = 4C_1T$ and $C_3 = 2C_1(2T^2+1)$.

(d) Note that (2.64) and the remarks at the beginning of this section imply that for each $k \in I_{N-1}$,

$$\begin{aligned} k\tau^2\sum_{n=1}^{k}\|\widetilde{G}^n\|_2^2 &\leq C_0k\tau^2\sum_{n=1}^{k+1}\left(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2\right) \leq 2C_0k\tau^2\sum_{n=1}^{k+1}\|\varepsilon^n\|_2^2 \leq 2C_0T\tau\sum_{n=1}^{k+1}\|\varepsilon^n\|_2^2 \\ &\leq 2C_0T\tau\sum_{n=1}^{k+1}\left(2\|\varepsilon^0\|_2^2 + 2T\tau\sum_{l=0}^{n-1}\|\delta_t^{(1)}\varepsilon^l\|_2^2\right) \\ &\leq 4C_0T^2\|\varepsilon^0\|_2^2 + 4C_0T^3\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2. \end{aligned} \tag{2.72}$$

Which is the stated in the Lemma. □

Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that $(u^n)_{n=0}^N$, $(v^n)_{n=0}^N$ and $(R^n)_{n=0}^N$ are sequences in $\mathcal{V}_h$. As in our last result, let $\varepsilon^n = v^n - u^n$ and $\widetilde{G}^n = \delta_{v,t}G(v^n) - \delta_{w,t}G(w^n)$ for each $n \in \overline{I}_{N-1}$.

Suppose also that

$$\mu_t\delta_t^2\varepsilon^n - \sum_{i=1}^{p}\mu_t\delta_{x_i}^{(\alpha_i)}\varepsilon^n + \gamma\delta_t^{(1)}\varepsilon^n + \widetilde{G}^n = R^n, \quad \forall n \in I_{N-1}. \tag{2.73}$$

Using this identity it follows that

$$\begin{aligned} \delta_t^{(2)}\varepsilon^{k+1} &= -\delta_t^{(2)}\varepsilon^k + \textstyle\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1} + \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^k - 2\left[\gamma\delta_t^{(1)}\varepsilon^k + \widetilde{G}^k - R^k\right], \\ \delta_t^{(2)}\varepsilon^k &= -\delta_t^{(2)}\varepsilon^{k-1} + \textstyle\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^k + \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k-1} - 2\left[\gamma\delta_t^{(1)}\varepsilon^{k-1} + \widetilde{G}^{k-1} - R^{k-1}\right]. \end{aligned} \tag{2.74}$$

Substituting $\delta_t^{(2)}\varepsilon^k$ from the second equation into the first one:

$$\begin{aligned} \delta_t^{(2)}\varepsilon^{k+1} &= -\delta_t^{(2)}\varepsilon^{k-1} + \textstyle\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1} - \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k-1} \\ &\quad -2\left[\gamma\left(\delta_t^{(1)}\varepsilon^k - \delta_t^{(1)}\varepsilon^{k-1}\right) + \left(\widetilde{G}^k - \widetilde{G}^{k-1}\right) - \left(R^k - R^{k-1}\right)\right], \end{aligned} \tag{2.75}$$

with this idea, we apply mathematical induction:

$$\begin{aligned} \delta_t^{(2)}\varepsilon^{k+1} = (-1)^k\delta_t^{(2)}\varepsilon^1 + \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1} + (-1)^{k+1}\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^1 \\ + 2\sum_{n=1}^k(-1)^n\left[\gamma\delta_t^{(1)}\varepsilon^n + \widetilde{G}^n - R^n\right], \quad \forall k \in I_{N-2}. \end{aligned} \tag{2.76}$$

Moreover, calculating the square of the Euclidean norm of $\delta_t^{(2)}\varepsilon^{k+1}$, multiplying by τ^2 and simplifying

$$\begin{aligned} \tau^2\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 &= \tau^2\left\|\delta_t^{(2)}\varepsilon^1 + \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1} + \sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^1 + 2\sum_{n=1}^k\left[\gamma\delta_t^{(1)}\varepsilon^n + \widetilde{G}^n - R^n\right]\right\|_2^2 \\ &\leq 5\tau^2\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 5\tau^2\left\|\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\right\|_2^2 + 5\tau^2\left\|\sum_{i=1}^p \delta_{x_i}^{(\alpha_i)}\varepsilon^1\right\|_2^2 \\ &\quad +5(2\gamma\tau)^2\left\|\sum_{n=1}^k \delta_t^{(1)}\varepsilon^n\right\|_2^2 + 5(2\tau)^2\left\|\sum_{n=1}^k \widetilde{G}^n - R^n\right\|_2^2 \\ &\leq 5\tau^2\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 5\tau^2 p\sum_{i=1}^p\left\|\delta_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\right\|_2^2 + 5\tau^2 p\sum_{i=1}^p\left\|\delta_{x_i}^{(\alpha_i)}\varepsilon^1\right\|_2^2 \\ &\quad +20\gamma^2\tau^2 k\sum_{n=1}^k\left\|\delta_t^{(1)}\varepsilon^n\right\|_2^2 + 20\tau^2\left[2k\sum_{n=1}^k\left\|\widetilde{G}^n\right\|_2^2 + 2k\sum_{n=1}^k\|R^n\|_2^2\right]. \end{aligned} \tag{2.77}$$

Using (2.63) we get

$$\begin{aligned} \tau^2\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 &\leq 5\tau^2\|\delta_t^{(2)}\varepsilon^1\|_2^2 + 5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2 + 5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\ &\quad +20\gamma^2 T\tau\sum_{n=1}^k\|\delta_t^{(1)}\varepsilon^n\|_2^2 + 40k\tau^2\sum_{n=1}^k\left(\|\widetilde{G}^n\|_2^2 + \|R^n\|_2^2\right). \end{aligned} \tag{2.78}$$

Then the inequalities at the beginning of this section, applying Lemma 2.13 (a), we readily obtain that

$$
\begin{aligned}
&\tau^2\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 \\
&\quad\leq 5\tau^2\left\|\frac{\delta_t^{(1)}\varepsilon^1-\delta_t^{(1)}\varepsilon^0}{\tau}\right\|_2^2+5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\
&\quad+20\gamma^2 T\tau\sum_{n=1}^k\|\delta_t^{(1)}\varepsilon^n\|_2^2+40T\tau\sum_{n=1}^k\|R^n\|_2^2+40\left(4C_0T^2\|\varepsilon^0\|_2^2+4C_0T^3\tau\sum_{n=0}^k\|\delta_t^{(1)}\varepsilon^n\|_2^2\right) \\
&\quad\leq 160C_0T^2\|\varepsilon^0\|_2^2+20\mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2+5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+5p\tau^2 g_h^{(\alpha)}\sum_{i=1}^p\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\
&\quad+40T\tau\sum_{n=1}^k\|R^n\|_2^2+20(8C_0T^2+\gamma^2)T\tau\sum_{n=0}^k\|\delta_t^{(1)}\varepsilon^n\|_2^2.
\end{aligned}
\tag{2.79}
$$

This inequality will be used in the following section to establish the stability and the convergence of the finite-difference method (2.33).

The following result will be useful to prove the stability and convergence properties of (2.33). It is obviously a discrete version of the well-known Gronwall inequality.

Lemma 2.14 (Pen-Yu [21])**.** *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative mesh functions, and suppose that there exists $C\geq 0$ such that*

$$\omega^k\leq\rho^k+C\tau\sum_{n=0}^{k-1}\omega^k,\quad\forall k\in I_{N-1}. \tag{2.80}$$

Then $\omega^n\leq\rho^n e^{Cn\tau}$ for each $n\in\overline{I}_N$. $\square$

2.6 Numerical results

The main numerical properties of the finite-difference method (2.33) as well as some illustrative computational simulations are presented in this stage. Here we show that our scheme is a consistent, stable and convergent technique under suitable conditions on the parameters of the model. In a first stage, we show that (2.33) is a second-order consistent technique, and that the discrete energy density (2.53) also provides a consistent approximation to the continuous Hamiltonian (2.8). For practical purposes we define the following continuous and discrete functionals:

$$\mathcal{L}u(x,t)=\frac{\partial^2 u}{\partial t^2}(x,t)-\sum_{i=1}^p\frac{\partial^{\alpha_i}u}{\partial|x_i|^{\alpha_i}}(x,t)+\gamma\frac{\partial u}{\partial t}(x,t)+G'(u(x,t)),\quad\forall(x,t)\in\Omega, \tag{2.81}$$

$$Lu_j^n=\mu_t\delta_t^{(2)}u_j^n-\sum_{i=1}^p\mu_t\delta_{x_i}^{(\alpha_i)}u_j^n+\gamma\delta_t^{(1)}u_j^n+\delta_{u,t}^{(1)}G(u_j^n),\quad\forall(j,n)\in J\times I_{N-2}. \tag{2.82}$$

Lemma 2.15 (Brouwer's fixed-point theorem)**.** *Let $\mathcal{V}_{\mathbb{R}}$ be a finite-dimensional vector space, and $\langle\cdot,\cdot\rangle$ an inner product on $\mathcal{V}$. Suppose that $f:\mathcal{V}_{\mathbb{R}}\to\mathcal{V}_{\mathbb{R}}$ is continuous, and that there is $\lambda>0$ such that $\langle f(w),w\rangle\geq 0$, for each $w\in\mathcal{V}$ with $\|w\|=\lambda$. There exists $w\in\mathcal{V}$ with $\|w\|\leq\lambda$ satisfying $f(w)=0$.*

In the following, for each $w \in \mathcal{V}_h$ and $j \in J$, we define

$$\delta^{(1)}_{w,v,t} G^n_j(w) = \begin{cases} \dfrac{2\left[G(\frac{1}{2}(w_j + v^n_j)) - G(\mu_t v^{n-1}_j)\right]}{w_j - v^{n-1}_j}, & \text{if } w_j \neq \mu_t v^{n-1}_j, \\ G'(\mu_t v^{n-1}_j), & \text{if } w_j = \mu_t v^{n-1}_j. \end{cases}$$

Define the vector

$$\delta^{(1)}_{w,v,t} G^n(w) = (\delta^{(1)}_{w,v,t} G^n_j(w))_{j\in J}.$$

Note that $\delta^{(1)}_{w,v,t} G^n(w)$ is a continuous operator on $\mathcal{V}_h$.

Theorem 2.16 (Solubility)**.** *If $G' \in L^\infty(\mathbb{R})$ then the method is solvable for any set of initial conditions.*

Proof. There exists $K > 0$ such that $\|\delta^{(1)}_{w,v,t} G^n(w)\|_2 \leq K$, for any $w \in \mathcal{V}$ and $n \in I_{N-1}$. Let $f : \mathcal{V}_h \to \mathcal{V}_h$ be the continuous function whose jth component is given by

$$f_j(w) = \frac{1}{\tau^2}(w_j - 2v^n_j + v^{n-1}_j) - \sum_{i\in I_p} \delta^{(\alpha)}_{x_i} v^n_j + \frac{\gamma}{2\tau}(w_j - v^{n-1}_j) + \delta^{(1)}_{w,v,t} G^n_j.$$

Using Cauchy–Schwarz and the lemmas, we obtain

$$\begin{aligned} \langle f(w), w\rangle \quad &\geq \quad \frac{1}{\tau^2}\|w\|_2 \left(\|w\|_2 - 2\|v^n\|_2 - \|v^{n-1}\|_2\right) + \sum_{i\in I_p} \langle w, -\delta^{(\alpha)}_{x_i} v^n\rangle \\ &\qquad + \frac{\gamma}{2\tau}\|w\|_2 \left(\|w\|_2 - \|v^{n-1}\|_2\right) - K\|w\|_2 \\ &\geq \quad \frac{1}{2\tau^2}\|w\|_2 \left[(2+\gamma\tau)\|w\| - 4\|v^n\| - (2+\gamma\tau)\|v^{n-1}\| - 2\tau^2 K\right] \\ &\qquad - \sum_{i\in I_p} \|\Lambda^{(\alpha)}_{x_i} w\|_2 \|\delta_{x_i} v^n\|_2 \\ &\geq \quad \frac{2+\gamma\tau}{2\tau^2}\|w\|_2 \left(\|w\|_2 - \lambda\right). \end{aligned}$$

Here,

$$\lambda = \|v^{n-1}\|_2 + \frac{(4 + g^{(\alpha)}_h)\|v^n\|_2 + 2\tau^2 K}{2+\gamma\tau}.$$

Clearly $\lambda > 0$, and $\langle f(w), w\rangle \geq 0$ for each $w \in \mathcal{V}_h$ with $\|w\|_2 = \lambda$. The conclusion follows from Brouwer's fixed-point theorem. □

Theorem 2.17 (Consistency)**.** *If $u \in \mathcal{C}^5(\overline{\Omega})$ then there exist constants $C, C' > 0$ which are independent of h and τ such that for each $j \in J$ and each $n \in I_{N-2}$,*

$$\left|Lu^n_j - \mathcal{L}u(x_j, t_n)\right| \quad \leq \quad C(\tau^2 + \|h\|_2^2), \tag{2.83}$$

$$\left|Hu^n_j - \mathcal{H}u(x_j, t_n)\right| \quad \leq \quad C'(\tau + \|h\|_2^2). \tag{2.84}$$

Proof. We employ here the usual arguments with Taylor polynomials and the identity (2.31). Using the hypotheses of continuous differentiability, there exist constants $C_1, C_{2,i}, C_3, C_4 \in \mathbb{R}$ for $i \in I_p$ such that

$$\begin{aligned}
\left|\mu_t \delta_t^{(2)} u_j^n - \frac{\partial^2 u}{\partial t^2}(x_j, t_{n+\frac{1}{2}})\right| &\leq C_1 \tau^2, && (2.85)\\
\left|\mu_t \delta_{x_i}^{(\alpha_i)} u_j^n - \frac{\partial^{\alpha_i} u}{\partial |x|^{\alpha_i}}(x_j, t_{n+\frac{1}{2}})\right| &\leq C_{2,i}(\tau^2 + h_i^2), && (2.86)\\
\left|\delta_t^{(1)} u_j^n - \frac{\partial u}{\partial t}(x_j, t_{n+\frac{1}{2}})\right| &\leq C_3 \tau^2, && (2.87)\\
\left|\delta_{u,t}^{(1)} G(u_j^n) - G'(u(x_j, t_{n+\frac{1}{2}}))\right| &\leq C_4 \tau^2, && (2.88)
\end{aligned}$$

for each $j \in J$ and each $n \in I_{N-2}$. The first inequality in the conclusion of this theorem is readily reached using the triangle inequality and defining $C = \max\{C_1, \gamma C_3, C_4\} \vee \max\{C_{2,i} : i \in I_p\}$. To establish the second inequality, note that the consistency of the forward-difference operators, the Mean Value Theorem and the smoothness of the function u guarantee that there exists a constant C_5 independent of τ such that

$$\begin{aligned}
\left|\delta_t^{(1)} u_j^n \delta_t^{(1)} u_j^{n-1} - \left(\frac{\partial u}{\partial t}(x_j, t_n)\right)^2\right| &\leq \left|\delta_t^{(1)} u_j^{n-1}\right| \left|\delta_t^{(1)} u_j^n - \frac{\partial u}{\partial t}(x_j, t_n)\right| \\
&+ \left|\frac{\partial u}{\partial t}(x_j, t_n)\right| \left|\delta_t^{(1)} u_j^{n-1} - \frac{\partial u}{\partial t}(x_j, t_n)\right| \leq C_5 \tau, \quad (2.89)
\end{aligned}$$

for each $j \in J$ and each $n \in I_{N-1}$. Likewise, there exist constants $C_{6,i}$ for each $i \in I_p$ such that

$$\left|u_j^n \delta_{x_i}^{(\alpha_i)} u_j^n - u(x_j, t_n) \frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x_j, t_n)\right| \leq C_{6,i} h_i^2, \tag{2.90}$$

for each $j \in I$ and each $n \in I_{N-1}$. The second inequality of the conclusion follows again using the triangle inequality and letting $C' = \frac{1}{2}(C_5 \vee \max\{C_{6,i} : i \in I_p\})$. □

We turn our attention to the stability and the convergence properties of (2.33). In the following, the constants C_1, C_2 and C_3 are as in Lemma 2.13, and (ϕ_v, ψ_v) and (ϕ_w, ψ_w) will denote two sets of initial conditions of (2.1).

Theorem 2.18 (Stability)**.** *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that τ and h satisfy*

$$\frac{5}{2} p \tau^2 g_h^{(\alpha)} < 1. \tag{2.91}$$

Let $\mathbf{v} = (v^n)_{n=0}^N$ and $\mathbf{w} = (w^n)_{n=0}^N$ be solutions of (2.33) for (ϕ_v, ψ_v) and (ϕ_w, ψ_w), respectively, and let $\varepsilon^n = v^n - w^n$ for each $n \in \overline{I}_N$. Then there exist constants $C_4, C_5 \in \mathbb{R}^+$ and $0 < \eta_0 < 1$ independent of $\mathbf{v}$ and $\mathbf{w}$ such that

$$\frac{1}{2}\|\delta_t^{(2)} \varepsilon^n\|_2^2 + (1-\eta_0) \sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)} \varepsilon^n\|_2^2 \leq C_4 \left(\|\varepsilon^0\|_2^2 + \mu_t \|\delta_t^{(1)} \varepsilon^0\|_2^2 + \sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)} \varepsilon^1\|_2^2\right) e^{C_5 n \tau}, \quad \forall n \in I_{N-1}. \tag{2.92}$$

Proof. Let η_0 satisfy $\frac{5}{2} p \tau^2 g_h^{(\alpha)} < \eta_0 < 1$. Obviously, the sequence $(\varepsilon^n)_{n=0}^N$ satisfies the initial-boundary-value problem

$$\mu_t\delta_t^{(2)}\varepsilon_j^n - \sum_{i=1}^{p}\mu_t\delta_{x_i}^{(\alpha_i)}\varepsilon_j^n + \gamma\delta_t^{(1)}\varepsilon_j^n + \delta_{v,t}^{(1)}G(v_j^n) - \delta_{w,t}G(w_j^n) = 0, \quad \forall (j,n)\in J\times I_{N-2},$$
$$\text{such that } \begin{cases} \varepsilon_j^0 = \phi_v(x_j) - \phi_w(x_j), & \forall j\in J,\\ \delta_t\varepsilon_j^0 = \psi_v(x_j) - \psi_w(x_j), & \forall j\in J,\\ \varepsilon_j^n = 0, & \forall (j,n)\in \partial J\times I_N. \end{cases} \tag{2.93}$$

For the sake of convenience, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)}G(v_j^n) - \delta_{w,t}G(w_j^n)$ for each $j\in J$ and each $n\in \overline{I}_{N-1}$. From the identities preceding Theorem 2.9 and those after the proof of Corollary 2.10, we readily obtain that

$$\begin{aligned} \left\langle \mu_t\delta_t^{(2)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\right\rangle &= \frac{1}{2}\delta_t^{(1)}\mu_t\|\delta_t^{(1)}\varepsilon^{n-1}\|_2^2 - \frac{\tau^2}{4}\delta_t^{(1)}\|\delta_t^{(2)}\varepsilon^n\|_2^2,\\ \langle -\mu_t\delta_{x_i}^{(\alpha_i)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle &= \frac{1}{2}\delta_t^{(1)}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^n\|_2^2, \quad \forall i\in I_{N-1},\\ |2\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle| &\le C_1\left(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\delta_t^{(1)}\varepsilon^n\|_2^2\right), \end{aligned} \tag{2.94}$$

for each $n\in I_{N-1}$ and for some $C_1\in\mathbb{R}^+$. Let $k\in I_{N-1}$. Taking the inner product of $\delta_t^{(1)}\varepsilon^n$ with both sides of the respective difference equation of (2.93)

$$\left\langle \mu_t\delta_t^{(2)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\right\rangle + \sum_{i=1}^{p}\langle -\mu_t\delta_{x_i}^{(\alpha_i)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle + \gamma\left\langle \delta_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\right\rangle + \langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{2.95}$$

substituting the identities above

$$\frac{1}{2}\delta_t^{(1)}\mu_t\|\delta_t^{(1)}\varepsilon^{n-1}\|_2^2 - \frac{\tau^2}{4}\delta_t^{(1)}\|\delta_t^{(2)}\varepsilon^n\|_2^2 + \frac{1}{2}\sum_{i=1}^{p}\delta_t^{(1)}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^n\|_2^2 + \gamma\|\delta_t^{(1)}\varepsilon^n\|_2^2 + \langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{2.96}$$

multiplying by 2τ on both sides and substituting the definition of $\delta_t^{(1)}v^n$

$$\begin{aligned} 0 &= \left(\mu_t\|\delta_t^{(1)}\varepsilon^n\|_2^2 - \mu_t\|\delta_t^{(1)}\varepsilon^{n-1}\|_2^2\right) - \frac{\tau^2}{2}\left(\|\delta_t^{(2)}\varepsilon^{n+1}\|_2^2 - \|\delta_t^{(2)}\varepsilon^n\|_2^2\right)\\ &+ \sum_{i=1}^{p}\left(\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{n+1}\|_2^2 - \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^n\|_2^2\right) + 2\tau\gamma\|\delta_t^{(1)}\varepsilon^n\|_2^2 + 2\tau\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle, \end{aligned} \tag{2.97}$$

calculating then the sum of the resulting identity for all $n\in I_k$

$$\begin{aligned} 0 &= \left(\mu_t\|\delta_t^{(1)}\varepsilon^k\|_2^2 - \mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2\right) - \frac{\tau^2}{2}\left(\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 - \|\delta_t^{(2)}\varepsilon^0\|_2^2\right)\\ &+ \sum_{i=1}^{p}\left(\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 - \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2\right) + \sum_{n=1}^{k}\left[2\tau\gamma\|\delta_t^{(1)}\varepsilon^n\|_2^2 + 2\tau\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle\right], \end{aligned} \tag{2.98}$$

from the equation above we obtain

$$
\begin{aligned}
\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 &= -\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{k}\|_2^2 + \frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 - \frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^{0}\|_2^2 + \mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 \\
&\quad + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 - 2\tau\gamma\sum_{n=1}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2 - 2\tau\sum_{n=1}^{k}\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle \\
&\le \mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 + \frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 + 2\tau\sum_{n=1}^{k}\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle,
\end{aligned}
\tag{2.99}
$$

by definition of absolute value and applying Lemma 2.13 with $R^n = 0$ and simplifying algebraically yields

$$
\begin{aligned}
&\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\
&\quad \le \quad \mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 + \frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 + 2\tau\sum_{n=1}^{k}\left|\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle\right| \\
&\quad \le \quad \mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 + \Big(80C_0T^2\|\varepsilon^0\|_2^2 + 10\mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 \\
&\qquad + .\frac{5p}{2}\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 + \frac{5p}{2}\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\
&\qquad + 10(8C_0T^2+\gamma^2)T\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2) + \left(C_2\|\varepsilon^0\|_2^2 + C_3\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2\right) \\
&\quad \le \quad (80C_0T^2 + C_2)\|\varepsilon^0\|_2^2 + 11\mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 + (1+\eta_0)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2 \\
&\qquad + \frac{5p}{2}\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 + (C_3 + 10(8C_0T^2+\gamma^2)T)\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2 \\
&\quad \le \quad C_4\left(\|\varepsilon^0\|_2^2 + \mu_t\|\delta_t^{(1)}\varepsilon^{0}\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{1}\|_2^2\right) + \frac{5p}{2}\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \\
&\qquad + (C_3 + 10(8C_0T^2+\gamma^2)T)\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2 \\
&\quad = \quad \rho + \frac{5}{2}p\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 + \frac{C_5\tau}{2}\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2 \\
&\quad \le \quad \rho + \frac{5}{2}p\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 + C_5\tau\sum_{n=0}^{k}\left(\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{n}\|_2^2 + (1-\eta_0)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{n}\|_2^2\right) \\
&\quad \le \quad \rho + \frac{5}{2}p\tau^2 g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 + C_5\tau\sum_{n=0}^{k}\omega^n, \quad \forall k \in I_{N-1},
\end{aligned}
\tag{2.100}
$$

where

$$\begin{aligned}
C_4 &= \max\{C_2 + 80C_0T^2, 11, 1+\eta_0\}, && (2.101)\\
C_5 &= 2C_3 + 20(8C_0T^2 + \gamma^2)T, && (2.102)\\
\rho &= C_4\left(\|\varepsilon^0\|_2^2 + \mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2 + \sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2\right), && (2.103)\\
\omega^n &= \frac{1}{2}\|\delta_t^{(1)}\varepsilon^n\|_2^2 + (1-\eta_0)\sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^n\|_2^2, \quad \forall n \in I_{N-1}. && (2.104)
\end{aligned}$$

Subtracting the second term on the right-hand side of (2.100)

$$\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2 + \left(1 - \frac{5}{2}p\tau^2\right)\sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2 \quad\leq\quad \rho + C_5\tau\sum_{n=0}^k \omega^n, \quad \forall k \in I_{N-1}, \tag{2.105}$$

we note that the hypotheses of Lemma 2.14 are readily satisfied with $C = C_5$ and $\rho^k = \rho$ for each $k \in I_{N-1}$, whence the conclusion of Theorem 2.18 follows. □

Note that the inequality (2.91) is satisfied for sufficiently small values of τ and of the components of h. Finally, we tackle the problem of the convergence of the numerical method (2.33). The proof of the following result is similar to that of Theorem 2.18.

Theorem 2.19 (Convergence)**.** *Let $u \in \mathcal{C}^5(\overline{\Omega})$ be a solution of (2.1) with $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and let $(v^n)_{n=0}^N$ be a solution of (2.33) for the initial conditions (ϕ, ψ). Assume that $\epsilon^n = v^n - u^n$ for each $n \in \overline{I}_N$. If (2.91) holds then the method (2.33) is convergent of order $\mathcal{O}(\tau^2 + \|h\|^2)$.*

Proof. Let η_0 be as in the proof of Theorem 2.18, and let R_j^n be the truncation error at the point (x_j, t_n) for each $j \in J$ and each $n \in \overline{I}_N$. Then $(\epsilon^n)_{n=0}^N$ satisfies

$$\begin{gathered}
\mu_t\delta_t^{(2)}\epsilon_j^n - \sum_{i=1}^p \mu_t\delta_{x_i}^{(\alpha_i)}\epsilon_j^n + \gamma\delta_t^{(1)}\epsilon_j^n + \delta_{v,t}^{(1)}G(v_j^n) - \delta_{w,t}G(w_j^n) = R_j^n, \quad \forall (j,n) \in J \times I_{N-2},\\
\text{such that} \begin{cases} \epsilon_j^0 = \delta_t\epsilon_j^0 = 0, & \forall j \in J,\\ \epsilon_j^n = 0, & \forall (j,n) \in \partial J \times I_N. \end{cases}
\end{gathered} \tag{2.106}$$

Following the proof of Theorem 2.18, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)}G(v_j^n) - \delta_{w,t}G(w_j^n)$ for each $j \in J$ and each $n \in \overline{I}_{N-1}$. Proceeding as in the proof of that theorem

$$\begin{aligned}
&\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2 + \sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2\\
&\qquad \leq \mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2 + \sum_{i=1}^p \|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2 + \frac{\tau^2}{2}\|\delta_t^{(2)}\varepsilon^{k+1}\|_2^2 + 2\tau\sum_{n=1}^k \left|\langle \widetilde{G}^n - R^n, \delta_t^{(1)}\varepsilon^n\rangle\right|,
\end{aligned} \tag{2.107}$$

we readily obtain

$$
\begin{aligned}
&\frac{1}{2}\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2+\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2\\
&\quad\le \mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2+\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+\Big(80C_0T^2\|\varepsilon^0\|_2^2+10\mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2\\
&\qquad+.\frac{5p}{2}\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+\frac{5p}{2}\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2+20T\tau\sum_{n=1}^{k}\|R^n\|_2^2\\
&\qquad+10(8C_0T^2+\gamma^2)T\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2)+\left(2\tau\sum_{n=1}^{k}\|R^n\|_2^2+C_2\|\varepsilon^0\|_2^2+C_3\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\right)\\
&\quad\le(80C_0T^2+C_2)\|\varepsilon^0\|_2^2+11\mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2+(1+\eta_0)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+(20T+2)\tau\sum_{n=1}^{k}\|R^n\|_2^2\\
&\qquad+\frac{5p}{2}\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2+(C_3+10(8C_0T^2+\gamma^2)T)\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\\
&\quad\le C_4\left(\|\varepsilon^0\|_2^2+\mu_t\|\delta_t^{(1)}\varepsilon^0\|_2^2+\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^1\|_2^2+\tau\sum_{n=1}^{k}\|R^n\|_2^2\right)+\frac{5p}{2}\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2\\
&\qquad+(C_3+10(8C_0T^2+\gamma^2)T)\tau\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\\
&\quad=\rho^{k+1}+\frac{5}{2}p\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^{k+1}\|_2^2+\frac{C_5\tau}{2}\sum_{n=0}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2\\
&\quad\le\rho^{k+1}+\frac{5}{2}p\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^{k+1}\|_2^2+C_5\tau\left(\frac{1}{2}\|\delta_t^{(1)}\varepsilon^n\|_2^2+(1-\eta_0)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^n\|_2^2\right)\\
&\quad\le\rho^{k+1}+\frac{5}{2}p\tau^2g_h^{(\alpha)}\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^{k+1}\|_2^2+C_5\tau\sum_{n=0}^{k}\omega^n,\quad\forall k\in I_{N-1},
\end{aligned}
\tag{2.108}
$$

where C_5 is as before, and

$$
\begin{aligned}
C_4&=\max\{C_2+80C_0T^2,11,1+\eta_0,20T+2\}, &(2.109)\\
\rho^k&=C_4\left(\|\epsilon^0\|_2^2+\mu_t\|\delta_t^{(1)}\epsilon^0\|_2^2+\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^1\|_2^2+\tau\sum_{n=0}^{k-1}\|R^n\|_2^2\right),\quad\forall k\in I_{N-1}, &(2.110)\\
\omega^k&=\frac{1}{2}\|\delta_t^{(1)}\epsilon^k\|_2^2+(1-\eta_0)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^k\|_2^2,\quad\forall k\in I_{N-1}. &(2.111)
\end{aligned}
$$

Subtracting the second term of the right-hand side of (2.108)

$$
\|\delta_t^{(1)}\varepsilon^{k+1}\|_2^2+\left(1-\frac{5}{2}p\tau^2\right)\sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\varepsilon^{k+1}\|_2^2\quad\le\quad\rho^{k+1}+C_5\tau\sum_{n=0}^{k}\omega^n,\quad\forall k\in I_{N-1},\tag{2.112}
$$

then the hypotheses of Lemma 2.14 are satisfied. Using the conclusion of that result, the consistency property of our method and the homogeneous initial-boundary conditions of (2.106) we obtain that

$$\frac{1}{2}\|\delta_t^{(1)}\epsilon^k\|_2^2 + \sum_{i=1}^{p}\|\Lambda_{x_i}^{(\alpha_i)}\epsilon^k\|_2^2 \leq C_4 e^{C_5 k\tau}\tau\sum_{n=0}^{k-1}\|R^n\|_2^2 \leq C_6(\tau^2 + \|h\|_2^2)^2, \quad \forall k \in I_{N-1}. \tag{2.113}$$

Here $C_6 = C4C^2 e^{C_5 T} T$ and C is the constant of Theorem 2.17. The conclusion of the theorem readily follows from the last inequality. □

Finally, we provide some numerical approximations of the solution of problem (2.1) that show the capability of (2.33) to preserve the energy. The simulations were obtained using an implementation of our method in ©Matlab 8.5.0.197613 (R2015a) on a ©Sony Vaio PCG-5L1P laptop computer with Kubuntu 16.04 as operating system. In terms of computational times, we are aware that better results may be obtained with more modern equipment and more modest Linux/Unix distributions.

In a first stage, we consider undamped and damped one-dimensional forms of problem (2.1).

Example 2.20 (One-dimensional problem)*.* Let $0 < \omega < 1$. In this example, we let $G(u) = 1 - \cos u$ for all $u \in \mathbb{R}$, and use the exact solution of the classical sine-Gordon equation described by

$$\varphi(x,t) = 4\arctan\left(\frac{\sqrt{1-\omega^2}\cos\omega t}{\omega\cosh\sqrt{1-\omega^2}x}\right), \quad \forall (x,t) \in \mathbb{R} \times (\mathbb{R}^+ \cup \{0\}), \tag{2.114}$$

to prescribe the initial conditions. Computationally, we consider the domain $\Omega = (-30, 30) \times (0, 100)$, $h_1 = 0.5$ and $\tau = 0.05$. Figure 2.2 shows the numerical solution (left column) and the associated energy density (right column) of the problem (2.1) obtained using (2.33) and (2.53), respectively, for $\omega = 0.9$ and $\gamma = 0$. Various derivative orders were used, namely, $\alpha_1 = 2$ (top row), $\alpha_1 = 1.6$ (middle row) and $\alpha_1 = 1.2$ (bottom row). The insets of the graphs of the right column represent the discrete dynamics of the total energy (2.47) of the system. The results show that the discrete total energy is conserved, in agreement with the theory established in this chapter and numerical results obtained through an implicit nonlinear numerical method [51]. We have used different computational parameters and the results (not presented here in view of their redundancy) show that the discrete total energy is likewise conserved. This qualitative behavior is in agreement with Theorem 2.9. □

Example 2.21 (One-dimensional problem)*.* Consider now the same problem as in Example 2.20, but letting $\gamma = 0.05$. The results of the simulations are shown in Figure 2.3. Obviously, in this case the quantities E^n are not conserved in view of the presence of a nonzero damping term. These results are in qualitative agreement with Theorem 2.9 and with the numerical simulations obtained in [51]. □

We consider now the problem (2.1) in two spatial dimensions.

Example 2.22 (Two-dimensional problem)*.* Let $\Omega = (-5, 5) \times (-5, 5) \times (0, 10)$, and define $G(u) = 1 - \cos u$ for each $u \in \mathbb{R}$. Consider the two-dimensional form of (2.1) with the initial conditions obtained using by $\varphi(x^2 + y^2, t)$, where φ is defined by (2.114) with $\omega = 0.8$. Under these circumstances, Figure 2.4 shows snapshots of the approximate solution of (2.1) at the times (a) $t = 0.22$, (b) $t = 0.34$, (c) $t = 0.46$, (d) $t = 0.58$, (e) $t = 0.70$ and (f) $t = 0.82$. The model parameters employed in this example were $\alpha_1 = 1.8$, $\alpha_2 = 1.6$ and $\gamma = 0$. Numerically, we used the method (2.33) with $M_1 = M_2 = 100$ and $N = 500$. The solutions appear to follow almost a periodic behavior. Moreover, Figure 2.5 shows the dynamics of the energy for various values of α_1 and α_2, and different damping coefficients, namely,

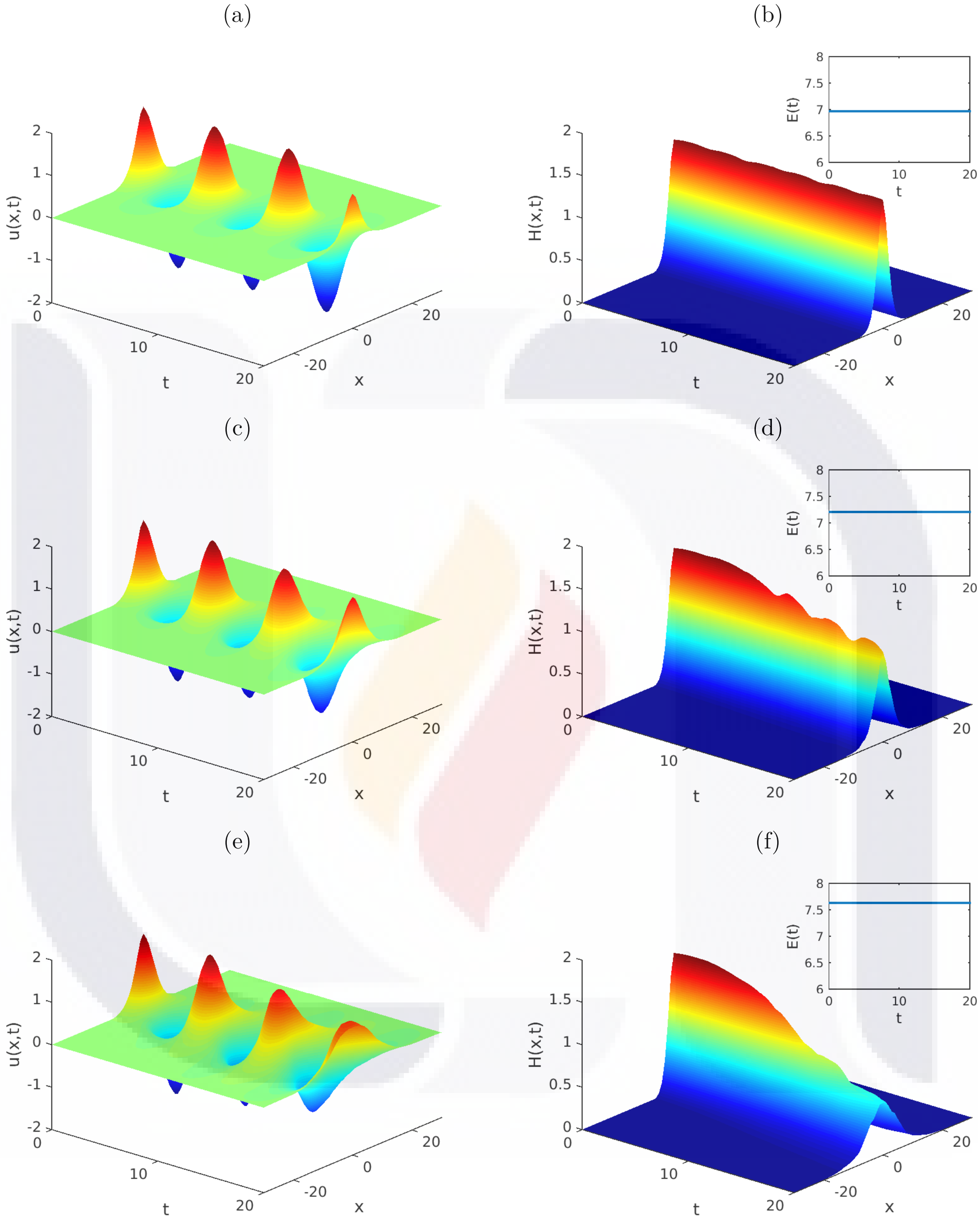

Figure 2.2: Graphs of the numerical solution (left column) and the associated energy density (right column) of the one-dimensional problem (2.1) with $G(u) = 1 - \cos u$ obtained using (2.33) and (2.53) on $\Omega = (-30, 30) \times (0, 100)$. The initial data were provided by (2.114) with $\omega = 0.9$, and the parameters employed were $\gamma = 0$, $h_1 = 0.5$ and $\tau = 0.05$. Various derivative orders were used, namely, $\alpha_1 = 2$ (top row), $\alpha_1 = 1.6$ (middle row) and $\alpha_1 = 1.2$ (bottom row). The insets of the graphs of the right column represent the discrete dynamics of the total energy (2.47) of the system.

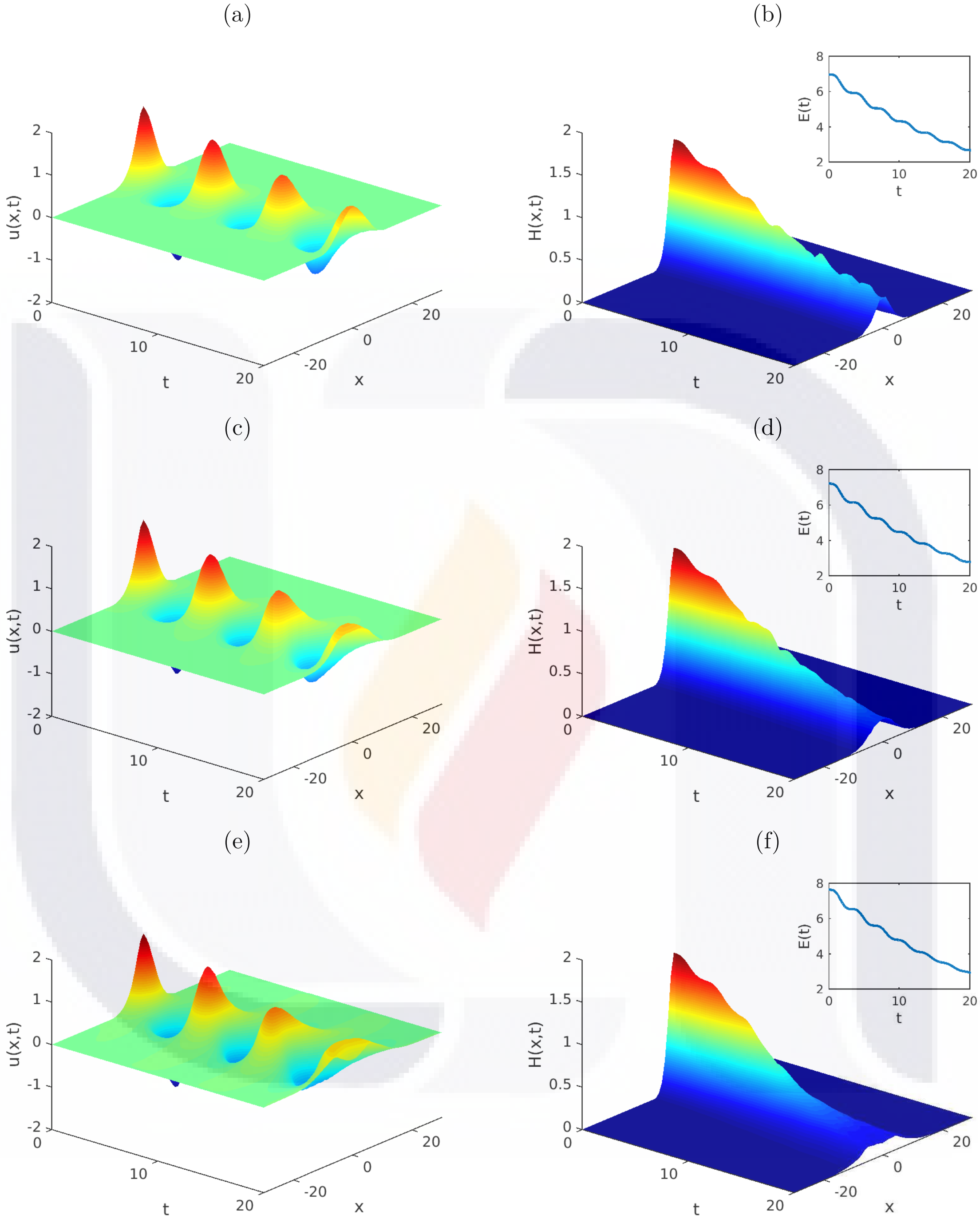

Figure 2.3: Graphs of the numerical solution (left column) and the associated energy density (right column) of the one-dimensional problem (2.1) with $G(u) = 1 - \cos u$ obtained using (2.33) and (2.53) on $\Omega = (-30, 30) \times (0, 100)$. The initial data were provided by (2.114) with $\omega = 0.9$, and the parameters employed were $\gamma = 0.05$, $h_1 = 0.5$ and $\tau = 0.05$. Various derivative orders were used, namely, $\alpha_1 = 2$ (top row), $\alpha_1 = 1.6$ (middle row) and $\alpha_1 = 1.2$ (bottom row). The insets of the graphs of the right column represent the discrete dynamics of the total energy (2.47) of the system.

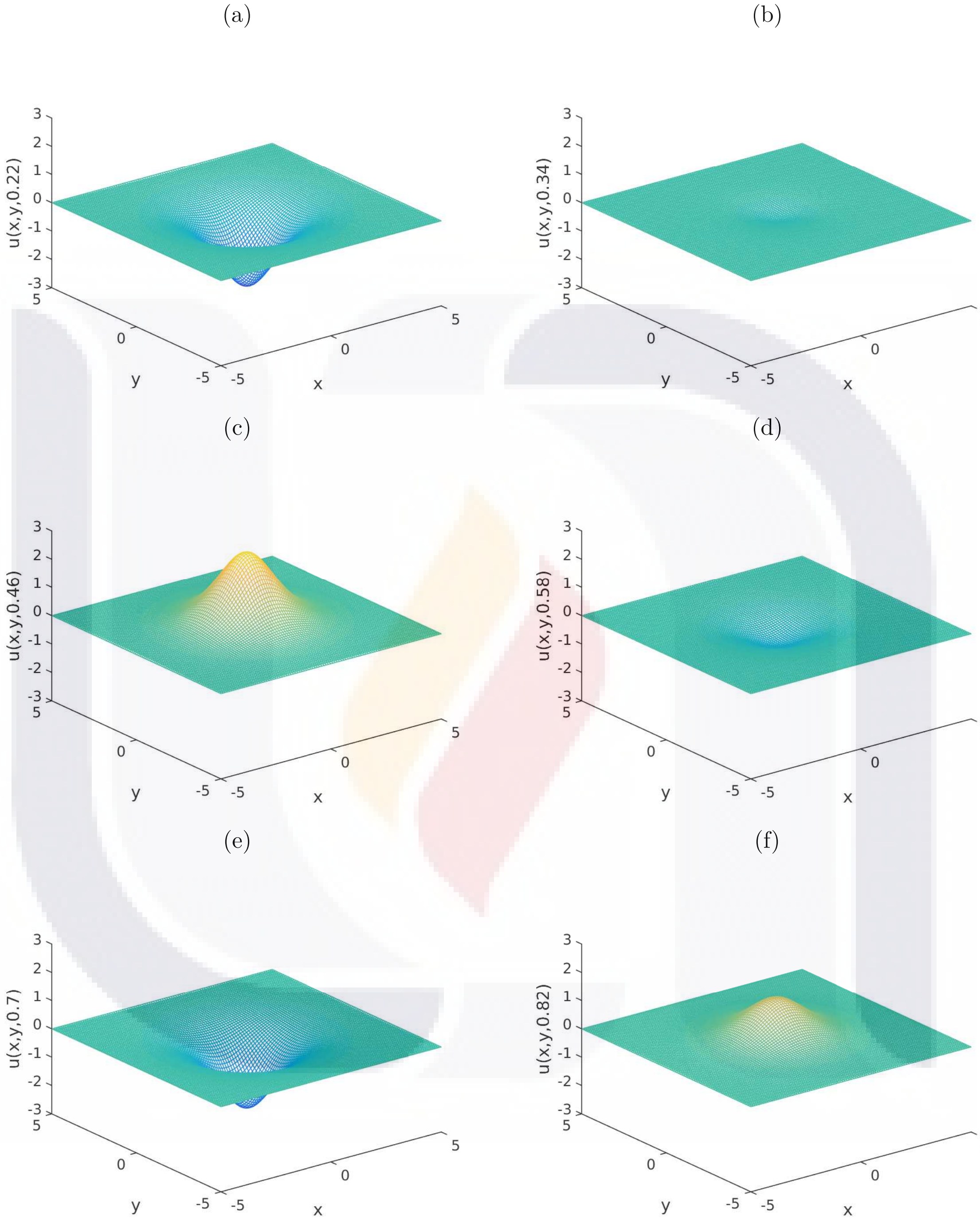

Figure 2.4: (Color online). Graphs of the approximate solution of (2.1) in two spatial dimensions at the times (a) $t = 0.22$, (b) $t = 0.34$, (c) $t = 0.46$, (d) $t = 0.58$, (e) $t = 0.70$ and (f) $t = 0.82$. The model parameters employed were $\alpha_1 = 1.8$, $\alpha_2 = 1.6$, $\gamma = 0$, $G(u) = 1 - \cos u$, $B = (-5, 5) \times (-5, 5)$ and $T = 10$. Meanwhile, the initial conditions were provided by $\varphi(x^2 + y^2, t)$, where φ is given by (2.114) with $\omega = 0.8$. Numerically, we used the method (2.33) with $M_1 = M_2 = 100$ and $N = 500$.

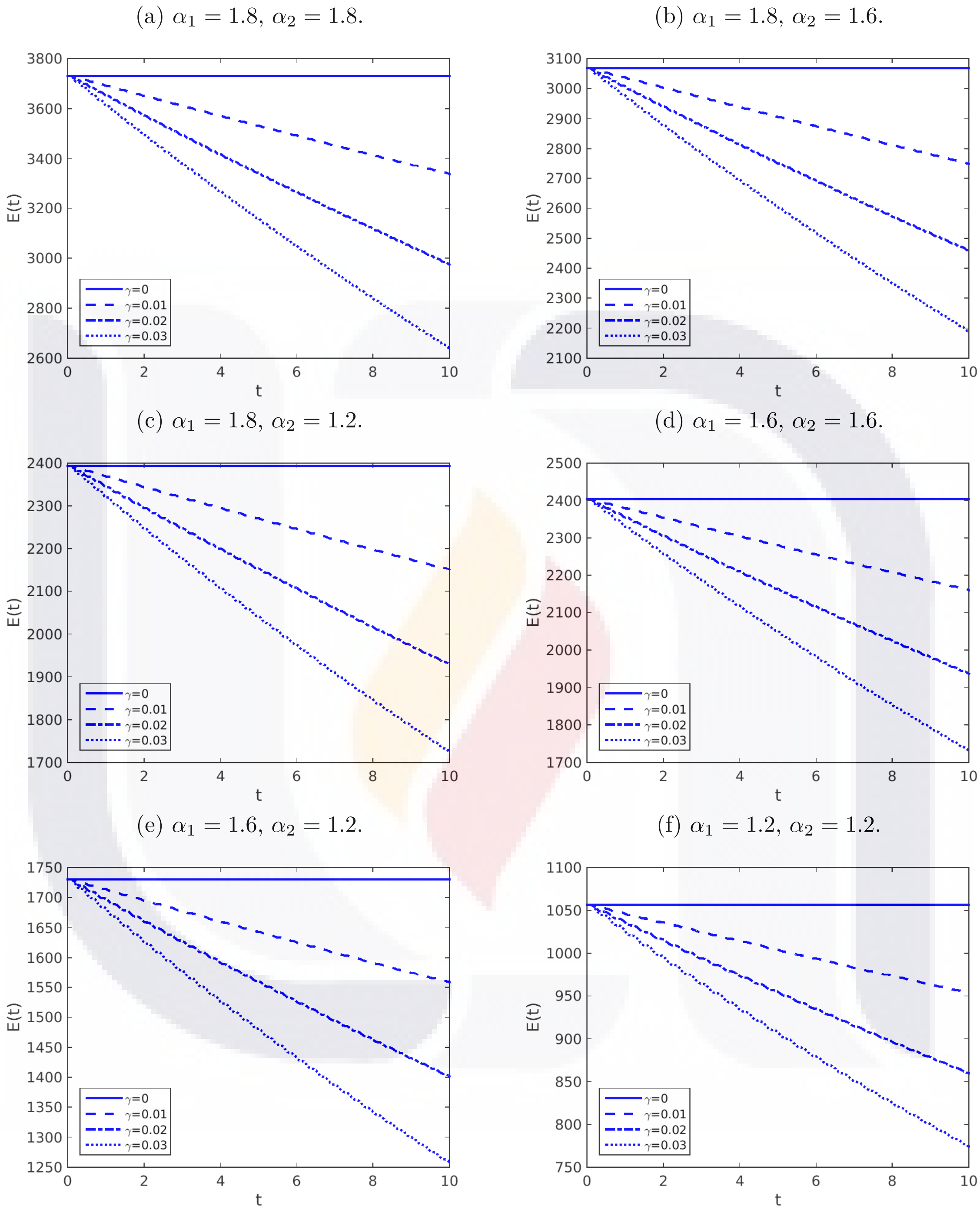

Figure 2.5: (Color online). Graphs of the energy dynamics of the solution of (2.1) in two spatial dimensions using various sets of the parameters α_1 and α_2, $G(u) = 1 - \cos u$, $B = (-5, 5) \times (-5, 5)$ and $T = 10$. The initial conditions were provided by $\varphi(x^2 + y^2, t)$, where φ is given by (2.114) with $\omega = 0.8$, and various damping coefficients were considered, namely, $\gamma = 0$ (solid), $\gamma = 0.01$ (dashed), $\gamma = 0.02$ (dashed-dotted) and $\gamma = 0.03$ (dotted). Numerically, we used the method (2.33) with $M_1 = M_2 = 100$ and $N = 500$.

$\gamma = 0$ (solid), $\gamma = 0.01$ (dashed), $\gamma = 0.02$ (dashed-dotted) and $\gamma = 0.03$ (dotted). The results illustrate the fact that the total energy of the system is conserved in the case when $\gamma = 0$, while the system is dissipative if $\gamma = 0$. The results are in agreement with the theorems derived in this chapter. $\square$

2.7 Computer implementation

In this appendix, we provide a Matlab code of the numerical method (2.33) for the one-dimensional scenario. It is worth noting that its implementation is straightforward, and it requires input and output constants, which refer to the computational and model parameters described in the manuscript. The code uses the functions `exact` and `G`, which are used to prescribe the initial conditions and the potential function G, respectively.

Input `(a,b,T,M,N,alpha,gama)`.
`a`$= a_1$, `b`$= b_1$, `T`$= t$,
`M`$= M_1$, `N`$= N$, `alpha`$= \alpha$, `gama`$= \gamma$.

Output `[x,u,H,t,E]`. The first three variables are vectors of lengths equal to $M+1$, while the last two are vectors of lengths $N+1$. These vectors are defined as
`x(j)`$= x_{j-1}$ for each $j \in \{1, \ldots, M+1\}$,
`u(j)`$= v_{j-1}^N$ for each $j \in \{1, \ldots, M+1\}$,
`H(j)`$= H_{j-1}^N$ for each $j \in \{1, \ldots, M+1\}$,
`t(n)`$= t_{n-1}$ for each $n \in \{1, \ldots, N+1\}$,
`E(n)`$= E_{n-1}$ for each $n \in \{1, \ldots, N+1\}$.

Code.

```
function [x,u,H,t,E]=fracwave(a,b,T,M,N,alpha,gama)
  function y=exact(t)
    y=4.*atan(sqrt(1-omega^2)*cos(omega.*t)./cosh((1-omega^2).*x)./omega);
  end

  function y=G(z)
    y=1-cos(z);
  end

  omega=0.8;
  h=(b-a)/M;
  tau=T/N;
  r=2*gama*tau;
  R=2*tau^2;

  x=a:h:b;
  t=0:tau:T;

  u1=exact(t(1));
```

```
  u2=exact(t(2));
  u3=exact(t(3));
  u=zeros(size(u1));
  E=zeros(size(t));
  H=u;

  g=zeros(size(u1));
  g(1)=tau^2*gamma(alpha+1)/gamma(0.5*alpha+1)^2/h^alpha;
  for k=1:M
    g(k+1)=(1-(alpha+1)/(0.5*alpha+k+1))*g(k);
  end

  frac1=zeros(size(u));
  for j=2:M
    for k=1:M+1
      frac1(j)=frac1(j)+g(abs(j-k)+1)*u2(k);
    end
  end
  for n=4:N
    for j=2:M
      frac2=0;
      for k=1:M+1
        frac2=frac2+g(abs(j-k)+1)*u3(k);
      end
      u(j)=u3(j)+u2(j)-u1(j)-(frac1(j)+frac2)-r*(u3(j)-u2(j))...
        -R*(G(u3(j))-G(u2(j)))/(u3(j)-u2(j));
      H(j)=0.5*(u3(j)-u2(j))*(u2(j)-u1(j))/tau/tau...
        +0.5*u2(j)*frac1(j)/tau/tau+G(u2(j));
      frac1(j)=frac2;
    end
    E(n)=h*sum(H);
    u1=u2;
    u2=u3;
    u3=u;
  end
end
```

3. High Order of Accuracy Scheme for the Fractional Wave Equation

In this final chapter, we consider a general class of damped wave equations in two spatial dimensions. The model considers the presence of Riesz space-fractional derivatives as well as a generic nonlinear potential. The system has an associated positive energy functional when damping is not present, in which case, the model is capable of preserving the energy throughout time. Meanwhile, the energy of the system is dissipated in the damped scenario. In this chapter, the Riesz space-fractional derivatives are approximated through second-order accurate fractional centered differences. A high-order compact difference scheme with fourth order accuracy in space and second order in time is proposed. Some associated discrete quantities are introduced to estimate the energy functional. We prove that the numerical method is capable of conserving the discrete variational structure under the same conditions for which the continuous model is conservative. The positivity of the discrete energy of the system is also discussed. The properties of consistency, solvability, stability and convergence of the proposed method are rigorously proved. We provide some numerical simulations that illustrate the agreement between the physical properties of the continuous and the discrete models.

3.1 Introduction

Derivatives and integrals of non-integer order are currently used in many applications in mechanics and physics. In particular, they have been employed in the description of anomalous kinetics and transport in walks [53, 54] and to obtain fractional analogues of equations of motion was proposed for sets of point-particles with long-range interactions [55, 32]. This procedure is helpful in the analysis of the dynamics of some discrete Hamiltonian systems of oscillators [31, 11]. On the other hand, the design of structure-preserving schemes to solve multi-dimensional problems has been the main topic of study in various reports [56, 57, 58, 59]. In a broad sense, 'structure preservation' refers to the capacity of a computational technique to preserve mathematical features of the relevant solutions of continuous systems. The condition of positivity, which is a natural requirement for problems in which the variables of interest are measured in absolute scales [60, 61, 62, 63], is one of those mathematical features. Boundedness is another characteristic in problems where there exist natural limitations of growth, particularly in models that describe the dynamics of populations under limited resources [64, 65]. Monotonicity is another property [44, 66].

3.2 Preliminaries

Throughout this Chapter, we let $T, \kappa \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+ \cup \{0\}$, and suppose that $a_i, b_i \in \mathbb{R}$ satisfy $a_i < b_i$, for each $i \in \{1, 2\}$. Let us define $B = (a_1, b_1) \times (a_2, b_2) \subseteq \mathbb{R}^2$, and let $\Omega = B \times (0, T) \subseteq \mathbb{R}^3$. We will employ the symbols $\overline{B}$, $\overline{\Omega}$ and ∂B to represent respectively the closures of B and Ω, and the boundary of B under the standard topology of $\mathbb{R}^3$. Throughout, we will observe the convention $x = (x_1, x_2) \in \mathbb{R}^2$. Assume that $G : \mathbb{R} \to \mathbb{R}$ is a function, and that $\phi, \psi : B \to \mathbb{R}$ satisfy $\phi(x) = \psi(x) = 0$ for each $x \in \partial B$. Moreover, suppose that G is nonnegative, that G'' is bounded and that $u : \overline{\Omega} \to \mathbb{R}$ is a sufficiently smooth function. We will assume that $u(x, t) = 0$ for each $t \in [0, T]$ and $x \in \mathbb{R}^2 \setminus \overline{B}$.

Definition 3.1. Let $\alpha > -1$ and suppose that n is a nonnegative integer such that $n - 1 < \alpha \leq n$. The *Riesz fractional derivatives* of u of order α with respect to x_1 and with respect to x_2 at the point (x, t) are respectively defined by

$$\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial |x_1|^\alpha} &= \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x_1^n} \int_{-\infty}^{\infty} \frac{u(\xi, x_2, t) d\xi}{|x_1 - \xi|^{\alpha+1-n}}, \quad \forall (x,t) \in \Omega, && (3.1) \\
\frac{\partial^\alpha u(x,t)}{\partial |x_2|^\alpha} &= \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x_2^n} \int_{-\infty}^{\infty} \frac{u(x_1, \xi, t) d\xi}{|x_2 - \xi|^{\alpha+1-n}}, \quad \forall (x,t) \in \Omega. && (3.2)
\end{aligned}$$

Here Γ is the usual Gamma function defined on $\mathbb{R} \setminus \{n \in \mathbb{Z} : n \leq 0\}$.

Throughout, assume that $1 < \alpha_i \leq 2$ for each $i \in \{1, 2\}$. In this chapter, we will investigate numerically the following two-dimensional initial-boundary-value problem

$$\begin{aligned}
&\frac{\partial^2 u(x,t)}{\partial t^2} - \kappa \left(\frac{\partial^{\alpha_1} u(x,t)}{\partial |x_1|^{\alpha_1}} + \frac{\partial^{\alpha_2} u(x,t)}{\partial |x_2|^{\alpha_2}} \right) + \gamma \frac{\partial u(x,t)}{\partial t} + G'(u(x,t)) = 0, \quad \forall (x,t) \in \Omega, \\
&\text{such that} \begin{cases} u(x,0) = \phi(x), & \forall x \in B, \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x), & \forall x \in B, \\ u(x,t) = 0, & \forall (x,t) \in \partial B \times (0,T). \end{cases}
\end{aligned} \tag{3.3}$$

Denote the Riesz derivative of u of order α_i with respect to x_i by $\mathcal{D}_{x_i}^{\alpha_i}$, for each $i \in \{1, 2\}$. For each such i, note that $-\mathcal{D}_{x_i}^{\alpha_i}$ has a unique square-root [50, 49], which is denoted by $\Xi_{x_i}^{\alpha_i}$ and satisfies $\langle -\mathcal{D}_{x_i}^{\alpha_i} u, v \rangle_x = \langle \Xi_{x_i}^{\alpha_i} u, \Xi_{x_i}^{\alpha_i} v \rangle_x$ for any two functions u and v. With these conventions, we use the following positive energy function [67]:

$$\mathcal{E}(t) = \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \frac{1}{2} \sum_{i=1}^{2} \left\| \Xi_{x_i}^{\alpha_i} u \right\|_{x,2}^2 + \| G(u) \|_{x,1}, \quad \forall t \in (0, T). \tag{3.4}$$

Theorem 3.2 (Macías-Díaz [68]). *If u is a solution of* (3.3) *then*

$$\mathcal{E}(t) = \mathcal{E}(0) - \gamma \int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 dt, \quad \forall t \in [0, T]. \tag{3.5}$$

As a consequence, $\mathcal{E}'(t) = -\gamma \| u_t \|_{x,2}^2$ for each $t \in (0, T)$, so that the system (3.3) *is conservative if* $\gamma = 0$.

3.3 Numerical method

For any natural number n, we let $I_n = \{1, \ldots, n\}$ and $\overline{I}_n = I_n \cup \{0\}$. For the remainder of this chapter, we let $h_1, h_2 \in \mathbb{R}^+$ be fixed step-sizes in the x_1- and x_2-directions, respectively, and let $\tau \in \mathbb{R}^+$ be the temporal step-size. Moreover, assume that $N = T/\tau$ and $M_i = (b_i - a_i)/h_i$ are positive integers for each $i \in I_2$. We will consider uniform partitions of $[a_i, b_i]$ for $i \in I_2$, and of $[0, T]$, respectively, given by

$$x_{i,j_i} = a_i + j_i h_i, \quad \forall i \in I_2, \forall j_i \in \overline{I}_{M_i}, \tag{3.6}$$

$$t_n = n\tau \quad \forall n \in \overline{I}_N. \tag{3.7}$$

Let $J = I_{M_1-1} \times I_{M_2-1}$ and $\overline{J} = \overline{I}_{M_1} \times \overline{I}_{M_2}$, and let $\partial J = \{j \in J : x_j \in \partial B\}$. Define $x_j = (x_{1,j_1}, x_{2,j_2})$ for each bi-index $j = (j_1, j_2) \in \overline{J}$. For each $(j, n) \in \overline{J} \times \overline{I}_N$, the symbol v_j^n will represent an approximation to the value $u_j^n = u(x_j, t_n)$.

We will use the discrete average operators

$$\mu_t u_j^n = \frac{u_j^{n+1} + u_j^n}{2} = u(x_j, t_n) + \mathcal{O}(\tau), \tag{3.8}$$

$$\mu_t^{(1)} u_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2} = u(x_j, t_n) + \mathcal{O}(\tau^2), \tag{3.9}$$

and the discrete difference operators

$$\delta_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau} = \frac{\partial u(x_j, t_n)}{\partial t} + \mathcal{O}(\tau), \tag{3.10}$$

$$\delta_t^{(1)} u_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau} = \frac{\partial u(x_j, t_n)}{\partial t} + \mathcal{O}(\tau^2), \tag{3.11}$$

$$\delta_t^{(2)} u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} = \frac{\partial^2 u(x_j, t_n)}{\partial t^2} + \mathcal{O}(\tau^2), \tag{3.12}$$

for each $(j, n) \in J \times I_{N-1}$. Also, the following estimates $G'(u(x_j, t_n))$ with an order of consistency equal to $\mathcal{O}(\tau^2)$:

$$\delta_{u,t}^{(1)} G(u_j^n) = \begin{cases} \dfrac{G(u_j^{n+1}) - G(u_j^{n-1})}{u_j^{n+1} - u_j^{n-1}}, & \text{if } u_j^{n+1} \neq u_j^{n-1}, \\ G'(u_j^n), & \text{if } u_j^{n+1} = u_j^{n-1}. \end{cases} \tag{3.13}$$

Definition 3.3. For any function $f : \mathbb{R} \to \mathbb{R}$, any $h > 0$ and any $\alpha > -1$, the *fractional centered difference* of order α of f at the point x is given by

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{3.14}$$

whenever the right-hand side of this expression converges. The coefficients of the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ are defined by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{3.15}$$

Lemma 3.4 (Çelik and Duman [52]). *If $1 < \alpha \leq 2$ then the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ satisfies*

(i) $g_0^{(\alpha)} \geq 0$,

(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ *for all* $k \geq 1$, *and*

(iii) $\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$.

As a consequence of Lemma 3.4, the series in the right-hand side of (3.14) converges absolutely for any bounded function $f \in L_1(\mathbb{R})$. With this notation, it is easy to see that any $f \in \mathcal{C}^5(\mathbb{R})$ for which all of its derivatives up to order five belong to $L_1(\mathbb{R})$, the number $-h^{-\alpha}\Delta_h^{\alpha} f(x)$ approximates quadratically the derivative of f of order α at the point x whenever $1 < \alpha \leq 2$ (see [52]). Under these circumstances, if $u \in C^5(\overline{B})$ then

$$\frac{\partial^{\alpha_i} u}{\partial |x_i|^{\alpha_i}}(x_j, t_n) = \delta_{x_i}^{(\alpha_i)} u_j^n + \mathcal{O}(h_i^2), \quad \forall i \in I_2, \forall (j,n) \in J \times \overline{I}_N, \tag{3.16}$$

where

$$\delta_{x_1}^{(\alpha_1)} u_j^n = -\frac{1}{h_1^{\alpha_1}} \sum_{k=0}^{M_1} g_{j_1-k}^{(\alpha_1)} u(x_{k,j_2}, t_n), \quad \forall (j,n) \in J \times \overline{I}_N, \tag{3.17}$$

$$\delta_{x_2}^{(\alpha_2)} u_j^n = -\frac{1}{h_2^{\alpha_2}} \sum_{k=0}^{M_2} g_{j_2-k}^{(\alpha_2)} u(x_{j_1,k}, t_n), \quad \forall (j,n) \in J \times \overline{I}_N. \tag{3.18}$$

For the remainder of this chapter, we will let $h = (h_1, h_2)$ and $h_* = h_1 h_2$. Let $\alpha = (\alpha_1, \alpha_2)$, and define the spatial mesh $R_h = \{x_j\}_{j \in J} \subseteq \mathbb{R}^2$. Let $\mathcal{V}_h$ be the vector space of all real functions on R_h. For any $u \in \mathcal{V}_h$ and $j \in J$, let $u_j = u(x_j)$.

Definition 3.5. Define respectively the *inner product* $\langle \cdot, \cdot \rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ and the *norm* $\| \cdot \|_1 : \mathcal{V}_h \to \mathbb{R}$ by

$$\langle u, v \rangle = h_* \sum_{j=1}^{M-1} u_j v_j, \qquad \|u\|_1 = h_* \sum_{j=1}^{M-1} |u_j|, \tag{3.19}$$

for any $u, v \in \mathcal{V}_h$. The *Euclidean norm* induced by $\langle \cdot, \cdot \rangle$ will be denoted by $\| \cdot \|_2$.

Definition 3.6. The *spatial compact difference operators* in the x_1- and the x_2-directions are the functions $\mathbb{A}_{x_1}^{(\alpha_1)}, \mathbb{A}_{x_2}^{(\alpha_2)} : \mathcal{V}_h \to \mathcal{V}_h$ defined, respectively, by

$$\mathbb{A}_{x_1}^{(\alpha_1)} \nu_j = \frac{\alpha_1}{24} \nu_{j_1-1,j_2} + \left(1 - \frac{\alpha_1}{12}\right) \nu_{j_1,j_2} + \frac{\alpha_1}{24} \nu_{j_1+1,j_2}, \quad \forall \nu \in \mathcal{V}_h, \forall j \in J, \tag{3.20}$$

$$\mathbb{A}_{x_2}^{(\alpha_2)} \nu_j = \frac{\alpha_2}{24} \nu_{j_1,j_2-1} + \left(1 - \frac{\alpha_2}{12}\right) \nu_{j_1,j_2} + \frac{\alpha_2}{24} \nu_{j_1,j_2+1}, \quad \forall \nu \in \mathcal{V}_h, \forall j \in J. \tag{3.21}$$

At the boundary of J, we conveniently define these operators as zero. Moreover, we introduce the linear operators $\mathbb{A}_h^{(\alpha)} = \mathbb{A}_{x_1}^{(\alpha_1)} \mathbb{A}_{x_2}^{(\alpha_2)}$ and $\Lambda_h = c(\mathbb{A}_{x_2}^{(\alpha_2)} \delta_{x_1}^{(\alpha_1)} + \mathbb{A}_{x_1}^{\alpha_1} \delta_{x_2}^{(\alpha_2)})$ on $\mathcal{V}_h$.

We will require the following results in the sequel.

Lemma 3.7 (Macías-Díaz [69]). *For each $i \in I_2$, there exists a unique positive self-adjoint (square root) operator $\Lambda_{x_i}^{(\alpha_i)} : \mathcal{V}_h \to \mathcal{V}_h$, such that $\langle -\delta_{x_i}^{(\alpha_i)} u, v \rangle = \langle \Lambda_{x_i}^{(\alpha_i)} u, \Lambda_{x_i}^{(\alpha_i)} v \rangle$, for each $u, v \in \mathcal{V}_h$.*

Define the constant $g_h^{(\alpha)} = 2h_*(g_0^{(\alpha_1)} h_1^{-\alpha_1} + g_0^{(\alpha_2)} h_2^{-\alpha_2})$. Using this convention, the following result summarizes some easy properties of the operators $\delta_{x_i}^{(\alpha_i)}$ and their respective square roots.

Lemma 3.8 (Macías-Díaz [67]). *Let $v \in \mathcal{V}_h$ and $i \in I_2$. Then*

(a) $\|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2g_0^{(\alpha_i)} h_* h_i^{-\alpha_i} \|v\|_2^2$,

(b) $\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 = \|\Lambda_{x_i}^{(\alpha_i)} \Lambda_{x_i}^{(\alpha_i)} v\|_2^2$,

(c) $\|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2g_0^{(\alpha_i)} h_* h_i^{-\alpha_i} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 4 \left(g_0^{(\alpha_i)} h_* h_i^{-\alpha_i} \right)^2 \|v\|_2^2$,

(d) $\displaystyle\sum_{i \in I_p} \|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq 2h_* \sum_{i \in I_p} g_0^{(\alpha_i)} h_i^{-\alpha_i} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq 4h_*^2 \|v\|_2^2 \sum_{i \in I_p} (g_0^{(\alpha_i)} h_i^{-\alpha_i})^2$ *and*

(e) $\displaystyle\sum_{i \in I_p} \|\delta_{x_i}^{(\alpha_i)} v\|_2^2 \leq g_h^{(\alpha)} \sum_{i \in I_p} \|\Lambda_{x_i}^{(\alpha_i)} v\|_2^2 \leq \left(g_h^{(\alpha)} \|v\|_2 \right)^2$.

It is important to recall that the compact operators satisfy $\|\mathbb{A}_{x_1}^{(\alpha_1)} v\|_2 \leq \|v\|_2$ and $\|\mathbb{A}_{x_2}^{(\alpha_2)} v\|_2 \leq \|v\|_2$, for each $v \in \mathcal{V}_h$. As a consequence, $\langle \Lambda_h v, w \rangle \leq c g_h^{(\alpha)} \|v\|_2 \|w\|_2$ for each $v, w \in \mathcal{V}_h$.

Lemma 3.9 (Zhao *et al.* [70]).

(1) *For any $v \in \mathcal{V}_h$ we have $\|\mathbb{A}_h^{(\alpha)} v\|_2 \leq \|v\|_2$ and $\frac{1}{3}\|v\|_2^2 \leq \langle \mathbb{A}_h^{(\alpha)} v, v \rangle \leq \|v\|_2^2$.*

(2) *There is a unique operator $\Lambda_h^{\frac{1}{2}} : \mathcal{V}_h \to \mathcal{V}_h$ such that $\langle -\Lambda_h v, v \rangle = \langle \Lambda_h^{\frac{1}{2}} v, \Lambda_h^{\frac{1}{2}} v \rangle = \|\Lambda_h^{\frac{1}{2}} v\|_2^2$, for each $v \in \mathcal{V}_h$.*

The following result will be useful to obtain a compact discretization of (3.3).

Lemma 3.10 (Li and Zeng [71]). *Let $1 < \alpha_i \leq 2$ for each $i \in I_2$. Let $u \in C_x^7(\Omega)$, and assume that all the spatial derivatives of u up to order seven belong to $L_1(\mathbb{R})$. Then*

$$\delta_{x_i}^{(\alpha_i)} u(x_i) = \mathbb{A}_{x_i}^{(\alpha_i)} \left[\frac{\partial^{\alpha_i} u(x,t)}{\partial |x_i|^{\alpha_i}} \right] + \mathcal{O}(h_i^4), \tag{3.22}$$

for each $(x,t) \in \Omega$ and $i \in I_2$.

Apply the operator $\mathbb{A}_h^{(\alpha)}$ on both sides of the partial differential equation of (3.3)

$$\mathbb{A}_h^{(\alpha)} \left(\frac{\partial^2 u(x,t)}{\partial t^2} - \kappa \left(\frac{\partial^{\alpha_1} u(x,t)}{\partial |x_1|^{\alpha_1}} + \frac{\partial^{\alpha_2} u(x,t)}{\partial |x_2|^{\alpha_2}} \right) + \gamma \frac{\partial u(x,t)}{\partial t} + G'(u(x,t)) \right) = \mathbb{A}_h^{(\alpha)} (0), \tag{3.23}$$

then rearrange the left side of the equation as follows, using the definition of $\mathbb{A}_h^{(\alpha)}$:

$$\begin{aligned}
&\mathbb{A}_h^{(\alpha)} \left(\frac{\partial^2 u(x,t)}{\partial t^2} - \gamma \frac{\partial u(x,t)}{\partial t} \right) - \kappa \left(\mathbb{A}_h^{(\alpha)} \frac{\partial^{\alpha_1} u(x,t)}{\partial |x_1|^{\alpha_1}} + \mathbb{A}_h^{(\alpha)} \frac{\partial^{\alpha_2} u(x,t)}{\partial |x_2|^{\alpha_2}} \right) + \mathbb{A}_h^{(\alpha)} G'(u(x,t)) \\
&\mathbb{A}_h^{(\alpha)} \left(\frac{\partial^2 u(x,t)}{\partial t^2} - \gamma \frac{\partial u(x,t)}{\partial t} \right) - \kappa \left(\mathbb{A}_{x_1}^{(\alpha_1)} \mathbb{A}_{x_2}^{(\alpha_2)} \frac{\partial^{\alpha_1} u(x,t)}{\partial |x_1|^{\alpha_1}} + \mathbb{A}_{x_1}^{(\alpha_1)} \mathbb{A}_{x_2}^{(\alpha_2)} \frac{\partial^{\alpha_2} u(x,t)}{\partial |x_2|^{\alpha_2}} \right) + \mathbb{A}_h^{(\alpha)} G'(u(x,t)) \\
&\mathbb{A}_h^{(\alpha)} \left(\frac{\partial^2 u(x,t)}{\partial t^2} - \gamma \frac{\partial u(x,t)}{\partial t} \right) - \kappa \left(\mathbb{A}_{x_2}^{(\alpha_2)} \mathbb{A}_{x_1}^{(\alpha_1)} \frac{\partial^{\alpha_1} u(x,t)}{\partial |x_1|^{\alpha_1}} + \mathbb{A}_{x_1}^{(\alpha_1)} \mathbb{A}_{x_2}^{(\alpha_2)} \frac{\partial^{\alpha_2} u(x,t)}{\partial |x_2|^{\alpha_2}} \right) + \mathbb{A}_h^{(\alpha)} G'(u(x,t)),
\end{aligned} \tag{3.24}$$

we use Lemma 3.10

$$\mathbb{A}_h^{(\alpha)}\left(\frac{\partial^2 u(x,t)}{\partial t^2}-\gamma\frac{\partial u(x,t)}{\partial t}\right)-\kappa\left(\mathbb{A}_{x_2}^{(\alpha_2)}\delta_{x_1}^{(\alpha_1)}u(x_1)+\mathbb{A}_{x_1}^{(\alpha_1)}\delta_{x_2}^{(\alpha_2)}u(x_2)\right)+\mathbb{A}_h^{(\alpha)}G'(u(x,t))=0, \tag{3.25}$$

and substitute the partial derivatives with respect to time by their second-order discrete approximations.

$$\mathbb{A}_h^{(\alpha)}\left(\delta_t^{(2)}v_j^n+\gamma\delta_t^{(1)}v_j^n\right)-\kappa\left(\mathbb{A}_{x_2}^{(\alpha_2)}\delta_{x_1}^{(\alpha_1)}\mu_t^{(1)}v_j^n+\mathbb{A}_{x_1}^{(\alpha_1)}\delta_{x_2}^{(\alpha_2)}\mu_t^{(1)}v_j^n\right)+\mathbb{A}_h^{(\alpha)}\delta_{v,t}^{(1)}G(v_j^n)=0. \tag{3.26}$$

In such way, we obtain the following scheme to approximate the solution of (3.3) on Ω:

$$\begin{gathered}\mathbb{A}_h^{(\alpha)}\left(\delta_t^{(2)}v_j^n+\gamma\delta_t^{(1)}v_j^n\right)-\Lambda_h\mu_t^{(1)}v_j^n+\mathbb{A}_h^{(\alpha)}\delta_{v,t}^{(1)}G(v_j^n)=0,\quad\forall(j,n)\in J\times I_{N-1},\\ \text{such that}\begin{cases}v_j^0=\phi(x_j), & \forall j\in J,\\ \delta_t v_j^0=\psi(x_j), & \forall j\in J,\\ v_j^n=0, & \forall(j,n)\in\partial J\times\overline{I}_N.\end{cases}\end{gathered} \tag{3.27}$$

This scheme is a three-level method, meaning that the approximation at the first time-level is required. In our implementation, we used side-by-side Taylor expansions with the approximations for the initial-boundary conditions in (3.27) and (3.3) in order to evaluate the approximation at the first time-level. In that way, we obtain

$$v_j^1=\phi(x_j)+\tau\psi(x_j)+\frac{c\tau^2}{2!}\left(\frac{\partial^{\alpha_1}u}{\partial|x_1|^{\alpha_1}}+\frac{\partial^{\alpha_2}u}{\partial|x_2|^{\alpha_1}}\right)\phi(x_j)+G'(\phi(x_j))-\gamma\phi(x_j),\quad\forall j\in J. \tag{3.28}$$

Let us represent the solutions of (3.27) by $(v^n)_{n=0}^N$, where we convey that $v^n=(v_j^n)_{j\in J}$ for each $n\in\overline{I}_N$. Moreover, let $\delta_{v,t}^{(1)}G(v^n)=(\delta_{v,t}^{(1)}G(v_j^n))_{j\in J}$. With this notation, the first thing we notice is:

$$\begin{aligned}\mathbb{A}_{x_2}^{(\alpha_2)}\mathbb{A}_{x_2}^{(\alpha_2)}\nu_j &= \mathbb{A}_{x_2}^{(\alpha_2)}\left(\frac{\alpha_2}{24}\nu_{j_1,j_2-1}+\left(1-\frac{\alpha_2}{12}\right)\nu_{j_1,j_2}+\frac{\alpha_2}{24}\nu_{j_1,j_2+1}\right)\\ &= \frac{\alpha_1\alpha_2}{24^2}\nu_{j_1-1,j_2-1}+\left(1-\frac{\alpha_1}{12}\right)\frac{\alpha_2}{24}\nu_{j_1,j_2-1}+\frac{\alpha_1\alpha_2}{24^2}\nu_{j_1+1,j_2-1}\\ &+ \frac{\alpha_1}{24}\left(1-\frac{\alpha_2}{12}\right)\nu_{j_1-1,j_2}+\left(1-\frac{\alpha_1}{12}\right)\left(1-\frac{\alpha_2}{12}\right)\nu_{j_1,j_2}+\frac{\alpha_1}{24}\left(1-\frac{\alpha_2}{12}\right)\nu_{j_1+1,j_2}\\ &+ \frac{\alpha_1\alpha_2}{24^2}\nu_{j_1-1,j_2+1}+\left(1-\frac{\alpha_1}{12}\right)\frac{\alpha_2}{24}\nu_{j_1,j_2+1}+\frac{\alpha_1\alpha_2}{24^2}\nu_{j_1+1,j_2+1},\end{aligned} \tag{3.29}$$

then we define $a=\frac{\alpha_1\alpha_2}{24^2}$, $b=\left(1-\frac{\alpha_1}{12}\right)\frac{\alpha_2}{24}$, $c=\frac{\alpha_1}{24}\left(1-\frac{\alpha_2}{12}\right)$ and $d=\left(1-\frac{\alpha_1}{12}\right)\left(1-\frac{\alpha_2}{12}\right)$, along with the matrices:

$$A=\begin{pmatrix}d & b & 0 & \cdots & 0\\ b & d & b & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & b & d & b\\ 0 & \cdots & 0 & b & d\end{pmatrix},\qquad B=\begin{pmatrix}c & a & 0 & \cdots & 0\\ a & c & a & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & c & a\\ 0 & \cdots & 0 & a & c\end{pmatrix}. \tag{3.30}$$

Ordering $\nu^n_{j_1,j_2}$ in a column vector lexicographically, $\mathbb{A}^{(\alpha_2)}_{x_2}\mathbb{A}^{(\alpha_2)}_{x_2}\nu^n = \mathbf{M}\nu^n$ with:

$$\mathbf{M} = \begin{pmatrix} A & B & 0 & \cdots & 0 \\ B & A & B & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & B & A & B \\ 0 & \cdots & 0 & B & A \end{pmatrix}. \tag{3.31}$$

It is easy to see that the system of difference equations of the method at the time t_n can be expressed in vector form as

$$\mathbf{M}\left(\delta^{(2)}_t v^n + \gamma\delta^{(1)}_t v^n\right) - \Lambda_h \mu^{(1)}_t v^n + \mathbf{M}\left(\delta^{(1)}_{v,t} G(v^n)\right) = 0, \quad \forall n \in I_{N-1}, \tag{3.32}$$

where $\mathbf{M}$ is a tridiagonal block matrix which is symmetric and positive-definite. As a consequence, $\mathbf{M}$ is nonsingular and its inverse $\mathbf{H}$ is a symmetric positive-definite real matrix, which means that it has a Cholesky decomposition $\mathbf{H} = \mathbf{R}\mathbf{R}^\top$. In light of these facts, the finite-difference scheme (3.32) can be reformulated as

$$\delta^{(2)}_t v^n + \gamma\delta^{(1)}_t v^n - \mathbf{R}\mathbf{R}^\top \Lambda_h \mu^{(1)}_t v^n + \delta^{(1)}_{v,t} G(v^n) = 0. \tag{3.33}$$

Theorem 3.11. *Let $(v^n)^N_{n=0}$ be solution of the system (3.27). For each $n \in I_{N-1}$, let*

$$E^n = \frac{1}{2}\|\delta_t v^n\|^2_2 + \frac{1}{2}\mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^n\|^2_2 + \mu_t \left\|G(v^n)\right\|_1. \tag{3.34}$$

If $n \in I_{N-1}$ then $\delta_t E^{n-1} = -\gamma\|\delta^{(1)}_t v^n\|^2_2$, so the quantities E^n are invariants of (3.27) when $\gamma = 0$. As a consequence,

$$E^n = E^0 - \gamma\tau \sum_{k=1}^{n} \left\|\delta^{(1)}_t v^k\right\|^2_2, \quad \forall n \in I_{N-1}. \tag{3.35}$$

Proof. The proof hinges on Lemma 3.9 and the following algebraic identities:

$$\begin{aligned}
\left\langle \delta^{(2)}_t v^n, \delta^{(1)}_t v^n \right\rangle &= \left\langle \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2}, \frac{v^{n+1} - v^{n-1}}{2\tau} \right\rangle \\
&= \frac{1}{2\tau}\left\langle \frac{v^{n+1} - v^n}{\tau} - \frac{v^n - v^{n-1}}{\tau}, \frac{v^{n+1} - v^n}{\tau} + \frac{v^n - v^{n-1}}{\tau} \right\rangle \\
&= \frac{1}{2\tau}\left\langle \delta_t v^n - \delta_t v^{n-1}, \delta_t v^n + \delta_t v^{n-1} \right\rangle \\
&= \frac{1}{2\tau}\left(\left\langle \delta_t v^n, \delta_t v^n\right\rangle - \left\langle \delta_t v^{n-1}, \delta_t v^n \right\rangle - \left\langle \delta_t v^n, \delta_t v^{n-1} \right\rangle\right. \\
&\quad \left. - \left\langle \delta_t v^{n-1}, \delta_t v^{n-1} \right\rangle\right) = \frac{1}{2\tau}\left(\|\delta_t v^n\|^2_2 - \|\delta_t v^{n-1}\|^2_2\right),
\end{aligned} \tag{3.36}$$

(3.37)

$$\begin{aligned}
\left\langle -\mathbf{R}\mathbf{R}^\top \Lambda_h \mu_t^{(1)} v^n, \delta_t^{(1)} v^n \right\rangle &= \left\langle -\Lambda_h \mathbf{R}\mathbf{R}^\top \mu_t^{(1)} v^n, \delta_t^{(1)} v^n \right\rangle \\
&= \left\langle \Lambda_h^{\frac{1}{2}} \mathbf{R}\mathbf{R}^\top \mu_t^{(1)} v^n, \Lambda_h^{\frac{1}{2}} \delta_t^{(1)} v^n \right\rangle = \left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} \mu_t^{(1)} v^n, \mathbf{R}\Lambda_h^{\frac{1}{2}} \delta_t^{(1)} v^n \right\rangle \\
&= \left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} \left(\frac{v^{n+1} + v^{n-1}}{2} \right), \mathbf{R}\Lambda_h^{\frac{1}{2}} \left(\frac{v^{n+1} - v^{n-1}}{2\tau} \right) \right\rangle \\
&= \frac{1}{4\tau} \left(\left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n+1}, \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n+1} \right\rangle - \left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n+1}, \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1} \right\rangle \right. \\
&\quad \left. + \left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}, \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n+1} \right\rangle - \left\langle \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}, \mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1} \right\rangle \right) \\
&= \frac{1}{2\tau} \left[\frac{\|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n+1}\|_2^2 + \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n}\|_2^2}{2} - \frac{\|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n}\|_2^2 - \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}\|_2^2}{2} \right] \\
&= \frac{1}{2\tau} \left[\mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n}\|_2^2 - \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}\|_2^2 \right], \qquad (3.38)
\end{aligned}$$

$$\begin{aligned}
\langle \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle &= \langle \frac{G(v^{n+1}) - G(v^{n-1})}{v^{n+1} - v^{n-1}}, \frac{v^{n+1} - v^n}{2\tau} \rangle \\
&= h_* \sum_{j \in I} \frac{G(v_j^{n+1}) - G(v_j^{n-1})}{v_j^{n+1} - v_j^{n-1}} \frac{v_j^{n+1} - v_j^{n-1}}{2\tau} \\
&= h_* \sum_{j \in I} \frac{G(v_j^{n+1}) - G(v_j^{n-1})}{2\tau} = \frac{1}{2\tau} \left[h_* \sum_{j \in I} G(v_j^{n+1}) - h_* \sum_{j \in I} G(v_j^{n-1}) \right] \\
&= \frac{1}{\tau} \left[\frac{\|G(v^{n+1})\|_1 + \|G(v^n)\|_1}{2} - \frac{\|G(v^n)\|_1 + \|G(v^{n-1})\|_1}{2} \right] \\
&= \frac{1}{\tau} \left[\mu_t \|G(v^n)\|_1 - \mu_t \|G(v^{n-1})\|_1 \right]. \qquad (3.39)
\end{aligned}$$

Let Θ^n be the real vector consisting of the left-hand sides of the difference equations in (3.27) for each $n \in I_{N-1}$, and suppose that $(v^n)_{n=0}^N$ is a solution of the method. Calculating the inner product of Θ^n with $\delta_t^{(1)} v^n$, using the identities above and simplifying

$$\begin{aligned}
\langle \Theta^n, \delta_t^{(1)} v^n \rangle &= \langle \delta_t^{(2)} v^n + \gamma \delta_t^{(1)} v^n - \mathbf{R}\mathbf{R}^\top \Lambda_h \mu_t^{(1)} v^n + \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle \\
&= \langle \delta_t^{(2)} v^n, \delta_t^{(1)} v^n \rangle + \langle \gamma \delta_t^{(1)} v^n, \delta_t^{(1)} v^n \rangle + \langle -\mathbf{R}\mathbf{R}^\top \Lambda_h \mu_t^{(1)} v^n, \delta_t^{(1)} v^n \rangle + \langle \delta_{v,t}^{(1)} G(v^n), \delta_t^{(1)} v^n \rangle \\
&= \frac{1}{2\tau} \left(\|\delta_t v^n\|_2^2 - \|\delta_t v^{n-1}\|_2^2 \right) + \gamma \|\delta_t^{(1)} v^n\|_2^2 + \frac{1}{2\tau} \left[\mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^n\|_2^2 - \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}\|_2^2 \right] \\
&\quad + \frac{1}{\tau} \left[\mu_t \|G(v^n)\|_1 - \mu_t \|G(v^{n-1})\|_1 \right] \\
&= \frac{1}{\tau} \left[\left(\frac{1}{2} \|\delta_t v^n\|_2^2 + \frac{1}{2} \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^n\|_2^2 + \mu_t \|G(v^n)\|_1 \right) \right. \\
&\quad \left. - \left(\frac{1}{2} \|\delta_t v^{n-1}\|_2^2 + \frac{1}{2} \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} v^{n-1}\|_2^2 + \mu_t \|G(v^{n-1})\|_1 \right) \right] + \gamma \|\delta_t^{(1)} v^n\|_2^2 \\
&= \delta_t E^{n-1} + \gamma \left\| \delta_t^{(1)} v^n \right\|_2, \qquad (3.40)
\end{aligned}$$

note that $0 = \langle 0, \delta_t^{(1)} v^n \rangle = \langle \Theta^n, \delta_t^{(1)} v^n \rangle = \delta_t E^{n-1} + \gamma \left\| \delta_t^{(1)} v^n \right\|_2^2$, whence the conclusion follows. $\square$

Theorem 3.11 is clearly the discrete counterpart of Theorem 3.2. Moreover, the quantities E^n are nonnegative, for each $n \in I_{N-1}$. This is in obvious agreement with the expressions (3.4).

3.4 Numerical results

In this section, we prove the most important numerical properties of our method. Firstly, we establish next that the finite-difference method (3.27) is solvable. For each $w \in \mathcal{V}_h$ and $(j,n) \in J \times I_{N-1}$, let

$$\delta^{(1)}_{w,v,t} G^n_j(w) = \begin{cases} \dfrac{G(w_j) - G(v^{n-1}_j)}{w_j - v^{n-1}_j}, & \text{if } w_j \neq v^{n-1}_j, \\ G'(v^n_j), & \text{if } w_j = v^{n-1}_j. \end{cases} \tag{3.41}$$

Note that $\delta^{(1)}_{w,v,t} G^n(w) = (\delta^{(1)}_{w,v,t} G^n_j(w))_{j\in J}$ is a continuous operator on $\mathcal{V}_h$ when G is differentiable on all of $\mathbb{R}$.

Theorem 3.12. *Let G be differentiable on $\mathbb{R}$, with $G' \in L^\infty(\mathbb{R})$. If $2 + \tau\gamma - 3\tau^2 c g^{(\alpha)}_h > 0$ then the compact difference method* (3.27) *has a solution for any set of initial conditions.*

Proof. The approximations v^0 and v^1 exist, so let us assume that v^{n-1} and v^n have been obtained for some $n \in I_{N-1}$. Since $\delta^{(1)}_{w,v,t} G^n$ is continuous and G' is bounded then the operator $\delta^{(1)}_{w,v,t} G^n$ is likewise bounded in $\mathcal{V}$. Let $f : \mathcal{V}_h \to \mathcal{V}_h$ be the continuous function whose jth component $f_j : \mathcal{V}_h \to \mathbb{R}$ is given by

$$f_j(w) = \mathbb{A}^{(\alpha)}_h \left(\frac{w - 2v^n_j + v^{n-1}_j}{\tau^2} + \gamma \frac{w - v^{n-1}_j}{2\tau} \right) - \Lambda_h \left(\frac{w + v^{n-1}_j}{2} \right) + \mathbb{A}\delta^{(1)}_{w,v,t} G^n_j(w), \quad \forall w \in \mathcal{V}_h, \forall j \in J. \tag{3.42}$$

Using the Cauchy–Schwarz inequality and the results of Section 3.3

$$\begin{aligned}
\langle f(w), w\rangle &= \langle \mathbb{A}^{(\alpha)}_h \frac{w - 2v^n_j + v^{n-1}_j}{\tau^2}, w\rangle + \langle \gamma \frac{w - v^{n-1}_j}{2\tau}, w\rangle - \langle \Lambda_h \left(\frac{w + v^{n-1}_j}{2} \right), w\rangle + \langle \mathbb{A}\delta^{(1)}_{w,v,t} G^n_j(w), w\rangle \\
&= \frac{1}{\tau^2} \langle \mathbb{A}^{(\alpha)}_h w - 2v^n_j + v^{n-1}_j, w\rangle + \frac{\gamma}{2\tau} \langle w - v^{n-1}_j, w\rangle \\
&\quad - \frac{1}{2} \langle \Lambda_h \left(w + v^{n-1}_j \right), w\rangle + \langle \mathbb{A}\delta^{(1)}_{w,v,t} G^n_j(w), w\rangle \\
&\geq \|w\|_2 \left[\frac{1}{\tau^2} \left(\|w\|_2 - 2\|v^n\|_2 - \|v^{n-1}\|_2 \right) \right] + \|w\|_2 \left[\frac{\gamma}{2\tau} \left(\|w\|_2 - \|v^{n-1}\|_2 \right) \right] \\
&\quad - \|w\|_2 \left[\frac{1}{2} \left(\|\Lambda_h w\|_2 + \|\Lambda_h v^{n-1}\|_2 \right) \right] - \|w\|_2 \left[\|\mathbb{A}\delta^{(1)}_{w,v,t} G^n_j(w)\|_2 \right] \\
&\geq \|w\|_2 \left[\frac{1}{\tau^2} \left(\frac{1}{3}\|w\|_2 - 2\|v^n\|_2 - \|v^{n-1}\|_2 \right) \right] + \|w\|_2 \left[\frac{\gamma}{2\tau} \left(\frac{1}{3}\|w\|_2 - \|v^{n-1}\|_2 \right) \right] \\
&\quad - \|w\|_2 \left[\frac{c g^{(\alpha)}_h}{2} \left(\|w\|_2 + \|v^{n-1}\|_2 \right) \right] - \|w\|_2 \left[\|\mathbb{A}\delta^{(1)}_{w,v,t} G^n_j(w)\|_2 \right],
\end{aligned} \tag{3.43}$$

it follows that there exists $K \geq 0$ such that

$$\begin{aligned}\langle f(w), w\rangle \geq \|w\|_2 &\left[\frac{1}{\tau^2}\left(\frac{1}{3}\|w\|_2 - 2\|v^n\|_2 - \|v^{n-1}\|_2\right) + \frac{\gamma}{2\tau}\left(\frac{1}{3}\|w\|_2 - \|v^{n-1}\|_2\right)\right. \\ &\left. -\frac{cg_h^{(\alpha)}}{2}\left(\|w\|_2 + \|v^{n-1}\|_2\right) - K\right] \\ &= \frac{\|w\|_2}{6\tau^2}\left[6\left(\frac{1}{3}\|w\|_2 - 2\|v^n\|_2 - \|v^{n-1}\|_2\right) + 3\gamma\tau\left(\frac{1}{3}\|w\|_2 - \|v^{n-1}\|_2\right)\right. \\ &\left. -3\tau^2 cg_h^{(\alpha)}\left(\|w\|_2 + \|v^{n-1}\|_2\right) - 6\tau^2 K\right] \\ &= \frac{\|w\|_2}{6\tau^2}\left[\left(2 + \gamma\tau - 3\tau^2 cg_h^{(\alpha)}\right)\|w\|_2 - \left(12\|v^n\|_2 + \left(6 + 3\gamma\tau + 3\tau^2 cg_h\right)\|v^{n-1}\|_2 + 6\tau^2 K\right)\right] \\ &= \frac{2 + \tau\gamma - 3\tau^2 cg_h^{(\alpha)}}{6\tau^2}\|w\|_2\left[\|w\|_2 - \lambda\right], \quad \forall w \in \mathcal{V}_h,\end{aligned} \tag{3.44}$$

where

$$\lambda = \frac{12\|v^n\|_2 + (6 + 3\tau\gamma + 3\tau^2 cg_h^{(\alpha)})\|v^{n-1}\|_2 + 6\tau^2 K}{2 + \tau\gamma - 3\tau^2 cg_h^{(\alpha)}}. \tag{3.45}$$

Clearly, $\lambda > 0$ and $\langle f(w), w\rangle \geq 0$ for each $w \in \mathcal{V}_h$ with $\|w\|_2 = \lambda$. By Brouwer's fixed-point theorem [72], there is $v^{n+1} \in \mathcal{V}_h$ with $\|v^{n+1}\|_2 \leq \lambda$, such that $f(v^{n+1}) = 0$. The conclusion of the theorem follows now by induction. □

Theorem 3.13. *If $u \in \mathcal{C}^{7,4}_{x,t}(\overline{\Omega})$ then the scheme (3.27) has local truncation error of order $\mathcal{O}(\tau^2 + h_1^4 + h_2^4)$.*

Proof. The proof follows from a standard application of Taylor's theorem on each of the discrete operators of the numerical model (3.27), the smoothness of u, the triangle inequality and Lemma 3.10. □

Theorem 3.14. *If $(v^n)_{n=0}^N$ is solution of (3.27) then there is $C \in \mathbb{R}^+$ such that $\|v^n\|_2^2 \leq 2\|v^0\|_2^2 + 4T^2E^0$, for each $n \in I_{N-1}$.*

Proof. Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|v^n\|_2^2 &= \left\|v^0 + \tau\sum_{k=0}^{n-1}\frac{v^n - v^{n-1}}{\tau}\right\|_2^2 \leq 2\|v^0\|_2^2 + 2\tau^2\left\|\sum_{k=0}^{n-1}\delta_t v^k\right\|_2^2 \\ &\leq 2\|v^0\|_2^2 + 2n\tau^2\sum_{k=0}^{n-1}\|\delta_t v^k\|_2^2 = 2\|v^0\|_2^2 + 4T\tau\sum_{k=0}^{n-1}\frac{1}{2}\|\delta_t v^k\|_2^2 \\ &\leq 2\|v^0\|_2^2 + 4T\tau\sum_{k=0}^{n-1}\left(\frac{1}{2}\|\delta_t v^k\|_2^2 + \frac{1}{2}\mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}v^n\|_2^2 + \mu_t\|G(v^n)\|_1\right) \\ &\leq 2\|v^0\|_2^2 + 4T\tau\left(\sum_{k=0}^{n-1}E^k\right) \\ &\leq 2\|v^0\|_2^2 + 4T\tau\left(nE^0\right) \\ &\leq 2\|v^0\|_2^2 + 4T^2E^0,\end{aligned} \tag{3.46}$$

the last part of the proof since:

$$\begin{aligned} E^k &= E^0 - \gamma\tau \sum_{i=1}^{k} \left\| \delta_t^{(1)} v^k \right\|_2^2, \\ \sum_{k=0}^{n-1} E^k &= nE^0 - \gamma\tau \sum_{k=0}^{n-1} \sum_{i=1}^{k} \left\| \delta_t^{(1)} v^k \right\|_2^2 \leq nE^0, \end{aligned} \tag{3.47}$$

for each $n \in I_{N-1}$. The conclusion of this result is reached now. □

Lemma 3.15 (Pen-Yu [21])**.** *Let $(\omega^n)_{n=0}^N, (\rho^n)_{n=0}^N \subseteq \mathcal{V}_h$ be nonnegative functions, and let $C \geq 0$ be such that*

$$\omega^n \leq \rho^n + C\tau \sum_{k=0}^{n-1} \omega^k, \quad \forall n \in \overline{I}_N. \tag{3.48}$$

Then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in \overline{I}_N$.

Lemma 3.16 (Macías-Díaz [68])**.** *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and let $(u^n)_{n=0}^N$, $(v^n)_{n=0}^N$ and $(R^n)_{n=0}^N$ be sequences in $\mathcal{V}_h$. Let $\varepsilon^n = v^n - u^n$ and $\widetilde{G}^n = \delta_{v,t} G(v^n) - \delta_{w,t} G(w^n)$ for each $n \in \overline{I}_N$. There are $C_2, C_3 \in \mathbb{R}^+$ depending only on G with*

$$2\tau \left| \sum_{n=1}^{k} \langle R^n - \widetilde{G}^n, \delta_t^{(1)} \varepsilon^n \rangle \right| \leq 2\tau \sum_{n=0}^{k} \|R^n\|_2^2 + C_2 \|\varepsilon^0\|_2^2 + C_3\tau \sum_{n=0}^{k} \|\delta_t \varepsilon^n\|_2^2, \tag{3.49}$$

for each $k \in I_{N-1}$.

In the following, the constants C_1, C_2 and C_3 are as in the previous lemma.

Theorem 3.17 (Stability)**.** *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and suppose that $\tau, h \in \mathbb{R}^+$ satisfy $C_3\tau < 1$. Let $\mathbf{v} = (v^n)_{n=0}^N$ and $\mathbf{w} = (w^n)_{n=0}^N$ be solutions of (3.27) for (ϕ_v, ψ_v) and (ϕ_w, ψ_w), respectively, and let $\varepsilon^n = v^n - w^n$ for each $n \in \overline{I}_N$. Then there exist $C_4, C_5 \in \mathbb{R}^+$ independent of $\mathbf{v}$ and $\mathbf{w}$ such that*

$$\|\delta_t \varepsilon^n\|_2^2 \leq C_4 \left(\|\delta_t \varepsilon^0\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^0\|_2^2 + \|\varepsilon^0\|_2^2 \right) e^{C_5 n\tau}, \quad \forall n \in I_{N-1}. \tag{3.50}$$

Proof. Beforehand, let $\eta_0 \in \mathbb{R}^+$ satisfy $C_3\tau < \eta_0 < 1$. Note that $(\varepsilon^n)_{n=0}^N$ satisfies the initial-boundary-value problem

$$\begin{gathered} \delta_t^{(2)} \varepsilon^n + \gamma \delta_t^{(1)} \varepsilon^n - \mathbf{H}\Lambda_h \mu_t^{(1)} \varepsilon^n + \delta_{v,t}^{(1)} G(v^n) - \delta_{w,t}^{(1)} G(w^n) = 0, \quad \forall n \in I_{N-1}, \\ \text{such that } \begin{cases} \varepsilon_j^0 = \phi_v(x_j) - \phi_w(x_j), & \forall j \in J, \\ \delta_t \varepsilon_j^0 = \psi_v(x_j) - \psi_w(x_j), & \forall j \in J, \\ \varepsilon_j^n = 0, & \forall (j,n) \in \partial J \times \overline{I}_N. \end{cases} \end{gathered} \tag{3.51}$$

For the sake of convenience, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)} G(v_j^n) - \delta_{w,t} G(w_j^n)$ for each $(j,n) \in J \times I_{N-1}$. It is easy to see that

$$\left\langle \delta_t^{(2)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n \right\rangle = \frac{1}{2\tau}\left(\|\delta_t\varepsilon^n\|_2^2 - \|\delta_t\varepsilon^{n-1}\|_2^2\right), \quad \forall n \in I_{N-1}, \tag{3.52}$$

$$\left\langle -\mathbf{H}\Lambda_h\mu_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n \right\rangle = \frac{1}{2\tau}\left[\mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^n\|_2^2 - \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^{n-1}\|_2^2\right], \quad \forall n \in I_{N-1}, \tag{3.53}$$

$$|2\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle| \leq C_1\left(\|\delta_t\varepsilon^n\|_2^2 + \|\delta_t\varepsilon^{n-1}\|_2^2 + \|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^{n-1}\|_2^2\right), \quad \forall n \in I_{N-1}. \tag{3.54}$$

Letting $k \in I_{N-1}$, taking the inner product of $\delta_t^{(1)}\epsilon^n$ with both sides of the respective difference equation of (3.51)

$$\langle \delta_t^{(2)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle + \gamma\langle \delta_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle - \langle \mathbf{H}\Lambda_h\mu_t^{(1)}\varepsilon^n, \delta_t^{(1)}\varepsilon^n\rangle + \langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{3.55}$$

substituting the identities above

$$\frac{1}{2\tau}\left(\|\delta_t\varepsilon^n\|_2^2 - \|\delta_t\varepsilon^{n-1}\|_2^2\right) + \gamma\|\delta_t^{(1)}\varepsilon^n\|_2^2 + \frac{1}{2\tau}\left[\mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^n\|_2^2 - \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^{n-1}\|_2^2\right] + \langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{3.56}$$

multiplying by 2τ on both sides

$$\|\delta_t\varepsilon^n\|_2^2 - \|\delta_t\varepsilon^{n-1}\|_2^2 + 2\gamma\tau\|\delta_t^{(1)}\varepsilon^n\|_2^2 + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^n\|_2^2 - \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^{n-1}\|_2^2 + 2\tau\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{3.57}$$

calculating then the sum of the resulting identity for all $n \in I_k$

$$\|\delta_t\varepsilon^k\|_2^2 - \|\delta_t\varepsilon^0\|_2^2 + 2\gamma\tau\sum_{i=1}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2 + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^k\|_2^2 - \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^0\|_2^2 + 2\tau\sum_{i=1}^{k}\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle = 0, \tag{3.58}$$

applying Lemma 3.16 with $R^n = 0$ and simplifying algebraically, we obtain

$$\begin{aligned}
\|\delta_t\varepsilon^k\|_2^2 &= \|\delta_t\varepsilon^0\|_2^2 - 2\gamma\tau\sum_{i=1}^{k}\|\delta_t^{(1)}\varepsilon^n\|_2^2 - \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^k\|_2^2 \\
&\quad + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^0\|_2^2 - 2\tau\sum_{i=1}^{k}\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle \\
&\leq \|\delta_t\varepsilon^0\|_2^2 + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^0\|_2^2 + 2\tau\left|\sum_{i=1}^{k}\langle \widetilde{G}^n, \delta_t^{(1)}\varepsilon^n\rangle\right| \\
&\leq \|\delta_t\varepsilon^0\|_2^2 + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^0\|_2^2 + C_2\|\varepsilon^0\|_2^2 + C_3\tau\sum_{n=0}^{k}\|\delta_t\varepsilon^n\|_2^2.
\end{aligned} \tag{3.59}$$

Subtract $C_3\tau\|\delta_t\varepsilon^k\|_2^2$ on both ends of this inequality and notice then that the left-hand side that results satisfies that $(1-\eta_0)\|\delta_t\varepsilon^k\|_2^2 \leq (1-c_3\tau)\|\delta_t\varepsilon^k\|_2^2$.

$$(1-\eta_0)\|\delta_t\varepsilon^k\|_2^2 \leq (1-c_3\tau)\|\delta_t\varepsilon^k\|_2^2 \leq \|\delta_t\varepsilon^0\|_2^2 + \mu_t\|\mathbf{R}\Lambda_h^{\frac{1}{2}}\varepsilon^0\|_2^2 + C_2\|\varepsilon^0\|_2^2 + C_3\tau\sum_{n=0}^{k-1}\|\delta_t\varepsilon^n\|_2^2. \tag{3.60}$$

Dividing next by $1-\eta_0$, by making $C_4 = max\{\frac{1}{1-\eta_0}, \frac{C_2}{1-\eta_0}\}$ and $C_5 = \frac{C_3}{1-\eta_0}$ we get that:

$$\|\delta_t \varepsilon^k\|_2^2 \leq C_4 \left(\|\delta_t \varepsilon^0\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^0\|_2^2 + \|\varepsilon^0\|_2^2 \right) + C_5 \tau \sum_{n=0}^{k-1} \|\delta_t \varepsilon^n\|_2^2. \tag{3.61}$$

The conclusion of this theorem is reached now using Lemma 3.15. □

Under the assumptions of Theorem 3.17, we readily establish the uniqueness of the numerical solutions.

Corollary 3.18. *Let $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and assume that $C_3\tau < 1$ holds. If v and w are solutions of* (3.27) *corresponding to the same set of initial data then $v = w$.*

The proof of the following result is similar to that of Theorem 3.17. We provide only an abridged proof.

Theorem 3.19. *Let $u \in \mathcal{C}^{7,4}_{x,t}(\overline{\Omega})$ be a solution of* (3.3) *with $G \in \mathcal{C}^2(\mathbb{R})$ and $G'' \in L^\infty(\mathbb{R})$, and let $(v^n)_{n=0}^N$ be a solution of* (3.27)*. If $C_3\tau < 1$ then* (3.27) *is convergent of order $\mathcal{O}(\tau^2 + h_1^4 + h_2^4)$.*

Proof. Let $\eta_0 \in \mathbb{R}^+$ be as in the proof of Theorem 3.17, and let R_j^n be the truncation error at the point (x_j, t_n), for each $(j,n) \in \overline{J} \times \overline{I}_N$. Let $\epsilon^n = v^n - u^n$ for each $n \in \overline{I}_N$. Then $(\epsilon^n)_{n=0}^N$ satisfies

$$\begin{gathered}
\delta_t^{(2)} \epsilon^n + \gamma \delta_t^{(1)} \epsilon^n - \mathbf{H}\Lambda_h \mu_t^{(1)} \epsilon^n + \delta_{v,t}^{(1)} G(v^n) - \delta_{u,t}^{(1)} G(u^n) = R^n, \quad \forall n \in I_{N-1}, \\
\text{such that} \begin{cases} \epsilon_j^0 = \delta_t \epsilon_j^0 = 0, & \forall j \in \overline{J}, \\ \epsilon_j^n = 0, & \forall (j,n) \in \partial J \times \overline{I}_N. \end{cases}
\end{gathered} \tag{3.62}$$

Following the proof of Theorem 3.17, let $\widetilde{G}_j^n = \delta_{v,t}^{(1)} G(v_j^n) - \delta_{u,t} G(u_j^n)$ for each $(j,n) \in J \times I_{N-1}$. Proceeding as in the proof of that theorem

$$\begin{aligned}
\|\delta_t \varepsilon^k\|_2^2 &= \|\delta_t \varepsilon^0\|_2^2 - 2\gamma\tau \sum_{i=1}^k \|\delta_t^{(1)} \varepsilon^n\|_2^2 - \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^k\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^0\|_2^2 - 2\tau \sum_{i=1}^k \langle \widetilde{G}^n - R^n, \delta_t^{(1)} \varepsilon^n \rangle \\
&\leq \|\delta_t \varepsilon^0\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^0\|_2^2 + 2\tau \left| \sum_{i=1}^k \langle \widetilde{G}^n - R^n, \delta_t^{(1)} \varepsilon^n \rangle \right| \\
&\leq \|\delta_t \varepsilon^0\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \varepsilon^0\|_2^2 + C_2 \|\varepsilon^0\|_2^2 + 2\tau \sum_{n=0}^k \|R^n\|_2^2 + C_3 \tau \sum_{n=0}^k \|\delta_t \varepsilon^n\|_2^2,
\end{aligned} \tag{3.63}$$

and defining C_4 and C_5 as follows $C_4 = max\{\frac{C_2}{1-\eta_0}, \frac{2}{1-\eta_0}\}$, $C_5 = \frac{C_3}{1-\eta_0}$, then

$$\|\delta_t \epsilon^k\|_2^2 \leq C_4 \left(\|\delta_t \epsilon^0\|_2^2 + \mu_t \|\mathbf{R}\Lambda_h^{\frac{1}{2}} \epsilon^0\|_2^2 + \|\epsilon^0\|_2^2 + \tau \sum_{n=0}^k \|R^n\|_2^2 \right) + C_5 \tau \sum_{n=0}^k \|\delta_t \epsilon^n\|_2^2, \quad \forall k \in \overline{I}_{N-1}. \tag{3.64}$$

Let C be the constant of Lemma 3.16, and let $C_6 = C_4 C^2 e^{C_5 T} T$. Lemmas 3.16 and 3.15, and the initial-boundary conditions in (3.62) imply now that $\|\delta_t \epsilon^k\|_2^2 \leq C_6(\tau^2 + h_1^4 + h_2^4)$ for each $k \in \overline{I}_{N-1}$. It

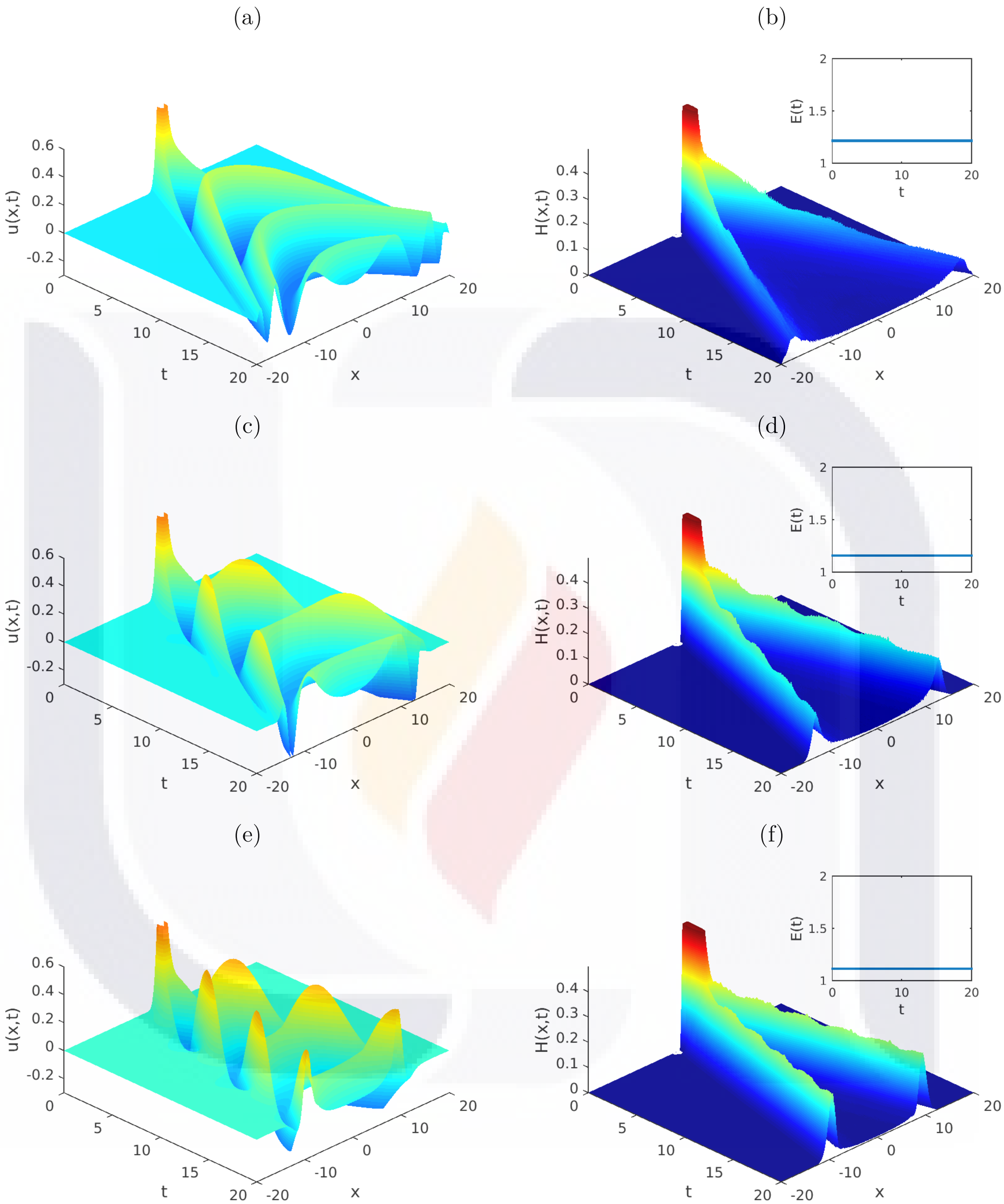

Figure 3.1: Numerical solution (left column) and numerical energy density (right column) of the one-dimensional problem (3.3) versus x and t over the domain $\Omega = (-20, 20) \times (0, 20)$. The set of initial conditions (3.66) were employed, together with $\gamma = 0$ and various orders of differentiation, namely, $\alpha_1 = 2$ (top row), $\alpha_1 = 1.7$ (middle row) and $\alpha_1 = 1.4$ (bottom row). The insets represent the corresponding dynamics of the total energy of the system. The approximations were obtained using the scheme (3.27) with $h_1 = 0.1$ and $\tau = 0.01$.

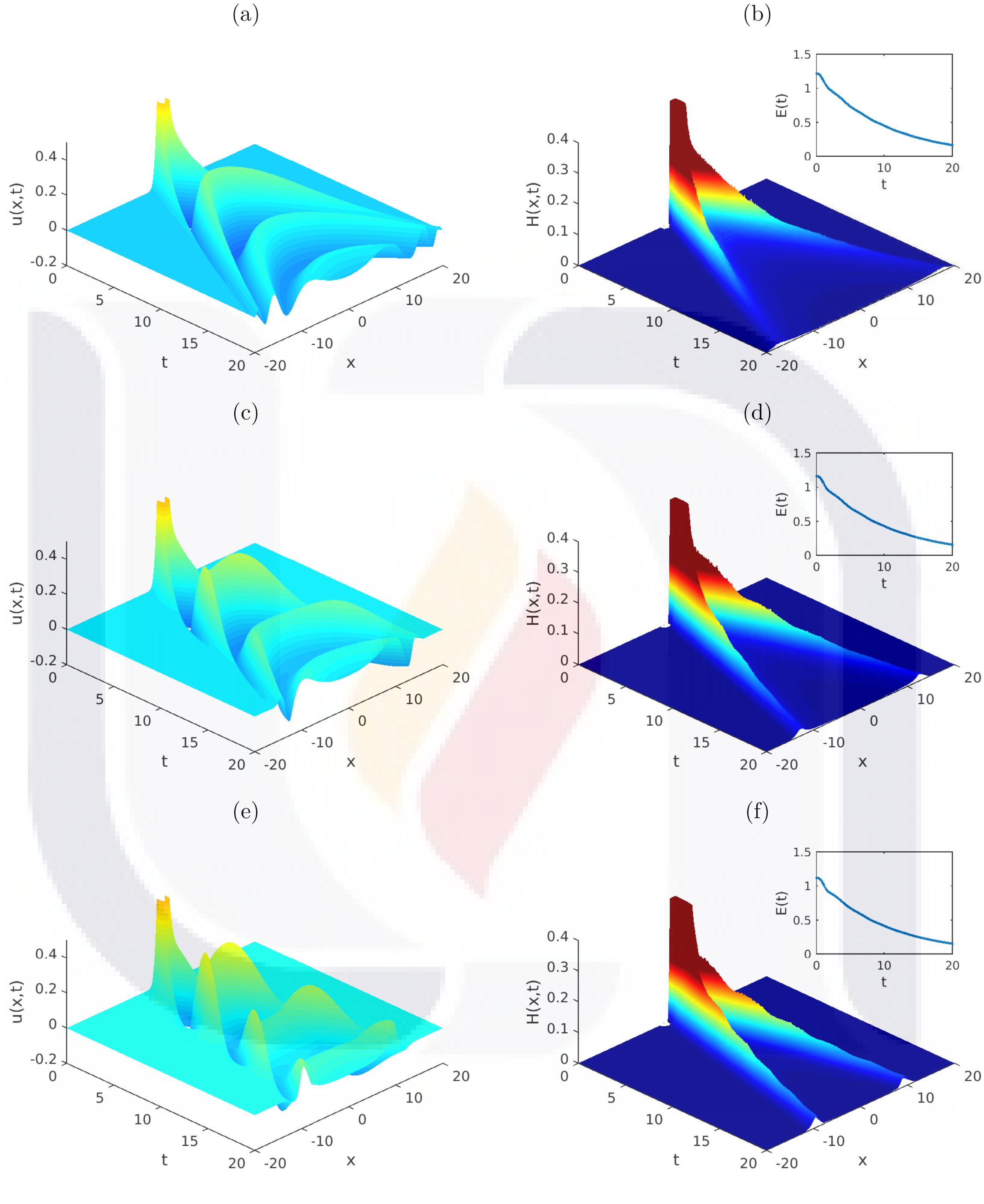

Figure 3.2: Numerical solution (left column) and numerical energy density (right column) of the one-dimensional problem (3.3) versus x and t over the domain $\Omega = (-20, 20) \times (0, 20)$. The set of initial conditions (3.66) were employed, together with $\gamma = 0.1$ and various orders of differentiation, namely, $\alpha_1 = 2$ (top row), $\alpha_1 = 1.7$ (middle row) and $\alpha_1 = 1.4$ (bottom row). The insets represent the corresponding dynamics of the total energy of the system. The approximations were obtained using the scheme (3.27) with $h_1 = 0.1$ and $\tau = 0.01$.

follows that

$$\|\epsilon^k\|_2^2 \leq 2\|\epsilon^0\|_2^2 + 2T\tau \sum_{n=0}^{k-1} \|\delta_t \epsilon^k\|_2^2 \leq 2T^2 C_6(\tau^2 + h_1^4 + h_2^4)^2, \quad \forall k \in I_N. \tag{3.65}$$

It follows that there is $K \geq 0$ such that $\|\epsilon^k\|_2 \leq K(\tau^2 + h_1^4 + h_2^4)$ for each $k \in I_N$, as proposed. □

Finally, we provide some illustrative simulations to show the energetic performance of our methodology.

Example 3.20. Consider the one-dimensional form of problem (3.3) and $G(u) = 1 - \cos u$, defined on the spatial interval $B = [-20, 20]$ and with $T = 20$. Clearly, the resulting problem is governed by a damped fractional sine-Gordon equation. As initial profile, we will use

$$\phi(x) = \exp(-x^2), \quad \forall x \in \overline{B}, \tag{3.66}$$

and zero initial velocities. Computationally, we will employ the method (3.27) with $h = 0.1$ and $\tau = 0.01$. Figure 3.1 shows the solution of the system when $\gamma = 0$ (left column) and the corresponding energy density (right column) versus x and t. Various values of α were used to that effect, namely, $\alpha = 2$ (top row), $\alpha = 1.7$ (middle row) and $\alpha = 1.4$ (bottom row). The insets depict the corresponding dynamics of the total energy of the system. The results for the non-fractional system are clearly in good agreement with those obtained in [73, 74]. Moreover, the energy of the system is conserved at all time, in agreement with Theorem 3.11. Figure 3.2 shows the same set of simulations using $\gamma = 0.1$. The results show that the method is a dissipative technique in this case, in agreement with Theorem 3.11.

Conclusions and discussions

Chapter 1 In this chapter, we considered the sine-Gordon wave equation, one of the classical nonlinear wave equations from quantum mechanics. The model under investigation is defined on the real line, and it considers the presence of a nonlinear potential function according to mathematical physics. We show here that the model possesses energy functionals which are preserved under suitable assumptions on the boundary conditions and the parameters of the model.

Since it is difficult to solve the equation under some conditions, we chose a numerical method to approximate a solution proposed by L. Vázquez, which is explicit and we proved that it conserves the energy, the method uses finite-difference operators of quadratic order, then we showed that the operator used to approximate the second derivate has a square root under some assumptions which helped us to prove that the discrete energy is conserved in concordance with the continuous case. We established that the method is consistent, stable and convergent.

Chapter 2 In this chapter, we investigated a fractional generalization of the sine-Gordon equation, considering damping, a general value for G and multiple spatial dimensions. This model is defined on a closed and bounded interval of $\mathbb{R}^{p+1}$, and it considers the presence of a general nonlinear potential function that generalizes many particular models from mathematical physics, including the sine-Gordon equation from Chapter 1. Moreover, we considered a space-fractional extension of the wave equation using Riesz fractional derivatives of orders in $(1, 2]$. We show here that the multidimensional model under investigation possesses energy functionals which are preserved under suitable assumptions on the boundary conditions and the parameters of the model. We designed a numerical technique that is capable to preserve the proposed discrete energy function.

The numerical method is based on the use of fractional centered differences, which provide second-order consistent approximations of fractional-order derivatives. Using operator theory, we show that the multidimensional discrete fractional Laplacian is a positive and self-adjoint operator, then the existence of a square root readily follows. This fact is employed then to propose a discrete energy functional of the numerical method which, under suitable conditions on the boundary conditions and the model parameters, is preserved at each discrete time. Additionally, the method is a second-order consistent discretization of the problem under investigation, and the simulations provided in this work show that the energy is conserved throughout time when the assumptions of the relevant theorems on energy preservation are satisfied. For the sake of convenience, a computer implementation of our method in the one-dimensional case is provided.

Chapter 3 In this chapter, we considered a two-dimensional damped fractional extension of the classical nonlinear two-dimensional damped wave equation. The model under investigation is defined on a closed and bounded interval of the real line, and it considers the presence of a general nonlinear potential function. The method provides fourth-order in space and second-order in time consistent approximations of fractional-order derivatives. The reason we use two spatial dimensions falls in the growing complexity every time we added a dimension, unlike Chapter 2 when we were able to provide a general scenario. The properties found in Chapter 2 are present in this method as well, with the difference that this method is implicit, we provided a proof of the existence of a solution under certain assumptions. Additionally, the method is a high-order consistent discretization of the problem under investigation, and the simulations provided in this work show that the energy is conserved throughout time when the assumptions of the relevant theorems on energy preservation are satisfied.

It is important to mention that the methodology proposed in this chapter has the advantage over the method of Chapter 2, of having a higher rate of convergence, we were able to do this by applying a compact operator to the equation. The conditions to guarantee the stability of the present scheme are also relatively flexible, and the positivity of the energy functionals are always guaranteed.

Bibliography

[1] Jean Le Rond D'Alembert. Recherches sur la courbe que forme une corde tendue mise en vibration. *Histoire de l'Académie royale des sciences et belles -lettres de Berlin*, pages 214–219, 1747.

[2] Goldstine Neumann. Numerical inverting of matrices of high order. *Bulletin of the American Mathematical Society*, 53(11):1021–1099, 1947.

[3] F Geniet and J Leon. Energy transmission in the forbidden band gap of a nonlinear chain. *Physical Review Letters*, 89(13):134102, 2002.

[4] F Geniet and J Leon. Nonlinear supratransmission. *Journal of Physics: Condensed Matter*, 15(17):2933, 2003.

[5] Ramaz Khomeriki, Stefano Lepri, and Stefano Ruffo. Nonlinear supratransmission and bistability in the Fermi-Pasta-Ulam model. *Physical Review E*, 70(6):066626, 2004.

[6] J Leon and A Spire. Gap soliton formation by nonlinear supratransmission in Bragg media. *Physics Letters A*, 327(5):474–480, 2004.

[7] Ramaz Khomeriki and Jerome Leon. Bistability in the sine-Gordon equation: The ideal switch. *Physical Review E*, 71(5):056620, 2005.

[8] Vasily E Tarasov and Elias C Aifantis. Non-standard extensions of gradient elasticity: Fractional non-locality, memory and fractality. *Communications in Nonlinear Science and Numerical Simulation*, 22(1):197–227, 2015.

[9] Georgy Alfimov, Teresa Pierantozzi, and Luis Vázquez. Numerical study of a fractional sine-Gordon equation. *Fractional differentiation and its applications*, 4:153–162, 2004.

[10] Rudolf Gorenflo, Francesco Mainardi, Daniele Moretti, Gianni Pagnini, and Paolo Paradisi. Discrete random walk models for space–time fractional diffusion. *Chemical physics*, 284(1):521–541, 2002.

[11] Vasily E Tarasov and George M Zaslavsky. Conservation laws and Hamilton's equations for systems with long-range interaction and memory. *Communications in Nonlinear Science and Numerical Simulation*, 13(9):1860–1878, 2008.

[12] RC Koeller. Applications of fractional calculus to the theory of viscoelasticity. *ASME, Transactions, Journal of Applied Mechanics(ISSN 0021-8936)*, 51:299–307, 1984.

[13] YZ Povstenko. Theory of thermoelasticity based on the space-time-fractional heat conduction equation. *Physica Scripta*, 2009(T136):014017, 2009.

[14] Enrico Scalas, Rudolf Gorenflo, and Francesco Mainardi. Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and its Applications*, 284(1):376–384, 2000.

[15] Walter G Glöckle and Theo F Nonnenmacher. A fractional calculus approach to self-similar protein dynamics. *Biophysical Journal*, 68(1):46–53, 1995.

[16] Victor Namias. The fractional order Fourier transform and its application to quantum mechanics. *IMA Journal of Applied Mathematics*, 25(3):241–265, 1980.

[17] V. G. Pimenov, A. S. Hendy, and R. H. De Staelen. On a class of non-linear delay distributed order fractional diffusion equations. *Journal of Computational and Applied Mathematics*, 318:433–443, 2017.

[18] Om P Agrawal. Formulation of Euler–Lagrange equations for fractional variational problems. *Journal of Mathematical Analysis and Applications*, 272(1):368–379, 2002.

[19] Zhang Fei and Luis Vázquez. Two energy conserving numerical schemes for the sine-Gordon equation. *Applied Mathematics and Computation*, 45(1):17–30, 1991.

[20] Edmond Bour. *Théorie de la déformation des surfaces*. 1862.

[21] Kuo Pen-Yu. Numerical methods for incompressible viscous flow. *Scientia Sinica*, 20:287–304, 1977.

[22] Fernanda PC Benetti, Ana C Ribeiro-Teixeira, Renato Pakter, and Yan Levin. Nonequilibrium stationary states of 3d self-gravitating systems. *Physical review letters*, 113(10):100602, 2014.

[23] Alessandro Campa, Thierry Dauxois, and Stefano Ruffo. Statistical mechanics and dynamics of solvable models with long-range interactions. *Physics Reports*, 480(3):57–159, 2009.

[24] Adriana Miele and Job Dekker. Long-range chromosomal interactions and gene regulation. *Molecular biosystems*, 4(11):1046–1057, 2008.

[25] Rudolf Hilfer. *Applications of fractional calculus in physics*. World Scientific, Singapore, 2000.

[26] Vasily E Tarasov and George M Zaslavsky. Fractional dynamics of systems with long-range interaction. *Communications in Nonlinear Science and Numerical Simulation*, 11(8):885–898, 2006.

[27] Vasily E Tarasov and George M Zaslavsky. Fractional dynamics of coupled oscillators with long-range interaction. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 16(2):023110, 2006.

[28] J E Macías-Díaz. Numerical study of the process of nonlinear supratransmission in Riesz space-fractional sine-Gordon equations. *Communications in Nonlinear Science and Numerical Simulation*, 46:89–102, 2017.

[29] J E Macías-Díaz. Persistence of nonlinear hysteresis in fractional models of Josephson transmission lines. *Communications in Nonlinear Science and Numerical Simulation*, 53:31–43, 2017.

[30] J E Macías-Díaz. Numerical simulation of the nonlinear dynamics of harmonically driven Riesz-fractional extensions of the Fermi–Pasta–Ulam chains. *Communications in Nonlinear Science and Numerical Simulation*, 55:248–264, 2018.

[31] Vasily E Tarasov. Fractional generalization of gradient and Hamiltonian systems. *Journal of Physics A: Mathematical and General*, 38(26):5929, 2005.

[32] Vasily E Tarasov. Continuous limit of discrete systems with long-range interaction. *Journal of Physics A: Mathematical and General*, 39(48):14895, 2006.

[33] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A fractional porous medium equation. *Advances in Mathematics*, 226(2):1378–1409, 2011.

[34] Juan Luis Vázquez. Nonlinear diffusion with fractional Laplacian operators. In *Nonlinear partial differential equations*, pages 271–298. Springer, 2012.

[35] Matteo Bonforte and Juan Luis Vázquez. A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains. *Archive for Rational Mechanics and Analysis*, 218(1):317–362, 2015.

[36] Luís Pinto and Ercília Sousa. Numerical solution of a time-space fractional Fokker Planck equation with variable force field and diffusion. *Communications in Nonlinear Science and Numerical Simulation*, 50:211–228, 2017.

[37] Li Zhu and Qibin Fan. Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW. *Communications in Nonlinear Science and Numerical Simulation*, 18(5):1203–1213, 2013.

[38] AH Bhrawy and MA Zaky. Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations. *Computers & Mathematics with Applications*, 73(6):1100–1117, 2017.

[39] Hammad Khalil and Rahmat Ali Khan. A new method based on legendre polynomials for solutions of the fractional two-dimensional heat conduction equation. *Computers & Mathematics with Applications*, 67(10):1938–1953, 2014.

[40] S Sahoo and S Saha Ray. Improved fractional sub-equation method for (3+ 1)-dimensional generalized fractional KdV–Zakharov–Kuznetsov equations. *Computers & Mathematics with Applications*, 70(2):158–166, 2015.

[41] Shiping Chen, Fawang Liu, and Kevin Burrage. Numerical simulation of a new two-dimensional variable-order fractional percolation equation in non-homogeneous porous media. *Computers & Mathematics with Applications*, 68(12):2133–2141, 2014.

[42] Yuesheng Luo, Xiaole Li, and Cui Guo. Fourth-order compact and energy conservative scheme for solving nonlinear Klein-Gordon equation. *Numerical Methods for Partial Differential Equations*, in press:doi:10.1002/num.22143, 2017.

[43] Daisuke Furihata. Finite-difference schemes for nonlinear wave equation that inherit energy conservation property. *Journal of Computational and Applied Mathematics*, 134(1):37–57, 2001.

[44] J E Macías-Díaz and J Villa-Morales. A deterministic model for the distribution of the stopping time in a stochastic equation and its numerical solution. *Journal of Computational and Applied Mathematics*, 318:93–106, 2017.

[45] Pengde Wang and Chengming Huang. An energy conservative difference scheme for the nonlinear fractional Schrödinger equations. *Journal of Computational Physics*, 293:238–251, 2015.

[46] J E Macías-Díaz. Existence and uniqueness of positive and bounded solutions of a discrete population model with fractional dynamics. *Discrete Dynamics in Nature and Society*, 2017, 2017.

[47] Helen Christodoulidi, Constantino Tsallis, and Tassos Bountis. Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics. *EPL (Europhysics Letters)*, 108(4):40006, 2014.

[48] Helen Christodoulidi, Tassos Bountis, Constantino Tsallis, and Lambros Drossos. Chaotic behavior of the Fermi–Pasta–Ulam β–model with different ranges of particle interactions. *Journal of Statistical Mechanics*, 12(12):123206, 2016.

[49] Nick Laskin. Fractional Schrödinger equation. *Physical Review E*, 66(5):056108, 2002.

[50] Avner Friedman. *Foundations of modern analysis*. Courier Corporation, New York, 1970.

[51] J E Macías-Díaz. A structure-preserving method for a class of nonlinear dissipative wave equations with Riesz space-fractional derivatives. *Journal of Computational Physics*, submitted for publication, 2017.

[52] Cem Çelik and Melda Duman. Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. *Journal of Computational Physics*, 231(4):1743–1750, 2012.

[53] Alexander I Saichev and George M Zaslavsky. Fractional kinetic equations: solutions and applications. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 7(4):753–764, 1997.

[54] Rudolf Gorenflo and Francesco Mainardi. Fractional diffusion processes: probability distributions and continuous time random walk. In *Processes with Long-Range Correlations*, pages 148–166. Springer, 2003.

[55] Nick Laskin and G Zaslavsky. Nonlinear fractional dynamics on a lattice with long range interactions. *Physica A: Statistical Mechanics and its Applications*, 368(1):38–54, 2006.

[56] Guo Ben-Yu, Pedro J Pascual, María J Rodriguez, and Luis Vázquez. Numerical solution of the sine-Gordon equation. *Applied Mathematics and Computation*, 18(1):1–14, 1986.

[57] K Djidjeli, WG Price, and EH Twizell. Numerical solutions of a damped sine-Gordon equation in two space variables. *Journal of Engineering Mathematics*, 29(4):347–369, 1995.

[58] Eitan Tadmor and Weigang Zhong. Energy-preserving and stable approximations for the two-dimensional shallow water equations. In *Mathematics and computation, a contemporary view*, pages 67–94. Springer, 2008.

[59] Maohua Ran and Chengjian Zhang. A linearly implicit conservative scheme for the fractional nonlinear Schrödinger equation with wave operator. *International Journal of Computer Mathematics*, 93(7):1103–1118, 2016.

[60] Ronald E Mickens and P M Jordan. A positivity-preserving nonstandard finite difference scheme for the damped wave equation. *Numerical Methods for Partial Differential Equations*, 20(5):639–649, 2004.

[61] Yue Cheng, Fengyan Li, Jianxian Qiu, and Liwei Xu. Positivity-preserving DG and central DG methods for ideal MHD equations. *Journal of Computational Physics*, 238:255–280, 2013.

[62] Sibusiso Mabuza, Dmitri Kuzmin, Sunčica Čanić, and Martina Bukač. A conservative, positivity preserving scheme for reactive solute transport problems in moving domains. *Journal of Computational Physics*, 276:563–595, 2014.

[63] Bernard Parent. Positivity-preserving dual time stepping schemes for gas dynamics. *Journal of Computational Physics*, 361:391–411, 2018.

[64] Wendi Qin, Deqiong Ding, and Xiaohua Ding. Two boundedness and monotonicity preserving methods for a generalized Fisher-KPP equation. *Applied Mathematics and Computation*, 252:552–567, 2015.

[65] Yanyan Yu, Weihua Deng, and Yujiang Wu. Positivity and boundedness preserving schemes for space–time fractional predator–prey reaction–diffusion model. *Computers & Mathematics with Applications*, 69(8):743–759, 2015.

[66] Jorge Macias-Diaz and J Villa. Simple numerical method to study traveling-wave solutions of a diffusive problem with nonlinear advection and reaction. *Numerical Methods for Partial Differential Equations*, 29(5):1694–1708, 2013.

[67] J E Macías-Díaz, AS Hendy, and RH De Staelen. A pseudo energy-invariant method for relativistic wave equations with riesz space-fractional derivatives. *Computer Physics Communications*, 224:98–107, 2018.

[68] J E Macías-Díaz. A structure-preserving method for a class of nonlinear dissipative wave equations with Riesz space-fractional derivatives. *Journal of Computational Physics*, 315(1):40–58, 2017.

[69] J E Macías-Díaz. An explicit dissipation-preserving method for Riesz space-fractional nonlinear wave equations in multiple dimensions. *Communications in Nonlinear Science and Numerical Simulation*, 59:67–87, 2018.

[70] Xuan Zhao, Zhi-zhong Sun, and Zhao-peng Hao. A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation. *SIAM Journal on Scientific Computing*, 36(6):A2865–A2886, 2014.

[71] Changpin Li and Fanhai Zeng. Finite difference methods for fractional differential equations. *International Journal of Bifurcation and Chaos*, 22(04):1230014, 2012.

[72] Georgios D Akrivis. Finite difference discretization of the cubic Schrödinger equation. *IMA Journal of Numerical Analysis*, 13(1):115–124, 1993.

[73] J E Macías-Díaz. On the solution of a Riesz space-fractional nonlinear wave equation through an efficient and energy-invariant scheme. *International Journal of Computer Mathematics*, accepted for publication:1–25, 2018.

[74] Vladimir G Ivancevic, Tijana T Ivancevic, et al. Sine-Gordon solitons, kinks and breathers as physical models of nonlinear excitations in living cellular structures. *Journal of Geometry and Symmetry in Physics*, 31:1–56, 2013.